PRACTICE PROBLEMS FOR THE MIDTERM (WITH HINTS)

MATH 249 WINTER 2015

1. Where in the complex plane are the following functions complex differentiable?
   (a) $f(x+iy) = x^2y^3 + ixy^2$.
   (b) $f(x+iy) = \sin^2(x+y) + i\cos^2(x+y)$.
   (c) $f(z) = \sin(z\bar{z})$.
   Hint: The Cauchy-Riemann equations, and the definition of complex derivative.
2. Show that if f is holomorphic in some convex open set $\Omega \subset \mathbb{C}$ and takes only *real* values then f must be constant in Ω. *Hint*: The Cauchy-Riemann equations.
3. Show that if f is a holomorphic function in some convex open set $\Omega \subset \mathbb{C}$ and $|f| = 1$ in Ω then f must be constant in Ω. *Hint*: The Cauchy-Riemann equations.
4. Find an example of a power series that converges pointwise but not uniformly in $\mathbb{D}$. *Hint*: $\sum z^n$.
5. Produce a power series whose convergence radius is 1, which converges uniformly in $\mathbb{D}$. *Hint*: Choose a_n in $\sum a_n z^n$ so that $\sum |a_n| < \infty$.
6. Determine the convergence radii of the following power series.
   (a) $(\log 2)^2 z^2 + (\log 3)^2 z^3 + \ldots + (\log n)^2 z^n + \ldots$.
   (b) $1 + z + z^2 + z^6 + \ldots + z^{n!} + \ldots$.
   (c) $1 + z + 3z^2 + 3z^3 + 3^2 z^4 + 3^2 z^5 + 3^3 z^6 + 3^3 z^7 + \ldots$.
   (d) $(\sin 1)z + (\sin 2)z^2 + \ldots + (\sin n)z^n + \ldots$.
   (e) $1 + z + 9z^2 + z^3 + \ldots + (2 + (-1)^n)^n z^n + \ldots$.
   Hint: Ratio test for (a) and the Cauchy-Hadamard formula for the rest.
7. By applying both the ratio test and the Cauchy-Hadamard formula to a suitable power series, show that
$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$
   Hint: Choose a power series so that the Cauchy-Hadamard formula leads to $\limsup \sqrt[n]{n}$.
8. Show that if f is expressible as
$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$
   for $|z| < R$, with all coefficients a_n *real*, then $\overline{f(z)} = f(\bar{z})$ for $|z| < R$. *Hint*: Truncate the series, and then apply $\overline{\lim w_n} = \lim \bar{w}_n$.
9. Develop $f(z) = \frac{1}{z^2+z-1}$ into a power series centred at 0. *Hint*: Partial fractions.

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10. Show that for every $r > 0$ there exists N such that for any $n \geq N$ the polynomial
$$p(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!},$$
does not have any zeroes in $D_r = \{z \in \mathbb{C} : |z| < r\}$. *Hint*: What happens to $p(z)$ as $n \to \infty$? Can you say it converges uniformly in D_r? Does the limit function has zeroes?
11. Show that if a power series $\sum a_n z^n$ converges to some function $f : \mathbb{C} \to \mathbb{C}$ *uniformly in* $\mathbb{C}$, then $a_n = 0$ for all but finitely many n, and hence f must be a polynomial. *Hint*: Suppose that $a_n \neq 0$ for infinitely many n. Uniform convergence implies that the partial sums f_n of $\sum a_n z^n$ form a sequence that is uniformly Cauchy in $\mathbb{C}$. Can you make $|f_n(z) - f_m(z)|$ large by choosing z large?