THE FUNDAMENTAL THEOREMS OF FUNCTION THEORY

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Contents

1. Contour integration

Let $\Omega \subset \mathbb{C}$ be an open set. A (topological) *curve* in Ω is a continuous map $\gamma : [a, b] \to \Omega$, and it is called a *closed curve* or a *loop* if $\gamma(a) = \gamma(b)$. Loops in Ω can also be defined as continuous maps $\gamma : S^1 \to \Omega$. The terms path, contour and arc are also used for a curve, sometimes with slight differences in meaning. We will not make any distinction between any of these terms. Non-self-intersecting curves are called *simple*, and simple closed curves are called *Jordan curves*.

If $\phi : [c, d] \to [a, b]$ is a monotone increasing surjective function, then we say that the curve $\gamma \circ \phi : [c, d] \to \Omega$ is equivalent to the original $\gamma : [a, b] \to \Omega$, and call the equivalence classes of curves under this equivalence relation *oriented curves*. Intuitively, given the *image* $|\gamma| = \gamma([a, b])$ of the curve γ, an oriented curve can be recovered upon identifying the initial and terminal points, and specifying how to traverse at self-intersection points. By abuse of language we call the particular representation $\gamma : [a, b] \to \Omega$ of the underlying oriented curve also an oriented curve. Note that one can take the interval $[a, b]$ to be, say, $[0, 1]$ at one's convenience. Now, the *inverse* or the *opposite* of γ is defined by reversing the orientation: $\gamma^{-1}(t) = \gamma(b + a - t)$ for $t \in [a, b]$. If $\gamma : [0, 1] \to \Omega$ and $\sigma : [1, 2] \to \Omega$ are two curves with $\gamma(1) = \sigma(1)$, then their *product* or *concatenation* $\gamma\sigma : [0, 2] \to \Omega$ is defined as $\gamma\sigma(t) = \gamma(t)$ for $t \in [0, 1]$ and $\gamma\sigma(t) = \sigma(t)$ for $t \in [1, 2]$. When the order of the operations are not important, the above operations on curves can suggestively be written in the additive notation as $-\gamma \equiv \gamma^{-1}$ and $\gamma + \sigma \equiv \gamma\sigma$.

The curve $\gamma : [a, b] \to \Omega$ is called *differentiable* if $\gamma \in \mathscr{C}^1([a, b])$, with $\gamma'(a) = \gamma'(b)$ for loops, where the derivatives at a and b are to be understood in the one-sided sense. The curve γ is called *piecewise differentiable* in Ω and written $\gamma \in \mathscr{C}^1_{\text{pw}}([a, b], \Omega)$ if γ is the concatenation of finitely many differentiable curves. We assume that differentiable and piecewise differentiable

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curves are oriented, which amounts to saying, e.g., for the case of differentiable curves that we allow continuously differentiable monotone increasing reparameterizations of curves.

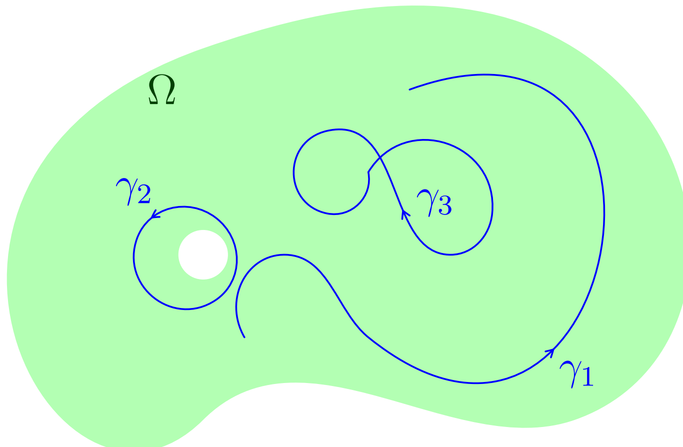

FIGURE 1. Examples of oriented curves.

Our goal in this section is to define an integral of a function $f : \Omega \to \mathbb{C}$ over a curve $\gamma : [a, b] \to \Omega$. To motivate the definition, we recall here a version of the fundamental theorem of calculus for real valued functions.

Theorem 1 (Fundamental theorem of calculus)**.** *(a) If* $g \in \mathscr{C}^1([a, b], \mathbb{R})$ *then*

$$\int_a^b g'(t)\,\mathrm{d}t = g(b) - g(a). \tag{1}$$

(b) If $f \in \mathscr{C}([a, b], \mathbb{R})$ *then the function*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \qquad a \leq x \leq b, \tag{2}$$

satisfies $F \in \mathscr{C}^1([a, b], \mathbb{R})$ *and* $F' = f$ *on* $[a, b]$*.*

Let us record here an immediate corollary that will be useful.

Corollary 2. *Let* $\phi \in \mathscr{C}^1([a, b], \mathbb{R})$*, and* $f \in \mathscr{C}([c, d], \mathbb{R})$ *with* $\phi([a, b]) \subset [c, d]$*. Then we have*

$$\int_a^b f(\phi(t))\phi'(t)\,\mathrm{d}t = \int_{\phi(a)}^{\phi(b)} f(x)\,\mathrm{d}x. \tag{3}$$

Proof. By hypothesis, the functions $f : [c, d] \to \mathbb{R}$ and $(f \circ \phi)\phi' : [a, b] \to \mathbb{R}$ are continuous, and therefore Riemann intergable. Let $F : [a, b] \to \mathbb{R}$ be defined by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \qquad a \leq x \leq b, \tag{4}$$

which, by the fundamental theorem of calculus, satisfies $F \in \mathscr{C}^1([a, b])$ and $F' = f$ in $[a, b]$. Then we have

$$\int_{\phi(a)}^{\phi(b)} f(x)\,\mathrm{d}x = \int_{\phi(a)}^{\phi(b)} F'(x)\,\mathrm{d}x = F(\phi(b)) - F(\phi(a)). \tag{5}$$

On the other hand, taking onto account the fact that

$$(F \circ \phi)'(t) = F'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t), \qquad a \leq t \leq b, \tag{6}$$

we infer

$$\int_a^b f(\phi(t))\phi'(t)\,\mathrm{d}t = \int_a^b (F\circ\phi)'(t)\,\mathrm{d}t = (F\circ\phi)(b) - (F\circ\phi)(a) \tag{7}$$

establishing the proof. □

Getting back to the main goal of this section, we want to require complex integration to have the property

$$\int_\gamma F'(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)), \tag{8}$$

where the left hand side is the yet-to-be-defined integral of $F' : \Omega \to \mathbb{C}$ over the curve $\gamma : [a,b] \to \Omega$. This property mimics the first part of the fundamental theorem of calculus, and basically asks complex integration to be an operation that inverts complex differentiation. Note that this is a very strong condition: At the very least (8) says that the integral of F' over γ depends *only* on the endpoints $\gamma(a)$ and $\gamma(b)$ of the curve, and it does not matter how the curve behaves between its endpoints.

