POWER SERIES

TSOGTGEREL GANTUMUR

Contents

1. Uniform convergence

In this section, we fix a nonempty set $G \subset \mathbb{C}$, and consider sequences of complex valued functions defined on G. By a *sequence of functions* we mean simply an assignment of a function $f_n : G \to \mathbb{C}$ to each index n, with the latter usually having positive integers as values. We denote this sequence by $\{f_1, f_2, \ldots\}$ or $\{f_n\}$, and consider it also as a collection of functions. So for example, $\{f_n\} \subset \mathscr{C}(G)$ would mean that every function in the sequence is continuous.

Definition 1. We say that a sequence $\{f_n\}$ *converges pointwise in* G to a function $f : G \to \mathbb{C}$, if for each $z \in G$, the number sequence $\{f_n(z)\}$ converges to $f(z)$, i.e., $f_n(z) \to f(z)$ as $n \to \infty$.

Example 2. Let $f_n(x) = \arctan(nx)$. Since $\arctan x \to \pm\frac{\pi}{4}$ as $x \to \pm\infty$, the sequence $\{f_n\}$ converges pointwise in $\mathbb{R}$ to f, where

$$f(x) = \begin{cases} \frac{\pi}{4} & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\frac{\pi}{4} & \text{for } x < 0. \end{cases} \tag{1}$$

Pointwise convergence is a very weak kind of convergence. For instance, as we have seen in the preceding example, the pointwise limit of a sequence of continuous functions is *not* necessarily continuous. The notion of uniform convergence is a stronger type of convergence that remedies this deficiency.

Definition 3. We say that a sequence $\{f_n\}$ *converges uniformly in* G to a function $f : G \to \mathbb{C}$, if for any $\varepsilon > 0$, there exists N such that $|f_n(z) - f(z)| \leq \varepsilon$ for any $z \in G$ and all $n \geq N$.

Remark 4. Let us introduce the *uniform norm*

$$\|g\|_G = \sup_{z \in G} |g(z)| \qquad \text{for} \quad g : G \to \mathbb{C}. \tag{2}$$

Then $f_n \to f$ uniformly in G if and only if $\|f_n - f\|_G \to 0$ as $n \to \infty$.

Exercise 5. Show that uniform convergence implies pointwise convergence.

Example 6. The sequence $\{f_n\}$ from Example 2 does *not* converge uniformly to f, since by continuity of the arctangent function, for any given n, however large, and for any given $\delta > 0$, there exists $x > 0$ small enough such that $\arctan(nx) < \delta$. So for instance, for any given n, there is $x > 0$ such that $|f(x) - f_n(x)| > \frac{\pi}{8}$.

Date: February 13, 2015.

Example 7. The sequence $\{f_n\}$ with $f_n(x) = \frac{1}{n}\sin x$ converges uniformly in $\mathbb{R}$ to 0.

Example 8. Let $f_n(z) = 1 + z + z^2 + \ldots + z^n$. Then the standard argument

$$f_n(z) - zf_n(z) = 1 + z + z^2 + \ldots + z^n - (z + z^2 + \ldots + z^n + z^{n+1}) = 1 - z^{n+1}, \tag{3}$$

implies that

$$f_n(z) = \frac{1 - z^{n+1}}{1 - z}. \tag{4}$$

We make a guess that $f_n(z)$ converges to $f(z) = \frac{1}{1-z}$, and compute

$$f(z) - f_n(z) = \frac{1}{1-z} - \frac{1 - z^{n+1}}{1 - z} = \frac{z^{n+1}}{1-z}, \tag{5}$$

which leads to

$$|f(z) - f_n(z)| = \frac{|z^{n+1}|}{|1-z|} \leq \frac{|z|^{n+1}}{|1-z|}. \tag{6}$$

Now if $|z| \leq r$ for some $r < 1$, then $|1 - z| \geq 1 - |z| \geq 1 - r$, and hence

$$|f(z) - f_n(z)| \leq \frac{r^{n+1}}{1-r}. \tag{7}$$

This shows that f_n converges to f uniformly in $\bar{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$, for any fixed $r < 1$.

Theorem 9 (Weierstrass 1861)**.** *Suppose that $\{f_n\} \subset \mathscr{C}(G)$ converges uniformly in G to a function $f : G \to \mathbb{C}$. Then $f \in \mathscr{C}(G)$.*

Proof. We will use the sequential criterion of continuity. Let $z \in G$, and let $\{z_k\} \subset G$ be a sequence converging to z. Then we have

$$f(z_k) - f(z) = f(z_k) - f_n(z_k) + f_n(z_k) - f_n(z) + f_n(z) - f(z), \tag{8}$$

for any n, and hence

$$|f(z_k) - f(z)| \leq |f(z_k) - f_n(z_k)| + |f_n(z_k) - f_n(z)| + |f_n(z) - f(z)|, \tag{9}$$

by the triangle inequality.

Let $\varepsilon > 0$ be given. Then by uniform convergence, there is N such that

$$|f(w) - f_n(w)| \leq \varepsilon, \qquad \text{for all} \quad w \in G, \quad \text{and for all} \quad n \geq N. \tag{10}$$

In particular,

$$|f(z_k) - f(z)| \leq 2\varepsilon + |f_n(z_k) - f_n(z)|, \qquad \text{for all} \quad k, \quad \text{and for all} \quad n \geq N. \tag{11}$$

Now we fix one such n, for example, put $n = N$, and use the continuity of f_n to imply the existence of K with the property that

$$|f_n(z_k) - f_n(z)| \leq \varepsilon, \qquad \text{for all} \quad k \geq K. \tag{12}$$

Finally, this means that

$$|f(z_k) - f(z)| \leq 3\varepsilon, \qquad \text{for all} \quad k \geq K, \tag{13}$$

and since $\varepsilon > 0$ was arbitrary, we infer that f is continuous at z. □

Exercise 10. Find a mistake in the following purported proof.

