ABEL'S FORMULA

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Let $y_1(x)$ and $y_2(x)$ be two solutions of

$$y'' + p(x)y' + q(x)y = 0, \tag{1}$$

and let

$$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x), \tag{2}$$

be their Wronskian. We can compute the derivative $W'(x)$ as

$$\begin{aligned} W'(x) &= y_1'(x)y_2'(x) + y_1(x)y_2''(x) - y_1''(x)y_2(x) - y_1'(x)y_2'(x) \\ &= y_1(x)y_2''(x) - y_1''(x)y_2(x). \end{aligned} \tag{3}$$

Since y_1 and y_2 are solutions of (1), we have

$$y_1''(x) = -p(x)y_1'(x) - q(x)y_1(x), \qquad \text{and} \qquad y_2''(x) = -p(x)y_2'(x) - q(x)y_2(x), \tag{4}$$

and substituting these into (3), we get

$$\begin{aligned} W'(x) &= y_1(x)[-p(x)y_2'(x) - q(x)y_2(x)] - [-p(x)y_1'(x) - q(x)y_1(x)]y_2(x) \\ &= -p(x)y_1(x)y_2'(x) - q(x)y_1(x)y_2(x) + p(x)y_1'(x)y_2(x) + q(x)y_1(x)y_2(x) \\ &= -p(x)y_1(x)y_2'(x) + p(x)y_1'(x)y_2(x) \\ &= -p(x)[y_1(x)y_2'(x) - y_1'(x)y_2(x)] \\ &= -p(x)W(x). \end{aligned} \tag{5}$$

In other words, the Wronskian satisfies the first order linear equation

$$W'(x) + p(x)W(x) = 0. \tag{6}$$

This fact is known as *Abel's theorem*. We can easily solve (6), and derive

$$W(x) = W(x_0)\exp\int_{x_0}^{x} p(t)\,\mathrm{d}t, \tag{7}$$

the formula known as *Abel's formula* or *Abel's identity*. In particular, if the coefficient $p(x)$ is constant, then

$$W(x) = W(x_0)e^{(x-x_0)p}. \tag{8}$$

An important consequence of Abel's formula is that *the Wronskian of two solutions of* (1) *is either zero everywhere, or nowhere zero.*

Example 1. We know that $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are solutions to $y'' + y = 0$. Since $p = 0$ in this case, in light of Abel's formula, the Wronskian $W(x)$ of y_1 and y_2 must be a constant. We confirm it by explicit computation:

$$W(x) = \cos x(\sin x)' - (\cos x)'\sin x = \cos^2 x + \sin^2 x = 1. \tag{9}$$

Example 2. The functions $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions to $y'' - 2y' + y = 0$. Since $p = -2$, we have $W(x) = ce^{2x}$ for some constant c. Explicit computation gives

$$W(x) = e^x(xe^x)' - (e^x)'e^x = e^x(e^x + xe^x) - xe^{2x} = e^{2x}, \tag{10}$$

so $c = 1$.

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