A clue to how to ensure (8) comes from the fundamental theorem of calculus itself. If we define $g(t) = F(\gamma(t))$ for $a \le t \le b$, then we have

$$\int_a^b g'(t)\,\mathrm{d}t = g(b) - g(a) = F(\gamma(b)) - F(\gamma(a)), \tag{9}$$

where g is considered as a pair of real valued functions defend on the interval $[a,b]$, and integration and differentiation of g are understood componentwise. Now the idea is basically to call the left hand side of (9) the integral of F' over γ. To write g' in terms of F' and possibly γ or γ', let us assume that F is complex differentiable, and that γ is differentiable. Then by definition, we have

$$F(\gamma(t+h)) - F(\gamma(t)) = \tilde{F}(\gamma(t+h))(\gamma(t+h) - \gamma(t)) = \tilde{F}(\gamma(t+h))\tilde{\gamma}(t)h, \tag{10}$$

where $\tilde{F}$ is continuous at $\gamma(t)$, and $\tilde{\gamma}$ is continuous at t, which yields

$$g'(t) = F'(\gamma(t))\gamma'(t), \tag{11}$$

and hence, in light of (9), we infer

$$\int_a^b F'(\gamma(t))\gamma'(t)\,\mathrm{d}t = F(\gamma(b)) - F(\gamma(a)). \tag{12}$$

Our intention is to define the left hand side to be the integral of F' over γ.

Definition 3. The integral of $f : \Omega \to \mathbb{C}$ over a curve $\gamma \in \mathscr{C}^1([a,b],\Omega)$ is defined by

$$\langle f, \gamma\rangle \equiv \int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t. \tag{13}$$

For piecewise differentiable curves the integral is defined via "linearity":

$$\langle f, \gamma_1 + \ldots + \gamma_n\rangle = \langle f, \gamma_1\rangle + \ldots + \langle f, \gamma_n\rangle. \tag{14}$$

Remark 4. (a) The integral $\langle f, \gamma\rangle$ is well-defined, e.g., if $f : \Omega \to \mathbb{C}$ is continuous.

(b) Let $f \in \mathscr{C}(\Omega)$ and $\gamma \in \mathscr{C}^1([a,b],\Omega)$. Let $\phi \in \mathscr{C}^1([c,d])$ with $\phi([c,d]) \subset [a,b]$. Then by Corollary 2 we have

$$\int_c^d f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)\,\mathrm{d}t = \int_{\phi(c)}^{\phi(d)} f(\gamma(t))\gamma'(\phi(t))\,\mathrm{d}t. \tag{15}$$

Putting $\phi(c) = a$ and $\phi(d) = b$, we infer

$$\langle f, \gamma\circ\phi\rangle = \langle f, \gamma\rangle, \tag{16}$$

which means that the integral $\langle f,\gamma\rangle$ is invariant under reparameterizations of the oriented curve γ. On the other hand, if $\phi(c)=b$ and $\phi(d)=a$, we have

$$\langle f,\gamma\circ\phi\rangle=-\langle f,\gamma\rangle, \tag{17}$$

and so in particular

$$\langle f,-\gamma\rangle=-\langle f,\gamma\rangle. \tag{18}$$

(c) If γ_1 and γ_2 are piecewise differentiable curves in Ω, and if $f\in\mathscr{C}(\Omega)$, then

$$\langle f,\gamma_1+\gamma_1\rangle=\langle f,\gamma_1\rangle+\langle f,\gamma_2\rangle. \tag{19}$$

(d) If $f\in\mathscr{C}(\Omega)$ and $\gamma\in\mathscr{C}^1([a,b],\Omega)$ then

$$|\langle f,\gamma\rangle|\leq\max_{a\leq t\leq b}|f(\gamma(t))|\cdot\int_a^b|\gamma'(t)|\,\mathrm{d}t. \tag{20}$$

Example 5. (a) Consider $f(z)=\bar{z}$ and $\gamma(t)=re^{it}$ for $0\leq t\leq 2\pi$, where $r>0$. By the chain rule (11), which we rewrite here as

$$\frac{\mathrm{d}}{\mathrm{d}t}g(\alpha(t))=\frac{\mathrm{d}g(z)}{\mathrm{d}z}\Big|_{z=\alpha(t)}\cdot\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t}, \tag{21}$$

we have

$$\gamma'(t)=\frac{\mathrm{d}}{\mathrm{d}t}(re^{it})=\frac{\mathrm{d}}{\mathrm{d}z}(re^{z})\Big|_{z=it}\cdot\frac{\mathrm{d}(it)}{\mathrm{d}t}=rie^{it}. \tag{22}$$

Then noting that $f(\gamma(t))=f(re^{it})=re^{-it}$, we infer

$$\int_\gamma\bar{z}\,\mathrm{d}z=\int_0^{2\pi}re^{-it}\cdot rie^{it}\mathrm{d}t=r^2i\int_0^{2\pi}\mathrm{d}t=2\pi r^2 i. \tag{23}$$

(b) Let $f(z)=z$, and let γ be as in (a). Then we have

$$\int_\gamma z\,\mathrm{d}z=\int_0^{2\pi}re^{it}\cdot rie^{it}\mathrm{d}t=r^2i\int_0^{2\pi}e^{2it}\mathrm{d}t. \tag{24}$$

On the other hand, the chain rule (21) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{2it}=2ie^{2it}, \tag{25}$$

and therefore (24) can be continued as

$$\int_\gamma z\,\mathrm{d}z=r^2i\int_0^{2\pi}\frac{1}{2i}\Big(\frac{\mathrm{d}}{\mathrm{d}t}e^{2it}\Big)\mathrm{d}t=\frac{r^2e^{2it}}{2}\Big|_0^{2\pi}=\frac{r^2(e^{4\pi i}-1)}{2}=0. \tag{26}$$

(c) Now consider $f(z)=\frac{1}{z}$, with γ as above.

$$\int_\gamma\frac{\mathrm{d}z}{z}=\int_0^{2\pi}\frac{rie^{it}}{re^{it}}\mathrm{d}t=i\int_0^{2\pi}\mathrm{d}t=2\pi i. \tag{27}$$

Exercise 6. For each $n\in\mathbb{Z}$, compute the integral of z^n over the circle given by $\gamma(t)=re^{it}$, $0\leq t\leq 2\pi$, where $r>0$.

Basically by construction, we get the following result.

Theorem 7 (FTC for holomorphic functions)**.** *Let $F\in\mathscr{O}(\Omega)$ be a holomorphic function, and suppose that F' is continuous in Ω. Then for any $\gamma\in\mathscr{C}^1_{\mathrm{pw}}([a,b],\Omega)$ we have*

$$\int_\gamma F'(z)\,\mathrm{d}z=F(\gamma(b))-F(\gamma(a)). \tag{28}$$

Proof. The function $g=F\circ\gamma$ satisfies $g\in\mathscr{C}^1([a,b],\mathbb{R}^2)$ with $g'=(F'\circ\gamma)\cdot\gamma'$, and an application of the fundamental theorem of calculus (Theorem 1) finishes the job. □

Remark 8. The continuity hypothesis on F' is in fact superfluous, since it will turn out that holomorphic functions are infinitely often differentiable. However, the above form (with the continuity hypothesis) will be used to prove that fact.