Claim: If $\{f_n\} \subset \mathscr{C}(G)$ converges pointwise in G to a function $f : G \to \mathbb{C}$, then f is continous in G.

Proof: Let $z \in G$, and let $\{z_k\} \subset G$ be a sequence converging to z. Then as in the preceding proof, we have

$$|f(z_k) - f(z)| \leq |f(z_k) - f_n(z_k)| + |f_n(z_k) - f_n(z)| + |f_n(z) - f(z)|. \tag{14}$$

Since f_n converges pointwise to f, both $|f(z_k)-f_n(z_k)|$ and $|f_n(z)-f(z)|$ tend to 0 as $n\to\infty$. Furthermore, $|f_n(z_k)-f_n(z)|\to 0$ as $k\to\infty$, because the function f_n is continuos. Hence by choosing k and n large enough, we can make the right hand side of (14) arbitrarily small, which means that f is continuous at z.

The definition of uniform convergence involves the limit function f, so we cannot apply this definition without having a candidate for the limit f. However, we will want to construct new functions as the limits of convergent sequences, and in most cases there will not be any limit candidates available. Therefore it is of interest to develop tools that can turn some easily verifiable properties of a sequence into the existence of a limit. In complex analysis, Cauchy's criterion is the main device of such kind.

Definition 11. A sequence $\{f_n\}$ is called *uniformly Cauchy in G*, if for any $\varepsilon>0$, there exists N such that $|f_n(z)-f_m(z)|\leq\varepsilon$ for all $z\in G$ and all $n,m\geq N$.

Remark 12. In terms of the uniform norm, the sequence $\{f_n\}$ being uniformly Cauchy in G is equivalent to the assertion that $\|f_n-f_m\|_G\to 0$ as $n,m\to\infty$.

Exercise 13. Show that if a sequence is uniformly convergent then it is uniformly Cauchy.

Theorem 14 (Cauchy's criterion)**.** *If $\{f_n\}$ is uniformly Cauchy in G, then there exists a function $f:G\to\mathbb{C}$, such that $f_n\to f$ uniformly in G.*

Proof. Let $z\in G$, and let $f_n(z)=a_n+ib_n$, with a_n and b_n real. Since f_n is uniformly Cauchy in G, for any $\varepsilon>0$, there exists N such that $|f_n(z)-f_m(z)|\leq\varepsilon$ whenever $n,m\geq N$. In other words, the complex number sequence $\{f_n(z)\}$ is Cauchy. Then the identity

$$|f_n(z)-f_m(z)|^2=|a_n-a_m|^2+|b_n-b_m|^2, \tag{15}$$

implies that both $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences of numbers, and hence there exist two real numbers a and b, such that $a_n\to a$ and $b_n\to b$ as $n\to\infty$. We set $f(z):=a+ib$. Since $z\in G$ was arbitrary, this procedure defines a function $f:G\to\mathbb{C}$, and by construction, $f_n\to f$ pointwise in G. It remains to show that this convergence is uniform.

Let $\varepsilon>0$. Then there exists N such that $|f_n(z)-f_m(z)|\leq\varepsilon$ for all $z\in G$ and all $n,m\geq N$. Now for any given $z\in G$, by pointwise convergence, there exists $m\geq N$ so large that $|f_m(z)-f(z)|\leq\varepsilon$. Therefore we have

$$|f_n(z)-f(z)|\leq|f_n(z)-f_m(z)|+|f_m(z)-f(z)|\leq 2\varepsilon, \tag{16}$$

whenever $n\geq N$ and $z\in G$, implying that $f_n\to f$ uniformly in G. □

Example 15. Consider $f_n(z)=1+z+\frac{z^2}{2}+\ldots+\frac{z^n}{n}$, which is a partial sum of the power series $\sum_n\frac{z^n}{n}$. Let us apply Cauchy's criterion to $\{f_n\}$. For $n\geq m$, we have

$$f_n(z)-f_m(z)=\frac{z^{m+1}}{m+1}+\frac{z^{m+2}}{m+2}+\ldots+\frac{z^n}{n}, \tag{17}$$

leading to the bound

$$|f_n(z)-f_m(z)|\leq\frac{|z|^{m+1}}{m+1}+\frac{|z|^{m+2}}{m+2}+\ldots+\frac{|z|^n}{n}\leq|z|^{m+1}(1+|z|+\ldots+|z|^{n-m-1}). \tag{18}$$

Now if we fix some $r<1$, and assume that $|z|\leq r$, then

$$|f_n(z)-f_m(z)|\leq r^{m+1}(1+r+\ldots+r^{n-m-1})\leq\frac{r^{m+1}}{1-r}, \tag{19}$$

which shows that $\{f_n\}$ is uniformly Cauchy in the disk $\bar{D}_r$, and hence there is a function $f:\bar{D}_r\to\mathbb{C}$ such that $f_n\to f$ uniformly in $\bar{D}_r$. Finally, since each f_n is continuous in $\bar{D}_r$, by Theorem 9, we conclude that $f\in\mathscr{C}(\bar{D}_r)$.

Recall that $D_r(z) = \{w \in \mathbb{C} : |w - z| < r\}$ is the open disk of radius r with the centre z.

Definition 16. A sequence $\{f_n\}$ of functions $f_n : G \to \mathbb{C}$ is called *locally uniformly convergent* in G if for each $z \in G$ there is $r > 0$ such that $\{f_n\}$ converges uniformly in $G \cap D_r(z)$. Similarly, $\{f_n\}$ is called *locally uniformly Cauchy*, if for each $z \in G$ there is $r > 0$ such that $\{f_n\}$ is uniformly Cauchy in $G \cap D_r(z)$.

