# ELEMENTARY FUNCTIONS

TSOGTGEREL GANTUMUR

Contents



## 1. Constant functions

Before delving into the study of elementary functions, we prove here a simple preliminary lemma on constant functions. Recall that $D_r(c) = \{z \in \mathbb{C} : |z - c| < r\}$.

**Lemma 1.** *Let $f \in \mathscr{O}(D_r(c))$ and $f' = 0$ in $D_r(c)$, where $r > 0$ and $c \in \mathbb{C}$. Then $f$ is constant in $D_r(c)$.*

*Proof.* Let $f(x + iy) = u(x, y) + iv(x, y)$. Since $f$ is holomorphic in $D_r(c)$, the partial derivatives of $u$ and $v$ exist in $D_r(c)$, and the (complex) derivative $f'(x + iy)$ is represented by multiplication by the Jacobian matrix

$$J(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x}(x, y) & \frac{\partial u}{\partial y}(x, y) \\ \frac{\partial v}{\partial x}(x, y) & \frac{\partial v}{\partial y}(x, y) \end{pmatrix}. \tag{1}$$

Now $f' = 0$ implies that the Jacobian matrix must vanish everywhere in $D_r(c)$. This means that $u$ and $v$ are constant along every horizontal line and along every vertical line. Since every point in $D_r(c)$ can be joined to the centre $c$ by a polygonal path consisting of only horizontal and vertical line segments, we conclude that $u$ and $v$ are equal to their values at $c$, and hence they must be constant in $D_r(c)$. $\square$

**Remark 2.** In the preceding proof, it was not necessary to consider paths consisting of only horizontal and vertical line segments. We could have joined the centre $c$ with any point in the disk by a straight line segment, and used directional derivatives instead of partial derivatives. Moreover, the region in which the result holds can be vastly generalized; The result holds in an open set $\Omega$ if any two points in $\Omega$ can be joined by a polygonal path, i.e., if $\Omega$ is *path-connected*.

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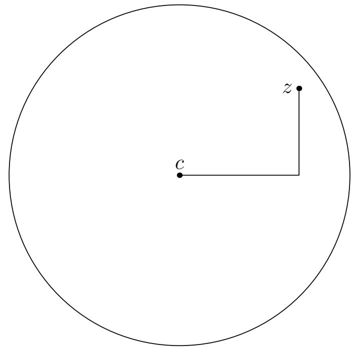



Figure 1. Any point in a disk can be connected to the centre by a polygonal path consisting of only horizontal and vertical line segments.

## 2. The exponential

We look for the complex exponential as a solution of the problem

$$f' = f, \qquad f(0) = 1,$$

and we look for it in the form of a power series $f(z) = \sum a_n z^n$. Formally differentiating the power series we find $a_n = a_{n-1}/n = \ldots = a_0/n!$, and the condition $f(0) = 1$ gives $a_0 = 1$. Thus the *complex exponential* is given by the power series

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \tag{2}$$

whose convergence radius (e.g. by the ratio test) is $\infty$, and so in particular $\exp \in \mathscr{O}(\mathbb{C})$.

Let us make some simple observations.

- By construction, we have
$$\frac{\mathrm{d}}{\mathrm{d}z} \exp = \exp \quad \text{in } \mathbb{C}, \qquad \text{and} \quad \exp(0) = 1. \tag{3}$$
- For $a \in \mathbb{C}$, let $g(z) = \exp(z)\exp(a - z)$. Then we have
$$g'(z) = \exp(z)\exp(a - z) - \exp(z)\exp(a - z) = 0, \tag{4}$$
for all $z \in \mathbb{C}$, which, by $g(0) = \exp(a)$ and by Lemma 1, implies that
$$\exp(z)\exp(a - z) = \exp(a) \qquad \text{for} \quad a, z \in \mathbb{C}. \tag{5}$$
- Putting $a = w + z$, we get the *law of addition*
$$\exp(z + w) = \exp(z)\exp(w) \qquad \text{for} \quad z, w \in \mathbb{C}. \tag{6}$$
- Putting $a = 0$, we infer
$$\exp(-z)\exp(z) = 1 \qquad \text{and so} \qquad \exp(z) \neq 0 \qquad \forall z \in \mathbb{C}.$$
- Therefore $\exp : \mathbb{C} \to \mathbb{C}^\times$ is a *group homomorphism*, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ is the multiplicative group of $\mathbb{C}$.
- By considering $g(z) = f(z)\exp(-z)$, one can show that the only function satisfying $f' = f$ in $\mathbb{C}$ with $f(0) = 1$ is the complex exponential.