Corollary 9. *In the setting of the preceding theorem, if γ is a closed curve, then we have*

$$\langle F', \gamma \rangle = 0. \tag{29}$$

In particular, for any polynomial p and any $\gamma \in \mathscr{C}^1_{\mathrm{pw}}(S^1, \mathbb{C})$, we have

$$\langle p, \gamma \rangle = 0. \tag{30}$$

Proof. For the first assertion, Theorem 7 and the condition $\gamma(a) = \gamma(b)$ give

$$\int_\gamma F'(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)) = 0. \tag{31}$$

For the second assertion, since p is a polynomial, it is *integrable in* $\mathbb{C}$, meaning that there is $F \in \mathscr{O}(\mathbb{C})$ such that $F' = p$ in $\mathbb{C}$. Then we apply the first assertion to finish the proof. □

2. Goursat's theorem

For a set $U \subset \mathbb{C}$ that is not open, the notation $f \in \mathscr{O}(U)$ means that f is holomorphic in an open neighbourhood of U. Let us denote by $[a, b]$ the (oriented) line segment with the initial point $a \in \mathbb{C}$ and the terminal point $b \in \mathbb{C}$. Given three points $a, b, c \in \mathbb{C}$, the *triangular loop* $[a, b, c]$ is defined to be the oriented loop $[a, b] + [b, c] + [c, a]$. The set of points that are strictly inside the loop $[a, b, c]$ forms an open set $\tau \subset \mathbb{C}$, which we call an *open triangle*. In this setting, the loop $[a, b, c]$ is called the *boundary of* τ, and written $\partial\tau = [a, b, c]$. Note that there is an ambiguity in the orientation of $\partial\tau$, since the loop $[a, c, b]$, whose orientation is the opposite of that of $[a, b, c]$, gives rise to the same open triangle τ. The default convention is to orient $\partial\tau$ in such a way that τ lies on the left of $\partial\tau$, but it is always a good idea to explicitly mention the chosen orientation to avoid confusion. Finally, the *closure* of τ, denoted by $\bar{\tau}$, is the union of τ and $\partial\tau$, the latter taken as a set.

The following theorem was proved by Édouard Goursat (1858-1936) in 1883. This is an improvement over Cauchy's theorem, in which Cauchy assumed that not only f is holomorphic, but also the derivative f' is continuous. While the original formulation by Goursat employs rectangles, the following "triangular" version is due to Alfred Pringsheim (1850-1941).

Theorem 10. *Let $\tau \subset \mathbb{C}$ be an open triangle, and let $f \in \mathscr{O}(\bar{\tau})$. Then $\langle f, \partial\tau \rangle = 0$.*

Proof. Let us subdivide τ into 4 congruent triangles $\tau_1, \tau_2, \tau_3, \tau_4$ by connecting the midpoints of the edges of τ. All lengths of the smaller triangles are measured as half the corresponding length of the original triangle τ. Moreover we have

$$\langle f, \partial\tau \rangle = \sum_{1 \le j \le 4} \langle f, \partial\tau_j \rangle. \tag{32}$$

Let τ_m be a triangle among the 4 triangles that gives the largest contribution to the sum, and call it $\tau^{(1)}$, that is, τ_m (with some m between 1 and 4) satisfies $|\langle f, \partial\tau_m \rangle| \ge |\langle f, \partial\tau_j \rangle|$ for any $1 \le j \le 4$. Then we have

$$|\langle f, \partial\tau \rangle| \le 4|\langle f, \partial\tau^{(1)} \rangle|. \tag{33}$$

Now subdividing $\tau^{(1)}$ into 4 still smaller triangles, and repeating this procedure, we get

$$|\langle f, \partial\tau \rangle| \le 4^n|\langle f, \partial\tau^{(n)} \rangle|, \tag{34}$$

with any length of $\tau^{(n)}$ being 2^{-n} part of the corresponding length of τ. In particular, if c_n is a point[1] in $\tau^{(n)}$, then the sequence $\{c_n\}$ is Cauchy, so $c_n \to c$ for some $c \in \overline{\tau}$. Since f is holomorphic in a neighbourhood of $\overline{\tau}$, by definition we have

$$f(z) = f(c) + \lambda(z - c) + o(2^{-n}), \qquad z \in \partial\tau^{(n)}, \tag{35}$$

with some constant $\lambda \in \mathbb{C}$. We calculate the integral of f over the boundary of $\tau^{(n)}$ to be

$$\langle f, \partial\tau^{(n)}\rangle = \langle f(c) + \lambda(z - c), \partial\tau^{(n)}\rangle + o(2^{-n} \cdot 2^{-n}) = o(4^{-n}), \tag{36}$$

where the integral vanish by Corollary 9, and we have taken into account that the perimeter of $\partial\tau^{(n)}$ is of the order $O(2^{-n})$. Substituting this into (34) establishes the proof. □

It is possible to slightly relax the hypothesis of Goursat's theorem, so that only holomorphy in the interior and continuity up to the boundary are assumed. The argument is a continuity argument that can be used to strengthen many of the theorems that follow.

Corollary 11. *Let $\tau \subset \mathbb{C}$ be an open triangle, and let $f \in \mathscr{O}(\tau) \cap C(\overline{\tau})$. Then $\langle f, \partial\tau\rangle = 0$.*

Proof. Let a, b, c be the vertices of τ, and let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of points in τ such that $a_n \to a$, $b_n \to b$, and $c_n \to c$ as $n \to \infty$. As the closure of the triangle τ_n defined by $[a_n, b_n, c_n]$ is entirely in τ, Goursat's theorem applies to τ_n, meaning that $\langle f, \partial\tau_n\rangle = 0$. By uniform continuity, $\langle f, \partial\tau_n\rangle$ tends to $\langle f, \partial\tau\rangle$, hence $\langle f, \partial\tau\rangle = 0$. □

3. Local integrability

In what follows, by default Ω will always denote an open subset of $\mathbb{C}$.

Definition 12. A continuous function $f \in C(\Omega)$ is called *integrable* in Ω if there is $F \in \mathscr{O}(\Omega)$ such that $F' = f$ in Ω. It is called *locally integrable* in Ω if for any $z \in \Omega$ there exists an open neighbourhood U of z such that f is integrable in U.

In combination with Goursat's theorem, the theorem below implies that holomorphic functions are locally integrable. By a *closed triangle* we mean a set of the form $\bar{\tau}$, where $\tau \subset \mathbb{C}$ is an open triangle.

Theorem 13. *Let $D = D_r(c)$ be an open disk, and let $f \in \mathscr{C}(D)$ satisfy $\langle f, \partial\tau\rangle = 0$ for any closed triangle $\tau \subset \Omega$. Then f is integrable in D.*

Proof. Define $F(z) = \langle f, [c, z]\rangle$ for $z \in D$. We would like to show that $F' = f$ on D, or equivalently that

$$F(w) = F(z) + f(z)(w - z) + o(|w - z|). \tag{37}$$

From the definition of F we have $F(w) - F(z) = \langle f, [z, w]\rangle$, and taking into account that $w - z = \langle 1, [z, w]\rangle$, we infer

$$F(w) - F(z) - f(z)(w - z) = \langle f, [z, w]\rangle - f(z)\langle 1, [z, w]\rangle. \tag{38}$$

Now $f = f(z) + o(1)$ on $[z, w]$, so the right hand side is of order $o(|w - z|)$. □

In the subsequent sections, by a sequence of several theorems, we will prove that locally integrable functions are analytic, therefore also holomorphic. Hence local integrability is equivalent to holomorphy.