Example 17. The series $\sum_n \frac{z^n}{n}$ converges locally uniformly in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Exercise 18. Prove the following.

(a) If $\{f_n\}$ is locally uniformly convergent in G, then there is a function $f : G \to \mathbb{C}$ such that $f_n \to f$ pointwise in G.
(b) If $\{f_n\} \subset \mathscr{C}(G)$ converges to $f : G \to \mathbb{C}$ locally uniformly in G, then $f \in \mathscr{C}(G)$.
(c) If $\{f_n\}$ locally uniformly Cauchy in G, then $\{f_n\}$ converges locally uniformly in G.

2. Absolutely uniform convergence

Given a sequence $\{f_m\}$ of functions $f_m : G \to \mathbb{C}$, by defining $g_0 = f_0$ and $g_n = f_n - f_{n-1}$ for $n = 1, 2, \ldots$, we can write

$$f_m = f_0 + (f_1 - f_0) + \ldots + (f_m - f_{m-1}) = \sum_{n=0}^{m} g_n. \tag{20}$$

This leads to the *function series*

$$\sum_{n=0}^{\infty} g_n, \tag{21}$$

which can be understood as an overloaded notation for both

- the sequence $\{f_m\}$ (or, what is the same, the sequence $\{g_n\}$), and
- the limit $\lim\limits_{m\to\infty} f_m = \lim\limits_{m\to\infty} \sum_{n=0}^{m} g_n$, if it exists.

Thus when one says that the series $\sum_n g_n$ converges uniformly in G, one is referring to the sequence $\{f_m\}$. On the other hand, the equality $\sum_n g_n = f$, or the statement that the sum of the series $\sum_n g_n$ is f, would be referring to the limit $\lim\limits_{m\to\infty} f_m$.

Definition 19. We say that a series $\sum_n g_n$ of functions $g_n : G \to \mathbb{C}$ *converges absolutely uniformly in* G if the series $\sum_n |g_n|$ converges uniformly in G, that is, if the sequence $\{\tilde{f}_n\}$ with the terms $\tilde{f}_n = |g_1| + \ldots + |g_n|$ converges uniformly in G.

Lemma 20. *If a series is absolutely uniformly convergent, then it converges uniformly.*

Proof. Suppose that $\sum_n g_n$ is a series of functions $g_n : G \to \mathbb{C}$, converging absolutely uniformly in G. Let $f_n = g_1 + g_2 + \ldots + g_n$ and $\tilde{f}_n = |g_1| + |g_2| + \ldots + |g_n|$. We start with the inequality

$$|f_m(z) - f_n(z)| \leq |g_{n+1}(z)| + \ldots + |g_m(z)| = \tilde{f}_m(z) - \tilde{f}_n(z), \tag{22}$$

which is true for all $z \in G$ and $m \geq n$. By hypothesis, $\tilde{f}_n \to \tilde{f}$ uniformly in G for some $\tilde{f} : G \to \mathbb{C}$. Now let $\varepsilon > 0$, and let N be such that $|\tilde{f}_n(z) - \tilde{f}(z)| \leq \varepsilon$ for all $n \geq N$. Then for all $z \in G$ and $n, m \geq N$, we have

$$\tilde{f}_m(z) - \tilde{f}_n(z) = |\tilde{f}_m(z) - \tilde{f}_n(z)| \leq |\tilde{f}_m(z) - \tilde{f}(z)| + |\tilde{f}(z) - \tilde{f}_n(z)| \leq 2\varepsilon, \tag{23}$$

which implies that

$$|f_m(z) - f_n(z)| \leq 2\varepsilon, \tag{24}$$

for all $z \in G$ and $m \geq n \geq N$. This shows that the sequence $\{f_n\}$ is uniformly Cauchy, hence $f_n \to f$ uniformly in G for some $f : G \to \mathbb{C}$. $\square$

Remark 21. In the notation of the preceding proof, we have the estimate

$$|f(z)-f_n(z)| \le \tilde{f}(z)-\tilde{f}_n(z) = \sum_{k=n+1}^{\infty} |g_k(z)|, \tag{25}$$

for any $z \in G$ and any n. Indeed, let $\varepsilon > 0$ be arbitrary, and let m be a large integer so that $|f_m(z)-f(z)| \le \varepsilon$ and $|\tilde{f}_m(z)-\tilde{f}(z)| \le \varepsilon$. Then we have

$$\begin{aligned} |f_n(z)-f(z)| &\le |f_n(z)-f_m(z)| + |f_m(z)-f(z)| \le |\tilde{f}_n(z)-\tilde{f}_m(z)| + \varepsilon \\ &\le |\tilde{f}_n(z)-\tilde{f}(z)| + |\tilde{f}(z)-\tilde{f}_m(z)| + \varepsilon \le \tilde{f}(z)-\tilde{f}_n(z)| + 2\varepsilon, \end{aligned} \tag{26}$$

where we have taken into account that $|\tilde{f}_n(z)-\tilde{f}(z)| = \tilde{f}(z)-\tilde{f}_n(z)$, because $\{\tilde{f}_n(z)\}$ is a nondecreasing sequence. This shows that $|f(z)-f_n(z)| \le \tilde{f}(z)-\tilde{f}_n(z)$. Finally, the equality in (25) follows from

$$\begin{aligned} \sum_{k=n+1}^{\infty} |g_k(z)| &= \lim_{m\to\infty} \sum_{k=n+1}^{m} |g_k(z)| = \lim_{m\to\infty} (\tilde{f}_m(z)-\tilde{f}_n(z)) \\ &= \big(\lim_{m\to\infty} \tilde{f}_m(z)\big) - \tilde{f}_n(z) = f(z)-\tilde{f}_n(z), \end{aligned} \tag{27}$$

where we have used the additive property of the limits of real number sequences.

