Solutions to the problems from the midterm exam

Math 249 Winter 2015

1. Let f be a holomorphic function in a convex open set $\Omega \subset \mathbb{C}$ satisfying $g(\mathrm{Re} f)+h(\mathrm{Im} f) = 0$ in Ω, where g and h are real-valued differentiable functions of a real variable. (For example, in one of the versions, we have $g(u) = u$ and $h(v) = -v^3$.) Assume that $g' \neq 0$ everywhere or $h' \neq 0$ everywhere. Show that f must be constant in Ω.

   Solution: Let us write $f(x + iy) = u(x, y) + iv(x, y)$ with u and v real. Then we have

   $$g(u) + h(v) = 0 \qquad \text{in } \Omega.$$

   Differentiating this with respect to x and y, we get

   $$g'(u)\frac{\partial u}{\partial x} + h'(v)\frac{\partial v}{\partial x} = 0 \qquad \text{and} \qquad g'(u)\frac{\partial u}{\partial y} + h'(v)\frac{\partial v}{\partial y} = 0,$$

   which, in light of the Cauchy-Riemann equations, imply that

   $$g'(u)\frac{\partial u}{\partial x} - h'(v)\frac{\partial u}{\partial y} = 0 \qquad \text{and} \qquad g'(u)\frac{\partial u}{\partial y} + h'(v)\frac{\partial u}{\partial x} = 0.$$

   Now we square each of these equations, and sum them, to conclude

   $$\big(|g'(u)|^2 + |h'(v)|^2\big)\left(\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2\right) = 0.$$

   Because of the nonvanishing of at least one of g' or h', we have

   $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \qquad \text{everywhere in } \Omega,$$

   and moreover,

   $$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \qquad \text{everywhere in } \Omega,$$

   by the Cauchy-Riemann equations.

   What remains is to explain why the vanishing of the partial derivatives implies that u and v are constants. The basic reason for this is that Ω is path-connected. To show this from first principles, we may proceed as follows. Without loss of generality, assume that $0 \in \Omega$, and let $w \in \Omega$ be an arbitrary point. Then by convexity and openness of Ω, there is $\varepsilon > 0$ such that $D_\varepsilon(wt) \subset \Omega$ for all $t \in [0, 1]$. In other words, Ω contains an ε-neighbourhood of the straight line segment connecting 0 and w. It is now clear that we can join 0 and w by a "zigzag" path that consists of finitely many horizontal and vertical line segments, and from here we have $u(w) = u(0)$ and $v(w) = v(0)$. Since w was arbitrary, we conclude that f is constant in Ω.

2. Determine the convergence radii of the following power series.

(a) $\sum z^{2^n}$.

Solution: We can write the given series as $\sum a_k z^k$ with

$$a_k = \begin{cases} 1 & \text{if } k = 2^n \text{ for some integer } n, \\ 0 & \text{otherwise,} \end{cases}$$

from which it is obvious that $\limsup \sqrt[k]{a_k} = 1$, and by the Cauchy-Hadamard formula, we have the convergence radius $R = 1$.

(b) $\sum (\cos n) z^n$.

Solution: We have $|(\cos n) z^n| \leq |z|^n$, and so the series converges for $|z| < 1$. This means that the convergence radius satisfies $R \geq 1$. Now we want to show that $\cos n$ does *not* converge to 0 as $n \to \infty$. Informally speaking, $|\cos n| \approx 1$ when $n \approx \pi k$ for some $k \in \mathbb{Z}$. As a way of making it precise, for each $k \in \mathbb{N}$ there is an integer $n_k > 3k$ such that

$$\left|\pi - \frac{n_k}{k}\right| \leq \frac{1}{k}, \qquad \text{or} \qquad |n_k - \pi k| \leq 1.$$

Therefore we have

$$|\cos n_k| \geq \cos 1 > 0,$$

and so $\cos n \not\to 0$ as $n \to \infty$. This means that if $|z| \geq 1$ then $(\cos n) z^n$ does not converge to 0, implying that $R \leq 1$.

(c) $\sum (\log n + c^n) z^n$, where $c \in \mathbb{C}$ is a constant.