As a simple application of the theorem, we get Cauchy's theorem for disks.

Corollary 14. *Let $f \in \mathscr{O}(D)$, where $D = D_r(c)$ is an open disk. Then $\langle f, \gamma\rangle = 0$ for any piecewise differentiable loop $\gamma \in \mathscr{C}^1_{\mathrm{pw}}(S^1, D)$ lying in D.*

[1] For concreteness, e.g., we may take c_n to be the barycenter of $\tau^{(n)}$.

Proof. By the preceding theorem (in combination with Goursat's theorem) there is $F \in \mathscr{O}(\Omega)$ such that $F' = f$ on D. Then the fundamental theorem of calculus for holomorphic functions (Theorem 7) states that the integral of f over any piecewise differentiable closed curve must be zero. □

We can slightly extend the argument in the proof of Theorem 13 to get a criterion on (global) integrability.

Theorem 15. *A continuous function $f \in \mathscr{C}(\Omega)$ is integrable in Ω if and only if $\langle f, \gamma \rangle = 0$ for any $\gamma \in \mathscr{C}^1_{\mathrm{pw}}(S^1, \Omega)$.*

Proof. One direction is immediate from the fundamental theorem of calculus. For the other direction, assume that Ω is connected (otherwise we work in connected components of Ω one by one). Let $c \in \Omega$, and for $z \in \Omega$ define $F(z) = \langle f, \gamma \rangle$ with γ a piecewise differentiable curve connecting[2] c and z. The value $F(z)$ does not depend on the particular curve γ, since if σ is another curve connecting c and z, then $\gamma - \sigma$ is a piecewise differentiable loop in Ω, so that $\langle f, \gamma \rangle = \langle f, \sigma \rangle$ by hypothesis. Now noting that $F(w) - F(z) = \langle f, [z, w] \rangle$, the proof proceeds in exactly the same way as in the proof of Theorem 13. □

4. Cauchy's theorem for homotopic loops

Definition 16. Loops $\gamma_0, \gamma_1 \in \mathscr{C}(S^1, \Omega)$ are called (*freely*) *homotopic* to each other, and written $\gamma_0 \approx \gamma_1$, if there exists a continuous map $\Gamma : S^1 \times [0, 1] \to \Omega$ such that $\Gamma(t, 0) = \gamma_0(t)$ and $\Gamma(t, 1) = \gamma_1(t)$ for $t \in S^1$.

Free homotopy is an equivalence relation in the space of loops, and so this space is partitioned into (free) *homotopy classes.* The following theorem shows that at least in the piecewise differentiable case, the integral of a given holomorphic function over a loop depends only on the homotopy class the loop represents.

Theorem 17. *For $f \in \mathscr{O}(\Omega)$ and for piecewise differentiable loops $\gamma_0, \gamma_1 \in \mathscr{C}^1_{\mathrm{pw}}(S^1, \Omega)$ with $\gamma_0 \approx \gamma_1$, we have*

$$\langle f, \gamma_0 \rangle = \langle f, \gamma_1 \rangle. \tag{39}$$

Proof. Let us parametrize the circle S^1 by the interval $[0, 1]$, so the curves will be maps defined on $[0, 1]$. Let $\Gamma : [0, 1]^2 \to \Omega$ be a homotopy between γ_0 and γ_1. Since $[0, 1]^2$ is compact, Γ is uniformly continuous, and the image $|\Gamma| = \{\Gamma(t, s) : (t, s) \in [0, 1]^2\}$ is a compact subset of Ω. Fix $\varepsilon > 0$ such that $\varepsilon < \mathrm{dist}(|\Gamma|, \mathbb{C} \setminus \Omega)$. Obviously, f is integrable in any disk $D_\varepsilon(z)$ with $z \in |\Gamma|$. For a large integer n, let $z_{j,k} = \Gamma(\frac{j}{n}, \frac{k}{n})$ for $j = 0, \ldots, n$, and $k = 0, \ldots, n$. Let $Q_{j,k}$ be the closed quadrilateral with the vertices $z_{j,k}$, $z_{j+1,k}$, $z_{j+1,k+1}$, and $z_{j,k+1}$. Then for $k = 0$, we modify $Q_{j,k}$ so that the straight edge $[z_{j,k}, z_{j+1,k}]$ is replaced by the piece of γ_0 that lies between $z_{j,k}$ and $z_{j+1,k}$. Similarly, for $k = n - 1$, we modify $Q_{j,k}$ so that the straight edge $[z_{j+1,k+1}, z_{j,k+1}]$ is replaced by the piece of γ_1 that lies between $z_{j+1,k+1}$ and $z_{j,k+1}$. Thus in general $Q_{j,k}$ with $k = 0$ or $k = n - 1$ is going to be a quadrilateral with a curved edge. We choose n to be so large that $Q_{j,k} \subset D_\varepsilon(z_{j,k})$ for all j and k. Then note that

$$\langle f, \gamma_0 \rangle - \langle f, \gamma_1 \rangle = \sum_{j,k} \langle f, \partial Q_{j,k} \rangle, \tag{40}$$

where the contribution from any edge of $Q_{j,k}$ that does not coincide with an edge of either γ_0 or γ_1 is canceled due to the opposite orientations that a common edge inherits from neighbouring polygons. Moreover, each integral $\langle f, \partial Q_{j,k} \rangle$ is zero because f is integrable on $D_\varepsilon(z_{j,k}) \subset \Omega$ and $\partial Q_{j,k}$ is a polygonal loop in $D_\varepsilon(z_{j,k})$. The theorem is proven. □

[2]Any two points in a connected open planar set can be connected by a piecewise linear curve.

If a loop γ is homotopic to a constant path, i.e., $\gamma \approx \delta$ with $\delta : [a, b] \to \Omega$ such that $\delta \equiv z$ for some $z \in \Omega$, then γ is said to be *topologically trivial* or *null-homotopic*, and this fact is written as $\gamma \approx 0$.

Corollary 18. *If $\gamma \in \mathscr{C}^1_{\rm pw}(S^1, \Omega)$ is topologically trivial, then $\langle f, \gamma\rangle = 0$ for any $f \in \mathscr{O}(\Omega)$.*

A somewhat trivial way to ensure that a particular closed curve in Ω is topologically trivial is to simply require that every closed curve in Ω is topologically trivial.

Definition 19. A set $\Omega \subset \mathbb{C}$ is called *simply connected* if it is connected and every closed curve in Ω is topologically trivial.

Exercise 20. A *star-shaped* sets are characterized by the property that there is $c \in \Omega$ such that $z \in \Omega$ implies $[z, c] \subset \Omega$. For example, convex sets are star-shaped. Show that star-shaped sets are simply connected.

