PRACTICE PROBLEMS FOR THE FINAL EXAM

MATH 249 WINTER 2015

1. Compute the following integrals.
   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4+1}$.
   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x \, \mathrm{d}x}{x^2+1}$.
   (c) $\displaystyle\int_{-\infty}^{\infty} \frac{x \cos x \, \mathrm{d}x}{x^2+1}$.
   (d) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 x \, \mathrm{d}x}{x^2}$.

   Hint: The standard contour of the semicircle in the upper half plane should work, except (d), in which case an indented semicircle can be used. In (c) one needs to use Jordan's lemma or an equivalent argument.
2. Let $\Omega \subset \mathbb{C}$ be an open set, and suppose that $f \in \mathscr{C}(\Omega)$ satisfies
$$\int_{\partial D} f(z) \, \mathrm{d}z = 0,$$
for each disk D whose closure $\bar{D}$ is contained in Ω. Prove that f is holomorphic in Ω. *Hint*: Try to modify the proof of Theorem 34 (Morera's theorem) or that of Theorem 13 from the notes.
3. Prove that an isolated singularity of a holomorphic function f is removable if the real part of f is bounded from below. *Hint*: The trick with exponentiation might work.
4. Show that f and e^f cannot have a common pole.
5. Prove that the residue of $\exp(z + \frac{1}{z})$ at 0 is $\sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}$.
6. Let $f \in \mathscr{C}(\Omega)$ with Ω an open set, and let $f^n \in \mathscr{O}(\Omega)$, where n is a positive integer. Show that $f \in \mathscr{O}(\Omega)$. *Hint*: If $g = f^n$ is holomorphic then $g(z) = (z-c)^N h(z)$ for each c in the domain of g, and for some nonnegative integer N. What happens when you take the n-th root of g?
7. Let Ω be a convex open set, and let $u \in \mathscr{C}^2(\Omega)$ be a real valued function satisfying
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{in} \quad \Omega.$$
Show that there is a real valued function $v \in \mathscr{C}^2(\Omega)$ such that $f = u + iv \in \mathscr{O}(\Omega)$. *Hint*: Supposing that $f = u + iv$ is holomorphic, can you write f' in terms of the partial derivatives u_x and u_y only? If you can, turn the obtained expression $f' = G(u_x, u_y)$

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into a differential equation and solve for f. What condition on $G(u_x, u_y)$ do you need to verify in order to integrate it?

8. Let f be a function meromorphic in the open disk D_r, with only one pole at $z_0 \in D_r \setminus \{0\}$. Show that
$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0,$$
where $\{a_n\}$ are the coefficients of the Maclaurin series of f. *Hint*: Consider the Laurent decomposition of f in the annulus $\{\rho < |z| < r\}$, where $\rho = |z_0|$. Would this decomposition be valid in $D_r \setminus \{z_0\}$? Now expand f around 0 in its Maclaurin series.