Example 22. Absolutely uniform convergence is strictly stronger than the combination of uniform convergence and pointwise absolute convergence. For example, take the series

$$\sum_n \frac{(-1)^n x^n}{n} \qquad \text{for} \quad x \in (0,1). \tag{28}$$

Obviously, it converges absolutely for each $x \in (0,1)$. Let $f_n(x)$ be the n-th partial sum of (28). Then from the alternating nature of the series, we have

$$|f_m(x)-f_n(x)| \le \frac{x^{n+1}}{n+1} < \frac{1}{n+1} \qquad \text{for} \quad x \in (0,1), \quad \text{and for} \quad m \ge n, \tag{29}$$

which shows that the series is uniformly convergent in $(0,1)$. However, (28) does *not* converge absolutely uniformly in $(0,1)$, because

$$\sum_{n=m}^{k} \frac{|(-1)^n x^n|}{n} = \sum_{n=m}^{k} \frac{x^n}{n} \to \sum_{n=m}^{k} \frac{1}{n} \qquad \text{as} \quad x \to 1, \tag{30}$$

and the sum in the right hand side grows unboundedly with k, regardless of the value of m.

Theorem 23 (Weierstrass M-test, majorant test, or comparison test)**.** *Let the functions $g_n : G \to \mathbb{C}$ satisfy $|g_n(z)| \le a_n$ for all $z \in G$ and for each n, where $\sum_n a_n$ is a convergent series of real numbers. Then the function series $\sum_n g_n$ converges absolutely uniformly in G. In other words, if $\sum_n \|g_n\|_G < \infty$ then $\sum_n g_n$ converges absolutely uniformly in G.*

Proof. With $\tilde{f}_n = |g_1| + \ldots + |g_n|$, for $n > m$ and $z \in G$, we have

$$\tilde{f}_n(z) - \tilde{f}_m(z) = |g_{m+1}(z)| + \ldots + |g_n(z)| \le a_{m+1} + \ldots + a_n \le \sum_{k=m+1}^{\infty} a_k, \tag{31}$$

which tends to 0 when $m \to \infty$. Note that the bound on the right hand side does not depend on z. So $\{\tilde{f}_n\}$ is uniformly Cauchy in G, hence converges uniformly in G. □

Example 24. Since $|\sin x| \le 1$ for all $x \in \mathbb{R}$, and $\sum n^{-2} < \infty$, the series $\sum_n \frac{\sin(n^2 x)}{n^2}$ converges absolutely uniformly in $\mathbb{R}$.

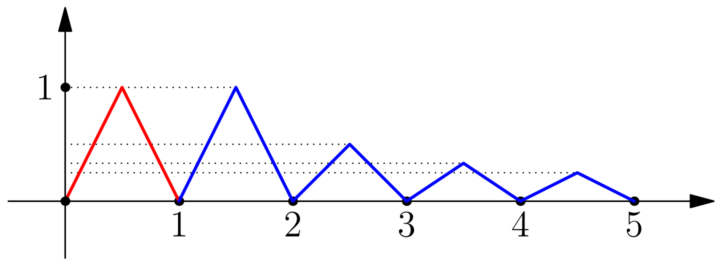

Figure 1. The blue curve is the graph of the 4-th partial sum of the series $\sum_n g_n$ from Example 25. Then each "blue tooth" represents an individual term g_n. The function ϕ is represented by the red curve in the interval $(0, 1)$, outside of which ϕ vanishes.

Example 25. The Weierstrass M-test is only a sufficient condition for absolutely uniform convergence. For example, let $\phi(x) = \max\{0, \min\{2x, 2 - 2x\}\}$, and let $g_n(x) = \frac{1}{n}\phi(x - n)$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then the series $\sum_n g_n$ converges absolutely uniformly in $\mathbb{R}$, because

$$0 \leq \sum_{n=m}^{k} g_n(x) \leq \frac{1}{m} \qquad \text{for any} \quad x \in \mathbb{R} \quad \text{and any} \quad k \geq m. \tag{32}$$

However, we have $\max\limits_{x\in\mathbb{R}} g_n(x) = \dfrac{1}{n}$, hence the M-test is not applicable here (see Figure 1).

Remark 26. Despite the observation in the preceding example, the Weierstrass M-test is applicable to all frequently occurring series in complex analysis. In fact, there is a notion of convergence, called *normal convergence*, that is by definition equivalent to the hypothesis of the Weierstrass M-test. Normal convergence is stronger than absolutely uniform convergence, hence simplifies many proofs. On the other hand, we do not lose generality, because complex analysis is basically built on normally convergent series anyways. Thus it appears that normal convergence is the optimal notion of convergence in complex analysis. In these notes we will not formally define normal convergence but will use it implicitly in some places.