In the following theorem, we construct a holomorphic (right) inverse of the exponential in the open disk $D_1(1)$. By using this inverse, we will also show that given $a \in \mathbb{C}^\times$ with $a = \exp \alpha$ for some $\alpha \in \mathbb{C}$, a holomorphic inverse of the exponential exists in the open disk $D_{|a|}(a)$. Recall here that $D_r(c) = \{z \in \mathbb{C} : |z - c| < r\}$.

**Theorem 3.** *(a) The power series*

$$\lambda(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n, \tag{7}$$

*converges in $D_1(1)$, and hence defines a function $\lambda \in \mathscr{O}(D_1(1))$. Moreover, we have*

$$\exp \lambda(z) = z \qquad \text{and} \qquad \lambda'(z) = \frac{1}{z} \qquad \text{for} \quad z \in D_1(1). \tag{8}$$

*(b) If $a \in \mathbb{C}^\times$ and $\exp \alpha = a$, then $\lambda_a(z) = \lambda(\frac{z}{a}) + \alpha$ satisfies*

$$\exp \lambda_a(z) = z \qquad \text{and} \qquad \lambda_a'(z) = \frac{1}{z} \qquad \text{for} \quad z \in D_{|a|}(a). \tag{9}$$

*In particular, $\lambda_a$ is holomorphic in $D_{|a|}(a)$.*

*Proof.* (a) By the ratio test, the convergence radius of (7) is 1, so (7) converges in $D_1(1)$. Then a termwise differentiation gives

$$\lambda'(z) = \sum_{n=1}^{\infty} (-1)^{n-1}(z-1)^{n-1} = \sum_{n=1}^{\infty} (1-z)^{n-1} = \frac{1}{1-(1-z)} = \frac{1}{z},$$

provided that $|1-z| < 1$, that is, $z \in D_1(1)$.

Now let $g(z) = z \exp(-\lambda(z))$. Then for $z \in D_1(1)$ we have

$$g'(z) = \exp(-\lambda(z)) - z \exp(-\lambda(z)) \lambda'(z) = 0,$$

meaning that $g(z) = g(1) = \exp(-\lambda(1)) = 1$ in $D_1(1)$.

(b) We have

$$\exp(\lambda_a(z)) = \exp\big(\lambda(\tfrac{z}{a})\big) \exp(\alpha) = \tfrac{z}{a} \cdot a = z, \tag{10}$$

and

$$\lambda_a'(z) = \lambda'\big(\frac{z}{a}\big) \cdot \frac{1}{a} = \frac{a}{z} \cdot \frac{1}{a} = \frac{1}{z}, \tag{11}$$

as long as $|\frac{z}{a} - 1| < 1$, that is, if $|z-a| < |a|$. This completes the proof. □

**Corollary 4.** *The map* $\exp : \mathbb{C} \to \mathbb{C}^\times$ *is surjective.*

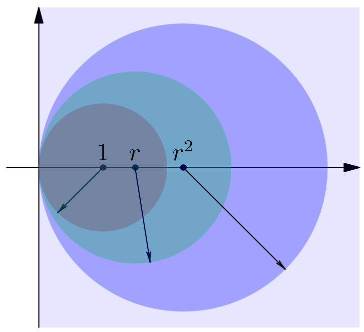
*Proof.* Let $A = \{\exp z : z \in \mathbb{C}\}$. By Theorem 3, we know that $D_1(1) \subset A$, and that $a \in A$ implies $D_{|a|}(a) \subset A$. Hence the proof is reduced to the following "game." Initially, the disk $D_1(1)$ is coloured blue, and the rest of the complex plane is white. At any stage in the game, we can choose a point $a$ in the blue region, and colour all the points in the disk $D_{|a|}(a)$ blue. The question is, by repeating this procedure, can we colour the entire set $\mathbb{C}^\times$ blue? It is not difficult to see that it is possible to do so.