Corollary 21. *If Ω is simply connected then $\langle f, \gamma\rangle = 0$ for $f \in \mathscr{O}(\Omega)$ and $\gamma \in \mathscr{C}^1_{\rm pw}(S^1, \Omega)$.*

Definition 22. Curves $\gamma_0, \gamma_1 \in \mathscr{C}([a, b], \Omega)$ are called *homotopic relative to their endpoints*, and written $\gamma_0 \approx_{\{a,b\}} \gamma_1$, if there exists a continuous map $\Gamma : [a, b] \times [0, 1] \to \Omega$ such that $\Gamma(t, 0) = \gamma_0(t)$ and $\Gamma(t, 1) = \gamma_1(t)$ for all $t \in [a, b]$, and $\Gamma(t, s) = \gamma_0(t)$ for all $s \in [0, 1]$ and $t \in \{a, b\}$.

Note that $\gamma_0 \approx_{\{a,b\}} \gamma_1$ implies in particular that $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$. Similarly to the free homotopy case, the space of curves that are "fixed" at their endpoints is partitioned into (relative) homotopy classes.

Corollary 23. *For $f \in \mathscr{O}(\Omega)$ and for piecewise differentiable curves $\gamma_0, \gamma_1 \in \mathscr{C}^1_{\rm pw}([a, b], \Omega)$ with $\gamma_0 \approx_{\{a,b\}} \gamma_1$, we have $\langle f, \gamma_0\rangle = \langle f, \gamma_1\rangle$.*

Proof. One can show that the curve $\gamma_0 - \gamma_1$ is a topologically trivial piecewise differentiable loop, by constructing a homotopy that, e.g., first follows the homotopy between γ_0 and γ_1 relative to the endpoints to collapse γ_0 onto γ_1, and then contracts γ_1 to a point. □

5. Evaluation of real definite integrals

Recall that Euler's main motivation for studying complex functions was to find a new way to integrate real functions. With the help of Cauchy's theorem, we can now make Euler's procedure precise. In this section, we want to look at a general method to treat improper Riemann integrals.

Definition 24. Given $f : (a, b) \to \mathbb{R}$ with $-\infty \le a < b \le \infty$, the *improper Riemann integral* of f over (a, b) is defined as

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\alpha\searrow a}\lim_{\beta\nearrow b}\int_\alpha^\beta f(x)\,\mathrm{d}x. \tag{41}$$

Moreover, for $f : (a, b) \cup (b, c) \to \mathbb{R}$, we define

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x. \tag{42}$$

Example 25. We have

$$\int_1^\infty \frac{\mathrm{d}x}{x^2} = \lim_{b\to\infty}\int_1^b \frac{\mathrm{d}x}{x^2} = \lim_{b\to\infty}\big(-\frac{1}{x}\big)\Big|_1^\infty = 1. \tag{43}$$

The trivial link between complex and real integrations is the observation that a complex contour integral reduces to a usual Riemann integral if the contour happens to be a real interval. Indeed, if $\gamma(t) = t$, with $a \leq t \leq b$, then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(t)\,\mathrm{d}t, \tag{44}$$

since $\gamma'(t) = 1$. Now we illustrate the method with an example.

Example 26. Let us compute the improper integral

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^2} = \lim_{R\to\infty} \int_0^R \frac{\mathrm{d}x}{1+x^2}. \tag{45}$$

First, we write

$$\int_0^R \frac{\mathrm{d}x}{1+x^2} = \frac{1}{2}\int_{-R}^R \frac{\mathrm{d}x}{1+x^2}. \tag{46}$$

Next, we consider the curve $\gamma_R^+(t) = Re^{it}$, $0 \leq t \leq \pi$, which is the upper half of the circle of radius R centred at the origin, and let γ_R be the concatenation of the real interval $[-R, R]$ and the semicircle γ_R^+. Then we have

$$\int_{-R}^R \frac{\mathrm{d}x}{1+x^2} = \int_{\gamma_R} \frac{\mathrm{d}z}{1+z^2} - \int_{\gamma_R^+} \frac{\mathrm{d}z}{1+z^2}. \tag{47}$$

Now we will evaluate the first integral in the right hand side. Since the function $f(z) = \frac{1}{1+z^2}$ is holomorphic in $\Omega = \mathbb{C} \setminus \{i, -i\}$, and the loop γ_R is freely homotopic in Ω to the loop $\gamma_\varepsilon(t) = i + \varepsilon e^{it}$, $0 \leq t \leq 2\pi$, we infer

$$\int_{\gamma_R} \frac{\mathrm{d}z}{1+z^2} = \int_{\gamma_\varepsilon} \frac{\mathrm{d}z}{1+z^2} = \int_0^{2\pi} \frac{i\varepsilon e^{it}\mathrm{d}t}{\varepsilon e^{it}(2i+\varepsilon e^{it})} = \int_0^{2\pi} \frac{i\mathrm{d}t}{2i+\varepsilon e^{it}}. \tag{48}$$

It is intuitively clear that the integrand is approximately $\frac{1}{2}$ when $\varepsilon > 0$ is small. To obtain a precise bound, note that

$$\Big|\frac{1}{2i+\varepsilon e^{it}} - \frac{1}{2i}\Big| = \Big|\frac{\varepsilon e^{it}}{2i(2i+\varepsilon e^{it})}\Big| \leq \frac{|\varepsilon e^{it}|}{4 - |2\varepsilon e^{it}|} \leq \frac{\varepsilon}{2}, \tag{49}$$

as long as $0 < \varepsilon \leq 1$, which shows that

$$\Big|\int_0^{2\pi} \frac{i\mathrm{d}t}{2i+\varepsilon e^{it}} - \int_0^{2\pi} \frac{\mathrm{d}t}{2}\Big| \leq \int_0^{2\pi} \frac{\varepsilon\mathrm{d}t}{2} = \pi\varepsilon. \tag{50}$$

since this is true for any small $\varepsilon > 0$, we conclude that

$$\int_{\gamma_R} \frac{\mathrm{d}z}{1+z^2} = \int_0^{2\pi} \frac{i\mathrm{d}t}{2i+\varepsilon e^{it}} = \int_0^{2\pi} \frac{\mathrm{d}t}{2} = \pi. \tag{51}$$

Finally, for the integral over the semicircle γ_R^+ in (47), we have

$$\Big|\int_{\gamma_R^+} \frac{\mathrm{d}z}{1+z^2}\Big| \leq \frac{1}{R^2-1}\cdot \pi R \to 0 \qquad \text{as} \quad R\to\infty, \tag{52}$$

and therefore the conclusion is

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^2} = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^R \frac{\mathrm{d}x}{1+x^2} = \frac{\pi}{2} - \frac{1}{2}\lim_{R\to\infty}\int_{\gamma_R^+} \frac{\mathrm{d}z}{1+z^2} = \frac{\pi}{2}. \tag{53}$$

Remark 27. The illustrated method works for any integral of the form

$$\int_{-\infty}^{\infty} \frac{p(x)\mathrm{d}x}{q(x)}, \tag{54}$$

where p and q are polynomials satisfying $\deg(q) \geq \deg(p) + 2$ and $q(x) \neq 0$ for $x \in \mathbb{R}$.

Exercise 28 (Jordan's lemma)**.** With γ_R^+ as in the preceding example, show that

$$\int_{\gamma_R^+} f(z)e^{i\alpha z}\mathrm{d}z \to 0 \qquad \text{as} \quad R \to \infty, \tag{55}$$

if $\alpha > 0$, and $|f(z)| \leq M(1 + |z|)^{-s}$ for $z \in \mathbb{C}$, with some constants M and $s > 0$.