Theorem 27 (Dirichlet 1837). *If $\sum_n g_n$ converges absolutely uniformly in G, then all its rearrangements converge (absolutely uniformly in G) to the same limit.*

Proof. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection, so that $\sum_n g_{\sigma(n)}$ is a rearrangement of $\sum_n g_n$. Fix $\varepsilon > 0$. Then there exists N such that

$$|g_n(z)| + \ldots + |g_{n+k}(z)| \leq \varepsilon \qquad \text{for all} \quad z \in G, \tag{33}$$

whenever $n \geq N$ and $k > 0$. Now let $f_n = g_1 + \ldots + g_n$ and $F_n = g_{\sigma(1)} + \ldots + g_{\sigma(n)}$. By the hypothesis of the theorem, $f_n \to f$ uniformly in G, for some $f : G \to \mathbb{C}$. Thus there is N' such that[1]

$$|f_n(z) - f(z)| \leq \varepsilon \qquad \text{for all} \quad z \in G, \tag{34}$$

whenever $n \geq N'$. We want show that $F_m - f_m$ gets smaller as m grows. To this end, let

$$M = \max\{\sigma^{-1}(n) : n = 1, 2, \ldots, N\}, \tag{35}$$

that is, the image of the interval $\{1, \ldots, M\}$ under σ covers the entire interval $\{1, \ldots, N\}$. Therefore for $m \geq M$, we have

$$|F_m(z) - f_m(z)| \leq |g_{N+1}(z)| + \ldots + |g_{N+k}(z)| \leq \varepsilon \qquad \text{for all} \quad z \in G, \tag{36}$$

[1] In fact, from the estimate (25) it follows that $N' = N$ would be sufficient.

for some k large enough. In combination with (34), this shows that

$$|F_m(z) - f(z)| \le 2\varepsilon \qquad \text{for all} \quad z \in G, \tag{37}$$

whenever $m \ge \max\{M, N'\}$, and so $F_m \to f$ uniformly in G.

We have proved that if a series converges absolutely uniformly, then its rearrangement converges *uniformly*. To show that the rearrangement converges *absolutely uniformly*, we simply apply what we have proved to the series $\sum_n |g_{\sigma(n)}|$, which is a rearrangement of $\sum_n |g_n|$. Since the latter converges absolutely uniformly, the former converges uniformly, meaning that $\sum_n g_{\sigma(n)}$ converges absolutely uniformly. □

The following is a simplification of the theorem we have proved in class. We impose a stronger hypothesis (which is related to normal convergence discussed in Remark 26), but the resulting theorem will nevertheless be sufficient for our purposes.

Theorem 28. *Let $f_{k,\ell} : G \to \mathbb{C}$ for $k, l \in \mathbb{N}$, and let $\sigma : \mathbb{N}^2 \to \mathbb{N}$ be a bijection. Define the sequence $\{g_n\}$ by $g_{\sigma(k,\ell)} = f_{k,\ell}$. Assume that $\{a_{k,\ell}\} \subset \mathbb{R}$ satisfy $|f_{k,\ell}(z)| \le a_{k,\ell}$ for all $z \in G$, $k, l \in \mathbb{N}$, and let $\{b_n\}$ be such that $b_{\sigma(k,\ell)} = a_{k,\ell}$. Then the following are equivalent:*
(a) $\sum_n b_n < \infty$.
(b) $\sum_\ell a_{k,\ell} < \infty$ for each k, and moreover $\sum_k(\sum_\ell a_{k,\ell}) < \infty$.
(c) $\sum_k a_{k,\ell} < \infty$ for each ℓ, and moreover $\sum_\ell(\sum_k a_{k,\ell}) < \infty$.
If any (hence all) of the above conditions is satisfied, then we have

$$\sum_\ell(\sum_k f_{k,\ell}) = \sum_k(\sum_\ell f_{k,\ell}) = \sum_n g_n. \tag{38}$$

Proof. First we prove the implication (a) ⇒ (b). Let $N = \sum_n b_n < \infty$. This obviously implies that for all $k \in \mathbb{N}$, $M_k = \sum_\ell a_{k,\ell} < \infty$. Let $\varepsilon > 0$ and let m_k be such that $\sum_{\ell > m_k} a_{k,\ell} \le 2^{-k}\varepsilon$. So for any m we have

$$\sum_{k \le m}(\sum_\ell a_{k,\ell}) \le \sum_{k \le m}(\sum_{\ell \le m_k} a_{k,\ell}) + 2\varepsilon \le N + 2\varepsilon. \tag{39}$$

For the other direction (b) ⇒ (a), we start with the definition of M_k and the condition $M = \sum_k M_k < \infty$. Then for any p we have

$$\sum_{n \le p} b_n \le \sum_{k \le m} \sum_{\ell \le m} a_{k,\ell} \le M, \tag{40}$$

where m is such that $\{\ell \le m\}^2 \supseteq \sigma^{-1}(\{n \le p\})$. The equivalence of (a) and (c) can be proven analogously.

Now we shall prove that $g = \sum_n g_n$ is equal to $f = \sum_k(\sum_\ell f_{k,\ell})$. To this end, let $\varepsilon > 0$, and let m be such that $\sum_{k>m}(\sum_\ell a_{k,\ell}) \le \varepsilon$. We also let m_k be such that $\sum_{\ell > m_k} a_{k,\ell} \le 2^{-k}\varepsilon$, and let $\tilde{f}_\varepsilon = \sum_{k \le m} \sum_{\ell \le m_k} f_{k,\ell}$. Then for $z \in G$ we have

$$|f(z) - \tilde{f}_\varepsilon(z)| \le \sum_{k>m}(\sum_\ell a_{k,\ell}) + \sum_{k \le m}(\sum_{\ell > m_k} a_{k,\ell}) \le 3\varepsilon. \tag{41}$$

Similarly, for sufficiently large p, the partial sum $\tilde{g}_p = \sum_{n \le p} g_n$ satisfies

$$|\tilde{g}_p(z) - \tilde{f}_\varepsilon(z)| \le \sum_{k>m}(\sum_\ell a_{k,\ell}) + \sum_{k \le m}(\sum_{\ell > m_k} a_{k,\ell}) \le 3\varepsilon, \tag{42}$$

and so we have

$$|f(z) - g(z)| \le |g(z) - \tilde{g}_p(z)| + 6\varepsilon. \tag{43}$$

Since $\tilde{g}_p(z) \to g(z)$ and both ε and z are arbitrary, we conclude that $f = g$. □

As an immediate application, we prove a result on the product of two series.