One possibility is as follows (Figure 2). Let $1 < r < 2$. Then $r \in D_1(1)$, and hence $D_r(r) \subset A$. In particular, $r^2 \in A$, and hence $D_{r^2}(r^2) \subset A$. By induction, this shows that $D_{r^n}(r^n) \subset A$ for any $n$, and since any $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ can be contained in $D_{r^n}(r^n)$ for sufficiently large $n$, we conclude that $\{z : \operatorname{Re} z > 0\} \subset A$.

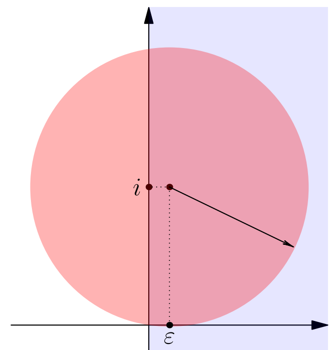
Let $a = i + \varepsilon$, where $\varepsilon > 0$ is a small real number. Then $i \in D_{|a|}(a)$, and hence $i \in A$, or $D_1(i) \subset A$. By considering the succession of points $\{ir^n\}$ with a constant $1 < r < 2$, we conclude that $D_{r^n}(ir^n) \subset A$ for all $n$, or $\{z : \operatorname{Im} z > 0\} \subset A$.

Finally, by considering the points $a = -1 + i\varepsilon$ and $a = -i + \varepsilon$, we get $-1 \in A$ and $-i \in A$. Then we repeat the same procedure with the sequences $\{-r^n\}$ and $\{-ir^n\}$ as the disk centres, to conclude that $\{z : \operatorname{Re} z < 0\} \subset A$ and $\{z : \operatorname{Im} z < 0\} \subset A$. □

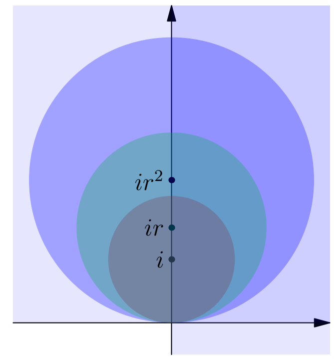
**Definition 5** (Euler 1748)**.** We define the *Euler number* by $e = \exp 1$.




(A) Starting with $D_1(1)$, and painting the disks $D_{r^n}(r^n)$ for a fixed $r \in (1,2)$ and for all $n = 1, 2, \ldots$, we can colour the right half plane $\{\mathrm{Re} z > 0\}$.




(B) Once we coloured the right half plane, it is easy to paint the point $i$.




(C) Once $i$ is coloured, we repeat the same procedure to paint the upper half plane $\{\mathrm{Im} z > 0\}$.

FIGURE 2. Illustration of the proof of Corollary 4.

**Remark 6** (Real exponential)**.** (a) From the definition (2) it is clear that if $x \in \mathbb{R}$ then $\exp x \in \mathbb{R}$. In particular, $e$ is real. We also have

$$\exp x = 1 + x + \frac{x^2}{2!} + \ldots \geq 1 + x \qquad \text{for} \quad x \geq 0, \tag{12}$$

which implies that $\exp x \to \infty$ as $x \to \infty$. Moreover, since $\exp(-x) = \frac{1}{\exp x}$, we have

$$0 < \exp(-x) \leq \frac{1}{1+x} \qquad \text{for} \quad x \geq 0, \tag{13}$$

and so in particular, $\exp x \to 0$ as $x \to -\infty$. We conclude that $\exp : \mathbb{R} \to (0, \infty)$ is surjective.