6. The Cauchy integral formula

In this section, we will prove the Cauchy integral formula, which may be considered as *the cornerstone* of complex analysis. Before stating the result, let us recall the notations $D_r(c) = \{z \in \mathbb{C} : |z - c| < r\}$ and $\bar{D}_r(c) = \{z \in \mathbb{C} : |z - c| \leq r\}$. Moreover, given a disk $D = D_r(c)$, we denote by ∂D the oriented curve given by $\gamma(t) = c + re^{it}$ for $0 \leq t \leq 2\pi$.

Theorem 29 (Cauchy 1831)**.** *Let $f \in \mathscr{O}(\Omega)$, and let $\bar{D}_r(c) \subset \Omega$ with $r > 0$. Then we have*

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial D_r(c)} \frac{f(z)\mathrm{d}z}{z - \zeta} \qquad \textit{for} \quad \zeta \in D_r(c). \tag{56}$$

Proof. The function

$$F(z) = \frac{f(z)}{z - \zeta}, \qquad z \in \Omega \setminus \{\zeta\}, \tag{57}$$

is holomorphic in $\Omega \setminus \{\zeta\}$, and with $\varepsilon > 0$ small, the loop γ_ε defined by $\gamma_\varepsilon(t) = \zeta + \varepsilon e^{it}$ for $0 \leq t \leq 2\pi$, is homotopic in $\Omega \setminus \{\zeta\}$ to the circle $\partial D_r(c)$. Hence we have

$$\int_{\partial D_r(c)} \frac{f(z)\mathrm{d}z}{z - \zeta} = \int_{\gamma_\varepsilon} \frac{f(z)\mathrm{d}z}{z - \zeta}. \tag{58}$$

By complex differentiability, there is a function $g : \Omega \to \mathbb{C}$, continuous at ζ, such that

$$f(z) = f(\zeta) + g(z)(z - \zeta). \tag{59}$$

Substituting this into (58), we get

$$\int_{\partial D_r(c)} \frac{f(z)\mathrm{d}z}{z - \zeta} = f(\zeta)\int_{\gamma_\varepsilon} \frac{\mathrm{d}z}{z - \zeta} + \int_{\gamma_\varepsilon} g(z)\mathrm{d}z. \tag{60}$$

The first integral in the right hand side leads to the familiar computation

$$\int_{\gamma_\varepsilon} \frac{\mathrm{d}z}{z - \zeta} = \int_0^{2\pi} \frac{i\varepsilon e^{it}\mathrm{d}t}{\varepsilon e^{it}} = 2\pi i. \tag{61}$$

As for the second integral, let $\delta > 0$ be such that $|g(z) - g(\zeta)| < 1$ whenever $|z - \zeta| < \delta$. Then for $0 < \varepsilon < \delta$, we have

$$\Big|\int_{\gamma_\varepsilon} g(z)\mathrm{d}z\Big| \leq (|g(\zeta)| + 1) \cdot 2\pi\varepsilon. \tag{62}$$

We conclude that

$$\Big|\int_{\partial D_r(c)} \frac{f(z)\mathrm{d}z}{z - \zeta} - 2\pi i f(\zeta)\Big| \leq (|g(\zeta)| + 1) \cdot 2\pi\varepsilon, \tag{63}$$

for any $0 < \varepsilon < \delta$, meaning that the left hand side is equal to 0. □

Corollary 30 (Mean value property)**.** *In the setting of the theorem, with* $\gamma(t) = c + re^{it}$ *for* $0 \leq t \leq 2\pi$, *we have*

$$f(c) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)\mathrm{d}z}{z-c} = \frac{1}{2\pi}\int_0^{2\pi} f(c+re^{it})\mathrm{d}t, \tag{64}$$

which shows that the value of f *at* c *is the average of* f *over the circle* $\partial D_r(c)$.

Corollary 31. *Let* $f \in \mathscr{O}(\Omega)$, $\zeta \in \Omega$, *and let* $\gamma \in \mathscr{C}^1_{\mathrm{pw}}(S^1, \Omega \setminus \{\zeta\})$ *be a loop homotopic to* $\partial D_\varepsilon(\zeta)$ *in* $\Omega \setminus \{\zeta\}$ *for some* $\varepsilon > 0$. *Then we have*

$$f(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)\mathrm{d}z}{z-\zeta}.$$

7. The Cauchy-Taylor theorem

Definition 32. A function $f : \Omega \to \mathbb{C}$ is called *(complex) analytic in* Ω, if for any $c \in \Omega$, one can develop f into a power series centred at c, with a nonzero convergence radius. The set of all analytic functions in Ω is denoted by $\mathscr{C}^\omega(\Omega)$.

By an n-fold differentiation of the power series

$$f(z) = \sum_{n=0}^\infty a_n(z-c)^n, \tag{65}$$

we infer that the coefficients are given by $a_n = f^{(n)}(c)/n!$. In other words, if $f \in \mathscr{C}^\omega(\Omega)$ and $c \in \Omega$, then the following *Taylor series* converges in a neighbourhood of c:

$$f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(c)}{n!}(z-c)^n. \tag{66}$$

Termwise differentiation of power series also implies that analytic functions are holomorphic, i.e., $\mathscr{C}^\omega(\Omega) \subset \mathscr{O}(\Omega)$. In fact, the converse $\mathscr{O}(\Omega) \subset \mathscr{C}^\omega(\Omega)$ is also true.

Theorem 33 (Cauchy 1841)**.** *Let* $f \in \mathscr{C}(\bar{D}_r(c))$ *with* $r > 0$, *and assume that*

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial D_r(c)} \frac{f(z)\,\mathrm{d}z}{z-\zeta} \qquad \text{for} \quad \zeta \in D_r(c). \tag{67}$$

Then the power series

$$f(\zeta) = \sum a_n(\zeta-c)^n, \tag{68}$$

with

$$a_n = \frac{1}{2\pi i}\int_{\partial D_r(c)} \frac{f(z)\,\mathrm{d}z}{(z-c)^{n+1}}, \tag{69}$$

converges in $D_r(c)$. *In particular, we have* $\mathscr{O}(\Omega) \subset \mathscr{C}^\omega(\Omega)$ *for open sets* $\Omega \subset \mathbb{C}$.