Corollary 29. *Let $f_k : G \to \mathbb{C}$ and $g_k : G \to \mathbb{C}$ satisfy $|f_k(z)| \leq a_k$ and $|g_k(z)| \leq b_k$ for all $z \in G$, and for $k \in \mathbb{N}$. Assume that $\sum_k a_k < \infty$ and $\sum_\ell b_\ell < \infty$. Then for every bijection $\sigma : \mathbb{N}^2 \to \mathbb{N}$, the series $\sum_n h_n$ with elements $h_{\sigma(k,\ell)} = f_k g_\ell$ converges absolutely uniformly in G to $(\sum_k f_k)(\sum_\ell g_\ell)$.*

Proof. We have

$$\sum_\ell a_k b_\ell = a_k \sum_\ell b_\ell < \infty, \tag{44}$$

and

$$\sum_k (\sum_\ell a_k b_\ell) = \sum_k a_k (\sum_\ell b_\ell) = (\sum_k a_k)(\sum_\ell b_\ell) < \infty. \tag{45}$$

Note that these imply

$$\sum_\ell f_k g_\ell = f_k (\sum_\ell g_\ell), \qquad \text{and} \qquad \sum_k (\sum_\ell f_k g_\ell) = (\sum_k f_k)(\sum_\ell g_\ell). \tag{46}$$

The proof is established upon employing Theorem 28(b) with $f_{k,\ell} = f_k g_\ell$. □

3. Power series

A *power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n (z-c)^n, \tag{47}$$

with the *coefficients* $a_n \in \mathbb{C}$ for $n = 0, 1, \ldots$, and the *centre* $c \in \mathbb{C}$.

Arguably the most important example of a power series is the geometric series $\sum_n z^n$ studied in Example 8. We know that this series converges for all z satisfying $|z| < 1$. On the other hand, if $|z| > 1$, then $|z^n| = |z|^n \not\to 0$ as $n \to \infty$, meaning that the series diverges. It turns out that this is basically happens in the general case; Given a power series, there is a special circle centred at c that separates convergence and divergence behaviours[2].

We start with the following important observation of Niels Henrik Abel (1802-1829).

Remark 30 (Abel 1826). (a) Suppose that (47) converges at some $z_0 \neq c$. Then it is necessary that $|a_n(z_0 - c)^n| = |a_n||z_0 - c|^n \to 0$ as $n \to \infty$. In particular, the sequence $\{|a_n||z_0 - c|^n\}$ is bounded, i.e., there is some constant M such that

$$|a_n| r^n \leq M \qquad \text{for all} \quad n, \tag{48}$$

where $r = |z_0 - c|$.

(b) Suppose that the coefficients of the series (47) satisfy the estimate (48) for some constants $r > 0$ and M. Let $0 < \rho < r$ and let $z \in D_\rho(c) = \{w \in \mathbb{C} : |w - c| < \rho\}$. Then

$$|a_n (z-c)^n| \leq |a_n| \rho^n \leq M \big(\frac{\rho}{r}\big)^n, \tag{49}$$

and since $\sum (\frac{\rho}{r})^n < \infty$, the *Weierstrass M-test is applicable* to (47) in the disk $D_\rho(c)$. Therefore the series (47) converges absolutely uniformly in $D_\rho(c)$.

(c) Combining (a) and (b) leads to the statement given in the caption of Figure 2.

[2]The geometric series diverges *on* the circle $|z| = 1$, but a general power series can converge on part of the aforementioned special circle. Note also that in some situations the "special" circle can degenerate into the point $\{0\}$ or it can cover the entire plane $\mathbb{C}$.

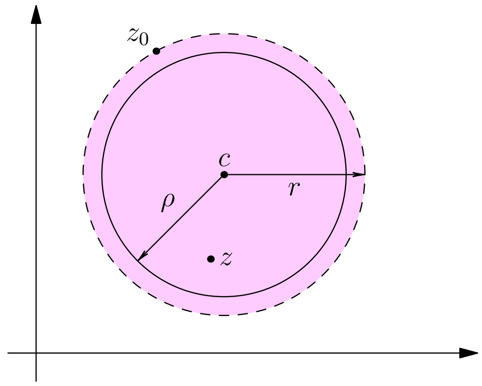

FIGURE 2. If the power series $\sum a_n(z-c)^n$ converges at $z = z_0$, then it converges at all points in the open disk $D_r(c)$ with $r = |z_0 - c|$. Moreover, the convergence is absolutely uniform in $D_\rho(c)$ for each $0 < \rho < r$. See Remark 30.

Definition 31. From (b) of the previous remark we see that it is important to find the largest value of r for which the estimate (48) holds. To this end, we let

$$A = \{r \geq 0 : \text{the sequence } \{|a_n|r^n\} \text{ is bounded}\}, \tag{50}$$

and define

$$R = \sup A, \tag{51}$$

which is called *convergence radius* of the power series $\sum a_n(z-c)^n$.

Example 32. (a) If $a_n = n^n$, the sequence $\{a_n r^n\}$ diverges to ∞ whatever the value of $r > 0$. Therefore we have $A = \{0\}$ and hence $R = 0$ in this case.

(b) If $a_n = n^{-n}$, the sequence $\{a_n r^n\}$ converges to 0 whatever the value of $r \geq 0$. Therefore we have $A = [0, \infty)$ and hence $R = \infty$.

(c) If $a_n = 2^n$, the sequence $\{a_n r^n\}$ is bounded for $r \leq \frac{1}{2}$ and unbounded for $r > \frac{1}{2}$. Therefore we have $A = [0, \frac{1}{2}]$ and hence $R = \frac{1}{2}$.