Solution: With $a_n = \log n + c^n$, an application of the ratio test leads to

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|\log(n+1) + c^{n+1}|}{|\log n + c^n|} = \left|\lim_{n\to\infty} \frac{\log(n+1) + c^{n+1}}{\log n + c^n}\right|. \tag{1}$$

Note that $|c|^n$ grows faster than $\log n$ if $|c| > 1$, and that $|c|^n$ does not grow at all if $|c| \leq 1$. Therefore we split the problem into two cases. First, assume that $|c| > 1$. Then we have

$$\lim_{n\to\infty} \frac{\log(n+1) + c^{n+1}}{\log n + c^n} = \lim_{n\to\infty} \frac{\frac{\log(n+1)}{c^n} + \frac{c^{n+1}}{c^n}}{\frac{\log n}{c^n} + \frac{c^n}{c^n}} = \frac{0 + c}{0 + 1} = c. \tag{2}$$

Now assume that $|c| \leq 1$. In this case, we have

$$\lim_{n\to\infty} \frac{\log(n+1) + c^{n+1}}{\log n + c^n} = \lim_{n\to\infty} \frac{\frac{\log(n+1)}{\log n} + \frac{c^{n+1}}{\log n}}{\frac{\log n}{\log n} + \frac{c^n}{\log n}} = \frac{1 + 0}{1 + 0} = 1. \tag{3}$$

Based on these computations, we conclude that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \begin{cases} |c| & \text{if } |c| > 1, \\ 1 & \text{if } |c| \leq 1, \end{cases} \tag{4}$$

or in other words,

$$R = \min\left\{1, \frac{1}{|c|}\right\}. \tag{5}$$

3. Given that the convergence radius of the power series $\sum a_n z^n$ is $R > 0$, determine the convergence radii of the following power series.

   (a) $\sum a_n z^{2n}$

   Solution: By assumption, the series $\sum a_n z^n$ converges whenever $|z| < R$ and diverges whenever $|z| > R$. Therefore the series $\sum a_n z^{2n} = \sum a_n (z^2)^n$ converges whenever $|z^2| < R$ and diverges whenever $|z^2| > R$. To repeat, the series $\sum a_n z^{2n}$ converges whenever $|z| < \sqrt{R}$ and diverges whenever $|z| > \sqrt{R}$. In other words, the convergence radius of $\sum a_n z^{2n}$ is $\sqrt{R}$.

   (b) $\sum n^2 3^n a_n z^n$

   Solution: As a preliminary observation, we claim that the convergence radius of $\sum n^2 a_n z^n$ is still R. We could have simply cited this result, but let us reproduce the argument here. First, denoting by R' the convergence radius of $\sum n^2 a_n z^n$, it is obvious that $R' \leq R$. Second, for any z with $|z| < R$, there is ρ and M with $|z| < \rho < R$ such that $|a_n| \leq M\rho^{-n}$ (Abel's observation). Then we have

$$\sum n^2 |a_n z^n| \leq \sum M n^2 \Big(\frac{|z|}{\rho}\Big)^n < \infty,$$

   which shows that $\sum n^2 a_n z^n$ converges, and thus $R' \geq R$. Hence the series $\sum n^2 a_n z^n$ converges whenever $|z| < R$ and diverges whenever $|z| > R$. This implies that the series $\sum n^2 3^n a_n z^n = \sum n^2 a_n (3z)^n$ converges whenever $|3z| < R$ and diverges whenever $|3z| > R$. In other words, the convergence radius of $\sum n^2 3^n a_n z^n$ is $\frac{R}{3}$.

   (c) $\sum a_n^2 z^n$

   Solution: Recall the definition

$$R = \sup A, \qquad \text{where} \qquad A = \{r \geq 0 : \sup_n |a_n| r^n < \infty\},$$

   for the convergence radius of $\sum a_n z^n$. On the other hands, for the convergence radius R' of $\sum a_n^2 z^n$, we have

$$R' = \sup B, \qquad \text{with} \qquad B = \{r \geq 0 : \sup_n |a_n|^2 r^n < \infty\},$$

   Now, obviously $r \in A$ implies $r^2 \in B$, because

$$\sup_n |a_n|^2 r^{2n} = \sup_n (|a_n| r^n)^2 = \big(\sup_n |a_n| r^n\big)^2.$$

   Moreover, $r \in B$ implies $\sqrt{r} \in A$, because

$$\sup_n |a_n| (\sqrt{r})^n = \sup_n \sqrt{|a_n|^2 r^n} = \sqrt{\sup_n |a_n^2| r^n}.$$

   Therefore we have $B = \{r^2 : r \in A\}$, and hence $R' = \sup B = (\sup A)^2 = R^2$.