Proof. Without loss of generality, let us assume $c = 0$, and start with the integral formula

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial D_r} \frac{f(z)\,\mathrm{d}z}{z-\zeta}, \qquad \text{for} \quad \zeta \in D_r. \tag{70}$$

This can be rewritten as

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial D_r} \left(\sum_{n=0}^\infty \frac{f(z)\zeta^n}{z^{n+1}}\right)\mathrm{d}z, \tag{71}$$

where we have used

$$\frac{1}{z-\zeta} = \frac{1}{z}\cdot\frac{1}{1-\zeta/z} = \frac{1}{z}\left(1 + \frac{\zeta}{z} + \ldots\right) = \sum_{n=0}^\infty \frac{\zeta^n}{z^{n+1}}. \tag{72}$$

Each term in the series under integral in (71) can be estimated as

$$\left|\frac{f(z)\zeta^n}{z^{n+1}}\right| \leq \frac{\|f\|_{D_r}}{r} \cdot \left(\frac{|\zeta|}{r}\right)^n, \qquad \text{where} \qquad \|f\|_{D_r} = \sup_{z\in D_r} |f(z)|, \tag{73}$$

so as a function of z, the series converges uniformly on ∂D_r. Therefore we can interchange the integral with the sum, resulting in

$$f(\zeta) = \frac{1}{2\pi i}\sum_{n=0}^{\infty} \zeta^n \int_{\partial D_r} \frac{f(z)\mathrm{d}z}{z^{n+1}}.$$

Now the individual term of the series satisfies

$$\left|\zeta^n \int_{\partial D_r} \frac{f(z)\mathrm{d}z}{z^{n+1}}\right| \leq 2\pi \|f\|_{D_r} \left(\frac{|\zeta|}{r}\right)^n,$$

implying that the series converges locally normally in D_r. □

8. Morera's theorem

The following result was proved by Giacinto Morera (1856-1907) in 1886.

Theorem 34. *A function $f \in \mathscr{C}(\Omega)$ is holomorphic if any of the following conditions holds.*
(a) f is locally integrable.
(b) $\langle f, \partial\tau\rangle = 0$ for any closed triangle $\tau \subset \Omega$.

Proof. Condition (a) follows from (b) by Theorem 13. Now suppose that (a) holds. Then by definition, each point in Ω has a neighbourhood U and $F \in \mathscr{O}(U)$ such that $F' = f$ on U. The Cauchy-Taylor theorem guarantees that $F \in \mathscr{C}^\omega(U)$, and by termwise differentiating we infer $f \in \mathscr{C}^\omega(U)$. This means that $f \in \mathscr{C}^\omega(\Omega)$, or in other words $f \in \mathscr{O}(\Omega)$. □

As an application, one can prove that the locally uniform limit of holomorphic functions is holomorphic. This is to be contrasted with the situation in the real differentiable case where the uniform limit of smooth functions is not smooth in general. The following theorem is often called the *Weierstrass convergence theorem.*

Theorem 35 (Weierstrass 1841)**.** *Let $\{f_k\} \subset \mathscr{O}(\Omega)$ be a sequence such that $f_k \to f$ locally uniformly in Ω for some function $f : \Omega \to \mathbb{C}$. Then $f \in \mathscr{O}(\Omega)$ and $f_k^{(n)} \to f^{(n)}$ locally uniformly in Ω, for each $n \in \mathbb{N}$.*

Proof. First of all we have $f \in \mathscr{C}(\Omega)$, without using complex analysis. Now let $\tau \subset \Omega$ be a closed triangle. Then since $\partial\tau$ is compact, f_n converges uniformly to f on $\partial\tau$, and so we have

$$\langle f, \partial\tau\rangle = \lim_{k\to\infty} \langle f_k, \partial\tau\rangle = 0,$$

implying that $f \in \mathscr{O}(\Omega)$ by Morera's theorem.

For the second part of the claim we employ the Cauchy estimates. Let $D_{2\delta}(a) \subset \Omega$ for some $\delta > 0$. Then since $f_k - f \in \mathscr{O}(\Omega)$, for $k \in \mathbb{N}$ the Cauchy estimate gives

$$\|f_k^{(n)} - f^{(n)}\|_{D_\delta(a)} \leq \frac{n!}{\delta^n}\|f_k - f\|_{D_{2\delta}(a)},$$

completing the proof. □

We close this section with a theorem that offers many different characterizations of holomorphic functions. This is an indication that the initial phase in the development of the theory is now complete. If we were building a rocket, at this point we have assembled it and are ready to start testing.

Theorem 36. *Let $\Omega \subset \mathbb{C}$ be open, and let $f \in \mathscr{C}(\Omega)$. Then the following are equivalent.*

(a) f *is holomorphic in* Ω, *i.e.,* $f \in \mathscr{O}(\Omega)$.
(b) For all closed triangles $\tau \subset \Omega$, *the integral of* f *over the boundary of* τ *is zero.*
(c) f *is locally integrable in* Ω.
(d) For all open disks D *with* $\bar{D} \subset \Omega$ *and for* $a \in D$, *one has* $f(a) = \dfrac{1}{2\pi i} \displaystyle\int_{\partial D} \frac{f(z)\mathrm{d}z}{z-a}$.
(e) f *is analytic in* Ω, *i.e.,* $f \in \mathscr{C}^\omega(\Omega)$.

Proof. The implication (e) ⇒ (a) follows from termwise differentiation of power series. The implication (a) ⇒ (b) is Goursat's theorem in §2, and (b) ⇒ (c) is the integrability theorem (Theorem 13) in §3. Then (a) ⇒ (d) is the argument leading to the Cauchy integral formula in §4 and §6, where we have also used the implication (a) ⇒ (c), and (d) ⇒ (e) is the Cauchy-Taylor theorem in §7. Using all of these, (c) ⇒ (a) is proven as Morera's theorem in §8. □

9. The Cauchy estimates

We have the following fundamental estimates on power series coefficients, which roughly says that the largest term in the series determines the maximum absolute value the power series can have in a given region.

Theorem 37 (Cauchy 1835). *Let* $f(z) = \sum a_n(z-c)^n$ *be convergent in an open neighbourhood of* $\bar{D}_\rho(c)$. *Then we have*

$$|a_n|\rho^n \leq \max_{|z-c|=\rho} |f(z)| \qquad \text{for } n = 0, 1, \ldots, \tag{74}$$

or equivalently,

$$|f^{(n)}(c)| \leq \frac{n!}{\rho^n} \max_{|z-c|=\rho} |f(z)| \qquad \text{for } n = 0, 1, \ldots. \tag{75}$$

Proof. It is immediate from the Cauchy-Taylor theorem (Theorem 33) that

$$|a_n| = \frac{1}{2\pi}\left| \int_{\partial D_\rho(c)} \frac{f(z)\,\mathrm{d}z}{(z-c)^{n+1}} \right| \leq \frac{1}{2\pi} \cdot 2\pi\rho \cdot \frac{1}{\rho^{n+1}}, \tag{76}$$

which is the desired estimate. □

Functions analytic in the entire complex plane $\mathbb{C}$ are called *entire functions*. In 1844, Cauchy proved that bounded entire functions are constant, but this theorem is now known as Liouville's theorem. The reason for this is attributed to Carl Borchardt (1817-1880), who learned the theorem from Joseph Liouville (1809-1882) in 1847, and then published it under the name Liouville's theorem in 1879.

Corollary 38 (Liouville's theorem). *If* $f \in \mathscr{O}(\mathbb{C})$, *and if there exists a constant* $M > 0$ *such that* $|f(z)| \leq M$ *for* $z \in \mathbb{C}$, *then* f *must be a constant function.*

Proof. By the Cauchy-Taylor theorem, the Taylor series of f centred at the origin converges uniformly on any closed disk. Applying the Cauchy estimates to this series on $\bar{D}_\rho$ with $\rho > 0$, we get the following bound on the n-th coefficient

$$|a_n| \leq \rho^{-n} \max_{|z|=\rho} |f(z)| \leq \rho^{-n} M.$$

Since ρ can be arbitrarily large, this estimate shows that $a_n = 0$ for all $n = 1, 2, \ldots$. □

Liouville's theorem can be used to prove the fundamental theorem of algebra.