(b) Let us compute the derivative of $\exp x$ with respect to $x \in \mathbb{R}$. Suppose that $f(x+iy) = u(x,y) + iv(x,y)$ is a complex differentiable function, with $u$ and $v$ real. Then we have

$$f'(x+iy) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y), \tag{14}$$

where $f'$ denotes the complex derivative of $f$. If $f(x) \in \mathbb{R}$ for $x \in \mathbb{R}$, then $f(x) = u(x,0)$ and $v(x,0) = 0$ for all $x$, implying that

$$f'(x) = \frac{\partial u}{\partial x}(x,0) = \frac{\mathrm{d}}{\mathrm{d}x} f(x). \tag{15}$$

Returning back to the exponential function, we infer

$$\frac{\mathrm{d}}{\mathrm{d}x} \exp x = \exp'(x) = \exp x > 0 \qquad \text{for all} \quad x \in \mathbb{R}. \tag{16}$$

Therefore, $\exp : \mathbb{R} \to (0, \infty)$ is strictly increasing, and hence a bijection. The inverse function $\log : (0, \infty) \to \mathbb{R}$ is called the *real logarithm.*

(c) For any $z \in \mathbb{C}$ and $n \in \mathbb{N}$, we have

$$\exp(nz) = \exp(z + \ldots + z) = \exp(z) \cdots \exp(z) = (\exp z)^n, \tag{17}$$

and

$$\exp(-nz) = \frac{1}{\exp(nz)} = \frac{1}{(\exp z)^n} = (\exp z)^{-n}, \tag{18}$$

showing that $\exp(nz) = (\exp z)^n$ for all $n \in \mathbb{Z}$. Putting $z = \frac{1}{n}$, we get $\exp 1 = (\exp \frac{1}{n})^n$, or $\exp \frac{1}{n} = e^{\frac{1}{n}}$. This implies that

$$\exp \tfrac{n}{m} = (\exp \tfrac{1}{m})^n = (e^{\frac{1}{m}})^n = e^{\frac{n}{m}} \qquad \text{for} \quad n \in \mathbb{Z}, \quad m \in \mathbb{N}. \tag{19}$$

Finally, by continuity of $\exp x$ and of $e^x$, we conclude that $\exp x = e^x$ for all $x \in \mathbb{R}$.

**Exercise 7.** With $\log : (0, \infty) \to \mathbb{R}$ denoting the real logarithm, prove the following.

- $\log(ab) = \log a + \log b$, for $0 < a, b < \infty$.
- $\log(a^x) = x \log a$, for $0 < a < \infty$ and $x \in \mathbb{R}$.

Our next task is to identify the kernel $\ker(\exp) = \{z \in \mathbb{C} : \exp z = 1\}$ of $\exp : \mathbb{C} \to \mathbb{C}^\times$.

**Lemma 8.** *We have* $|\exp z| = \exp(\operatorname{Re} z)$ *for* $z \in \mathbb{C}$.

*Proof.* For $y \in \mathbb{R}$, we have $\overline{(iy)^n} = (-iy)^n$, and so

$$\overline{\sum_{n=0}^{m} \frac{(iy)^n}{n!}} = \sum_{n=0}^{m} \frac{(-iy)^n}{n!} \qquad \text{for any} \quad m \in \mathbb{N}. \tag{20}$$

The right hand side of this equality converges to $\exp(-iy)$ as $m \to \infty$. Moreover, the left hand side converges to $\overline{\exp(iy)}$, because of the general rule $\lim \bar{z}_m = \overline{\lim z_m}$. To conclude, we have $\overline{\exp(iy)} = \exp(-iy)$, and hence $|\exp(iy)|^2 = \exp(iy)\overline{\exp(iy)} = \exp(iy)\exp(-iy) = 1$.

Finally, if $z = x + iy$ with $x, y \in \mathbb{R}$, then we have

$$|\exp z| = |\exp(x)\exp(iy)| = |\exp x||\exp(iy)| = \exp x, \tag{21}$$

completing the proof. $\square$

By this lemma, $|\exp z| = 1$ is equivalent to $\exp(\operatorname{Re} z) = 1$. Since $\exp : \mathbb{R} \to (0, \infty)$ is a bijection, $\exp(\operatorname{Re} z) = 1$ if and only if $\operatorname{Re} z = 0$.