(d) If $a_n = n2^n$, the sequence $\{a_n r^n\}$ is bounded for $r < \frac{1}{2}$ and unbounded for $r \geq \frac{1}{2}$. Therefore we have $A = [0, \frac{1}{2})$ and hence $R = \frac{1}{2}$.

By definition, the convergence radius R has the following characteristic properties.

- Given any $r < R$, there is M such that $|a_n| \leq M r^{-n}$ for all n.
- For any $r > R$, the sequence $\{|a_n|r^n\}$ is unbounded.

This leads to the following.

- Suppose that z satisfy $|z - c| \leq \rho < R$, and pick some r such that $\rho < r < R$. Then there is M such that $|a_n| \leq M r^{-n}$ for all n. This implies that

$$|a_n(z-c)^n| = |a_n||z-c|^n \leq M\big(\frac{\rho}{r}\big)^n, \tag{52}$$

  hence the Weierstrass M-test is applicable in the disk $D_\rho(c)$.
- If $|z - c| > R$, then $|a_n(z-c)^n| = |a_n|z-c|^n \not\to 0$ as $n \to \infty$, and so the power series $\sum a_n(z-c)^n$ diverges.

Therefore, the convergence radius of the power series $\sum a_n(z-c)^n$ can also defined as the (extended) real number $R \in [0, \infty]$ with the property that $\sum a_n(z-c)^n$ converges whenever $|z-c| < R$ and diverges whenever $|z-c| > R$. Note also that whenever $\rho < R$, the Weierstrass M-test is applicable in the disk $D_\rho(c)$, hence it converges absolutely uniformly in $D_\rho(c)$.

Definition 33. Given a real number sequence $\{s_n\} \subset \mathbb{R}$, we let

$$B = \{x \in \mathbb{R} : s_n > x \text{ for infinitely many } n\}, \tag{53}$$

and define the *limit supremum of* $\{s_n\}$ as

$$\limsup_{n\to\infty} s_n = \sup B. \tag{54}$$

Similarly, we define the *limit infimum*

$$\liminf_{n\to\infty} s_n = \inf C, \tag{55}$$

where

$$C = \{x \in \mathbb{R} : s_n < x \text{ for infinitely many } n\}. \tag{56}$$

Example 34. Let $s_n = (-1)^n + \frac{1}{n}$ for $n \in \mathbb{N}$. For any $x > 1$, there would only be finitely many n such that $s_n > x$. On the other hand, there are infinitely many n such that $s_n > 1$. Thus we have $B = (-\infty, 1]$ and

$$\limsup_{n\to\infty} s_n = 1. \tag{57}$$

On the other hand, we have $C = (-1, \infty)$, because $s_n > -1$ for all n and for any $x > -1$ there would be infinitely many n such that $s_n < x$. Therefore we conclude that

$$\liminf_{n\to\infty} s_n = -1. \tag{58}$$

Remark 35. Obviously, unless $B = \varnothing$, the limit supremum of $\{s_n\}$ is defined. If we agree to use the convention that $\sup \varnothing = -\infty$, then limit supremums are always defined. A similar discussion applies to limit infimums.

Exercise 36. Show that

$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty} \sup\{s_n, s_{n+1}, \ldots\}. \tag{59}$$

Remark 37. Let $s = \limsup s_n$. Then from Definition 33 we infer the following.

- For any $x < s$, there are infinitely many n such that $s_n > x$.
- Given any $x > s$, there exists N such that $s_n \leq x$ for all $n \geq N$.

By using a limit supremum, we can write down a formula for the convergence radius. This formula, which can be regarded as a version of the root test, was first published by Cauchy in 1821, and rediscovered in 1888 by Jacques Salomon Hadamard (1865-1963).

Theorem 38 (Cauchy-Hadamard formula)**.** *We have*

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}, \tag{60}$$

with the conventions $\frac{1}{\infty} = 0$ *and* $\frac{1}{0} = \infty$.

Proof. Without loss of generality, we may take $c = 0$. Let r be defined by

$$\frac{1}{r} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}. \tag{61}$$

By invoking the two properties we have derived in Remark 37, we will show that $r = R$.

- If $\frac{1}{|z|} < \frac{1}{r}$, then $\sqrt[n]{|a_n|} \geq \frac{1}{|z|}$, or $|a_n z^n| \geq 1$, for infinitely many n. Hence $\sum a_n z^n$ diverges, and so $r \geq R$.
- If $\frac{1}{|z|} > \frac{1}{\rho} > \frac{1}{r}$, then $\sqrt[n]{|a_n|} \leq \frac{1}{\rho}$ for all large n, hence $|a_n z^n| \leq (\frac{|z|}{\rho})^n$ for all large n. This implies that $\sum a_n z^n$ converges, and so $r \leq R$. $\square$

The Cauchy-Hadamard formula is a beautiful result, but we will not make much use of it in this course. More useful for us is the ratio test, which was discovered by Cauchy in 1821.

Theorem 39 (Ratio test)**.** *Provided that $a_n = 0$ for only finitely many n, one can estimate the convergence radius R of the power series (47) by*

$$\liminf_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \leq R \leq \limsup_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}. \tag{62}$$

In particular, if $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$ exists, then it is equal to R.

Proof. Without loss of generality, we will assume that $a_n \neq 0$ for all n (Equivalently, we only consider indices $n \geq M$ for some large M).

Let α be the limit infimum in (62) and suppose that $|z| < \rho < \alpha$. Then there exists N such that $\frac{|a_n|}{|a_{n+1}|} \geq \rho$ for all $n \geq N$, which leads to the estimate

$$|a_{n+1}| \leq \rho^{-1}|a_n| \leq \ldots \leq \rho^{N-1-n}|a_N| \qquad \text{for} \quad n \geq N, \tag{63}$$

or $|a_n| \leq |a_N|\rho^N\rho^{-n}$ for $n > N$. Thus we have

$$|a_n z^n| \leq |a_N|\rho^N \big(\frac{|z|}{\rho}\big)^n \qquad \text{for} \quad n > N, \tag{64}$$

and so $\sum a_n z^n$ converges. This means that $\alpha \leq R$.