Corollary 39. *Any nonconstant polynomial has at least one root in* $\mathbb{C}$.

Proof. Suppose that a polynomial p has no root. Then $f = \frac{1}{p} \in \mathscr{O}(\mathbb{C})$. If $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ with $a_n \neq 0$, then $|p(z)| \sim |a_n||z|^n$ for large z, meaning that f is bounded. By Liouville's theorem f must be constant, contradicting the hypothesis. □

Exercise 40. If $f \in \mathscr{O}(\mathbb{C})$, and if there exist $M > 0$ and s such that $|f(z)| \le M(1+|z|^s)$ for $z \in \mathbb{C}$, then f must be a polynomial of degree at most s.

10. The identity theorem

Recall that an *accumulation point* of a set $D \subset \mathbb{C}$ is a point $z \in \mathbb{C}$ such that any neighbourhood of z contains a point $w \neq z$ from D. We say that $z \in D$ is an *isolated point* if it is not an accumulation point of D. If all points of D are isolated D is called *discrete*.

Theorem 41 (Identity theorem). *Let $f \in \mathscr{C}^\omega(\Omega)$ with Ω a connected open set, and let one or both of the following conditions hold.*

(a) The zero set of f has an accumulation point in Ω.

(b) There is $c \in \Omega$ such that $f^{(n)}(c) = 0$ for all n.

Then $f \equiv 0$ in Ω.

Proof. By definition, connectedness of Ω means that if $\Omega = A \cup B$ for some open and disjoint sets A and B, then it is necessarily either $A = \Omega$ or $B = \Omega$.

Our strategy is to show that both $A = \{z \in \Omega : f^{(n)}(z) = 0\,\forall n\}$ and its complement $B = \Omega \setminus A$ are open, and that A is nonempty. This would establish that $A = \Omega$, and hence $f \equiv 0$ in Ω. It is easy to see that A is open, because $c \in A$ implies that $f \equiv 0$ in a small disk centred at c by a Taylor series argument. To see that B is open, we write it as $B = \bigcup_n B_n$ with $B_n = \{z \in \Omega : f^{(n)}(z) \neq 0\}$. Since B_n is the preimage of the open set $\mathbb{C} \setminus \{0\}$ under the continuous mapping $f^{(n)} : \Omega \to \mathbb{C}$, we infer that B_n is open, and thus B is open. Part (b) of the theorem is established, since $c \in A$ (and so $A \neq \varnothing$) by hypothesis.

For part (a), it remains to prove that A is nonempty. Let $c \in \Omega$ be an accumulation point of $\{z \in \Omega : f(z) = 0\}$, and suppose that $c \notin A$. Let n be the smallest integer such that $f^{(n)}(c) \neq 0$. Then we have $f(z) = (z-c)^n g(z)$ for some continuous function g with $g(c) \neq 0$. This implies the existence of a small open disk $D_\varepsilon(c)$ in which $f(z) = 0$ has only one solution $z = c$, contradicting that c is an accumulation point of the zero set of f. □

The following corollary records the fact that an analytic function is completely determined by its restriction to any non-discrete subset of its domain of definition. In other words, if it is at all possible to extend an analytic function (defined on a non-discrete set) to a bigger domain, then there is only one way to do the extension.

Corollary 42 (Uniqueness of analytic continuation). *Let $u, v \in \mathscr{C}^\omega(\Omega)$ with Ω a connected open set, and let $u \equiv v$ in a non-discrete set $D \subset \Omega$. Then $u \equiv v$ in Ω.*

11. The open mapping theorem

The Cauchy estimates (Theorem 37) can also be used to prove the open mapping theorem.

Theorem 43. *Let Ω be a connected open set, and suppose that $f \in \mathscr{O}(\Omega)$ is not a constant function. Then $f : \Omega \to \mathbb{C}$ is an open mapping, i.e., it sends open sets to open sets.*

Proof. Without loss of generality let us assume that $0 \in \Omega$ and that $f(0) = 0$. We will prove that a small disk centred at the origin will be mapped by f to a neighbourhood of the origin. Let $D_r \subset \Omega$ with $r > 0$, and let $w \notin f(D_r)$. Then the function $\phi(z) = \frac{1}{f(z)-w}$ is analytic in D_r. Choose $0 < \rho < r$ so small that $f(z) = 0$ has no solution with $|z| = \rho$, so that $\delta = \inf\limits_{|z|=\rho} |f(z)| > 0$. This is possible by the identity theorem since f is not constant and Ω is connected. Since $\rho < r$, the Taylor series of ϕ about 0 converges uniformly in the closed disk $\bar{D}_\rho$. Now we apply the Cauchy estimate to ϕ and get

$$|\phi(0)| \le \sup_{|z|=\rho} |\phi(z)| = \left(\inf_{|z|=\rho} |f(z) - w| \right)^{-1},$$

which, taking into account that $|\phi(0)| = |w|^{-1}$, is equivalent to

$$\inf_{|z|=\rho} |f(z) - w| \le |w|.$$

We have $|f(z) - w| \ge |f(z)| - |w| \ge \delta - |w|$ for $|z| = \rho$, therefore the above estimate gives $|w| \ge \delta/2$. It follows that $D_{\delta/2} \subset f(D_r)$. □

Thus for example, one cannot get a (nonzero-length) curve as the image of an open set under a holomorphic mapping. In particular, the only real-valued holomorphic functions defined on an open set in $\mathbb{C}$ are locally constant functions.

The open mapping theorem can be used to obtain a proof of maximum principles.

Corollary 44 (Maximum principle)**.** *Let $f \in \mathscr{O}(\Omega)$ with Ω an open subset of $\mathbb{C}$.*

(a) If Ω is connected and $|f(z)| = \sup_\Omega |f|$ at some $z \in \Omega$, then f is constant.

(b) If Ω is bounded and $f \in \mathscr{C}(\bar{\Omega})$, then we have $\sup_\Omega |f| \le \max_{\partial\Omega} |f|$.

Proof. The hypothesis in (a) says that $f(z)$ is a boundary point of the image $f(\Omega)$, since otherwise there would have to be a point in $f(\Omega)$ with absolute value strictly greater than $|f(z)|$. If f is not a constant, by the open mapping theorem $f(\Omega)$ cannot include any of its boundary points, leading to a contradiction.

For part (b), there is $z \in \bar{\Omega}$ with $|f(z)| = \sup_{\bar{\Omega}} |f|$ since Ω is bounded and f is continuous on $\bar{\Omega}$. If $z \in \partial\Omega$ then we are done; otherwise applying part (a) to the connected component of Ω that contains z concludes the proof. □

We end this section with two simple corollaries of the open mapping theorem.

Corollary 45 (Preservation of domains)**.** *If $\Omega \subset \mathbb{C}$ is connected open set and $f \in \mathscr{O}(\Omega)$ nonconstant, then $f(\Omega)$ is also a connected open set.*

Exercise 46. Let $f \in \mathscr{O}(\mathbb{C})$, and suppose that $|f(z)| \to \infty$ as $|z| \to \infty$. Then $f(\mathbb{C}) = \mathbb{C}$.