**Corollary 9.** *We have* $\{z \in \mathbb{C} : |\exp z| = 1\} = i\mathbb{R}$, *where* $i\mathbb{R} = \{ix : x \in \mathbb{R}\}$.

**Theorem 10.** *We have* $\ker(\exp) = iT\mathbb{Z}$ *for some constant* $T > 0$.

*Proof.* If $\exp z = 1$, then obviously $|\exp z| = 1$, and so by the preceding corollary, we have $K = \ker(\exp) \subset i\mathbb{R}$. It is clear that $0 \in K$, and so a natural question is whether or not $K$ has any nonzero element. To answer this question, we note that the surjectivity of $\exp : \mathbb{C} \to \mathbb{C}^\times$, combined with the preceding corollary, implies that there is $s \in \mathbb{R}$ with $\exp(is) = -1$, and so with $\exp(2is) = 1$. Obviously $s \neq 0$, and thus $K \neq \{0\}$.

We have the symmetry $K = -K$, because $\exp z = 1$ implies that $\exp(-z) = \frac{1}{\exp z} = 1$. Since $K$ contains numbers with positive imaginary parts, the number

$$T = \inf\{t > 0 : it \in K\}, \tag{22}$$

is well defined. In order to show that $iT \in K$, let $\{t_k\}$ be a sequence such that $\{it_k\} \subset K$ and $t_k \to T$ as $k \to \infty$. Then by continuity of the exponential, $\exp(it_k) \to \exp(iT)$ as $k \to \infty$. On the other hand, $\exp(it_k) = 1$ for all $k$, which implies that $\exp(iT) = 1$, that is, $iT \in K$. Furthermore, since $\exp(inT) = \exp(iT)^n = 1$ for $n \in \mathbb{Z}$, we conclude that $iT\mathbb{Z} \subset K$.

Now we wish to show that $T > 0$. To this end, we write

$$\exp z - 1 = z + \frac{z^2}{2!} + \ldots = z\big(1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots\big) = zg(z), \tag{23}$$

where we have introduced the function

$$g(z) = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}. \tag{24}$$

The convergence radius of the latter series is $\infty$, and so in particular we have $g \in \mathscr{O}(\mathbb{C})$. Since $g(0) = 1$, by continuity, $g$ has no zeroes in a small open disk centred at 0. Hence in that disk, the only solution to $\exp z = 1$ is $z = 0$, which means that $T > 0$.

Finally, suppose that $r \in \mathbb{R}$ with $ir \in K$. Then there is $n \in \mathbb{Z}$ such that $nT \leq r < (n+1)T$, or $0 \leq r - nT < T$. But $\exp(ir - inT) = \exp(ir)\exp(-inT) = 1$, hence $r - nT = 0$ by the minimal property of $T$. This proves that $K \subset iT\mathbb{Z}$. $\square$

**Remark 11.** Suppose that $w, z \in \mathbb{C}$ satisfy $\exp(z + w) = \exp(z)$. Then $\exp(z)\exp(w) = \exp(z)$, that is, we have $\exp(w) = 1$, meaning that $w \in iT\mathbb{Z}$. One implication of this is that the periods of the exponential function are precisely the numbers $inT$ with $n \in \mathbb{Z}$. Recall that $w \in \mathbb{C}$ is called a *period* of a function $f$ if $f(z + w) = f(z)$ for all $z \in \mathbb{C}$. Another, quite strong implication is that if $I$ is any half-open interval of length $T$, such as $I = [0, T)$, then the complex exponential restricted to the horizontal strip $\mathbb{R} + iI$ is bijective, where $\mathbb{R} + iI = \{x + iy : x \in \mathbb{R},\, y \in I\}$.