Now let β be the limit supremum in (62), and suppose that $|z| > \beta$. Then there exists N such that $\frac{|a_n|}{|a_{n+1}|} \leq |z|$ for all $n \geq N$, which leads to the estimate $|a_n z^n| \geq |a_N||z|^N$ for $n > N$. Since $a_N \neq 0$, the series $\sum a_n z^n$diverges, and hence $R \leq \beta$. □

Example 40. (a) For $a_n = n!$, we have $\frac{|a_n|}{|a_{n+1}|} = \frac{1}{n+1} \to 0$ as $n \to \infty$. Therefore the convergence radius of the series $\sum n! z^n$ is 0.

(b) For $a_n = \frac{1}{n!}$, we have $\frac{|a_n|}{|a_{n+1}|} = n+1 \to \infty$ as $n \to \infty$. Therefore the convergence radius of the series $\sum \frac{z^n}{n!}$ is ∞.

(c) For $a_n = (-1)^n n^3 2^n$, we have $\frac{|a_n|}{|a_{n+1}|} = \frac{n^3}{2(n+1)^3} \to \frac{1}{2}$ as $n \to \infty$. Therefore the convergence radius of the series $\sum (-1)^n n^3 2^n z^n$ is $\frac{1}{2}$.

(d) Let the sequence $\{a_n\}$ be given by $\{1, 1, 3, 3, 3^2, 3^2, 3^3, 3^3, \ldots\}$. Then the value of $\frac{|a_n|}{|a_{n+1}|}$ alternates between 1 and $\frac{1}{3}$, and hence $\liminf a_n = \frac{1}{3}$ and $\limsup a_n = 1$. Therefore the ratio test implies that the convergence radius of the series $\sum a_n z^n$ satisfies $\frac{1}{3} \leq R \leq 1$.

Exercise 41. Find the convergence radius of the series described in (d) of the preceding example. *Hint*: The Cauchy-Hadamard formula, or use Definition 31 directly.

Example 42. In Example 40 (b) we computed the radius of convergence to be $R = \infty$. Then the Cauchy-Hadamard formula must give

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = \frac{1}{R} = 0. \tag{65}$$

Since $n! > 0$, this implies that

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0, \qquad \text{or} \qquad \lim_{n\to\infty} \sqrt[n]{n!} = \infty, \tag{66}$$

which is a nontrivial conclusion.

Exercise 43. By applying both the ratio test and the Cauchy-Hadamard formula to a suitable power series, show that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1. \tag{67}$$

Now we turn to the question of termwise differentiating and integrating power series. One consequence of this is that any power series is holomoprhic in its disk of convergence.

Theorem 44. *Let $0 < R \leq \infty$ be the convergence radius of the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n. \tag{68}$$

Then both series

$$g(z) = \sum_{n=1}^{\infty} n a_n (z-c)^{n-1}, \qquad \textit{and} \qquad F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-c)^{n+1}, \tag{69}$$

have convergence radii equal to R, and there hold that

$$f' = g \qquad \textit{and} \qquad F' = f, \qquad \textit{in } D_R(c), \tag{70}$$

where in case $R = \infty$ it is understood that $D_\infty(c) = \mathbb{C}$.

Proof. Without loss of generality, we will assume that $c = 0$. It is obvious that the convergence radius R' of the power series representing g is at most R, that is, $R' \leq R$. To prove the other direction, let $|z| < r < R$. Then there is some M such that $|a_n| r^n \leq M$ for all n, which implies that

$$n|a_n||z|^n \leq n\big(\frac{|z|}{r}\big)^n. \tag{71}$$

Since $|z| < r$, we have $\sum n(\frac{|z|}{r})^n < \infty$, and so $R \leq R'$.

Now we will show that $f' = g$ in $D_R(c)$. To this end, we write

$$f(z+h) - f(z) = \sum_{n=0}^{\infty} a_n \left((z+h)^n - z^n\right) = h \sum_{n=0}^{\infty} a_n \sum_{j=0}^{n-1} (z+h)^j z^{n-1-j} =: h\lambda_z(h), \tag{72}$$

where $z \in D_R(c)$ is fixed and $h \in \mathbb{C}$ is such that $z + h \in D_R(c)$. Let $r < R$ be such that $|z| < r$, and let $h \in D_\delta$ with $\delta = r - |z|$. Then we have

$$\sum_{n=0}^{\infty} |a_n| \sum_{j=0}^{n-1} |z+h|^j |z|^{n-1-j} \leq \sum_{n=0}^{\infty} |a_n| n r^{n-1} < \infty, \tag{73}$$

implying that the series for λ_z converges absolutely uniformly in D_δ. In particular, we infer that λ_z is continuous in D_δ. Hence f is complex differentiable at z, with

$$f'(z) = \lambda_z(0) = g(z). \tag{74}$$

The claims about F follow from the above if we start with F instead of f. $\square$

Corollary 45. *In the setting of the preceding theorem, we have $f \in \mathscr{O}(D_R(c))$. Moreover, $f^{(n)} \in \mathscr{O}(D_R(c))$ for any n, so f is infinitely differentiable as a function $f : D_R(c) \to \mathbb{R}^2$.*

Example 46. (a) $f(z) = \sum n z^n$ is holomorphic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
(b) $f(z) = \sum \frac{z^n}{n!}$ is holomorphic in $\mathbb{C}$.