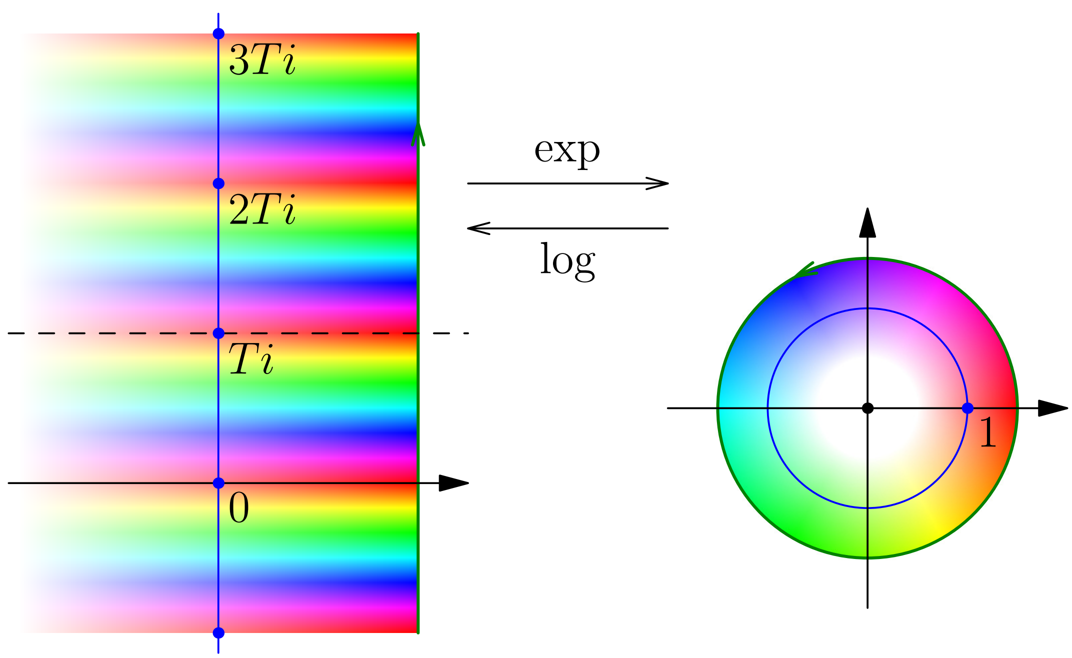



Figure 3. Mapping properties of the complex exponential.

## 3. The argument

In Corollary 9 we have established that $\exp^{-1}(S^1) = \{z \in \mathbb{C} : |\exp z| = 1\}$ is equal to the imaginary axis $i\mathbb{R}$. In view of Remark 11, this implies that the map $p : \mathbb{R} \to S^1$ defined by $p(t) = \exp(it)$ is surjective with the periods $T\mathbb{Z}$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Moreover, for any half-open interval $I$ of length $T$, the function $p : I \to S^1$ is bijective. In other words, every $z \in \mathbb{C}$ with $|z| = 1$ can be written uniquely as $z = p(t)$ with $t \in I$.

Supposing that $\exp(x + iy) = u(x, y) + iv(x, y)$, we have $p(t) = u(0, t) + iv(0, t)$, and hence

$$\begin{aligned} p'(t) &= \frac{\partial u}{\partial y}(0,t) + i\frac{\partial v}{\partial y}(0,t) = -\frac{\partial v}{\partial x}(0,t) + i\frac{\partial u}{\partial x}(0,t) \\ &= i\Big(\frac{\partial u}{\partial x}(0,t) + i\frac{\partial v}{\partial x}(0,t)\Big) = i\exp'(it) = i\exp(it) = ip(t), \end{aligned} \tag{25}$$

where we have used the Cauchy-Riemann equations in the second step, and $\exp'$ denotes the complex derivative of exp. From this we infer that $|p'(t)| = |\exp(it)| = 1$, meaning that $p : [0,T) \to S^1$ is an arc length parameterization of the unit circle $S^1$. Therefore, the arclength of the unit circle is equal to $T$. Moreover, since $p'(0) = i$, we have $\operatorname{Im} p(t) > 0$ for all sufficiently small $t > 0$, see Figure 4(a).

**Definition 12.** We define the number $\pi$ by $\pi = \dfrac{T}{2}$.

We will also use the shorthand notation $e^z = \exp(z)$ for the exponential function. Thus every $z \in \mathbb{C}^\times$ can be written as $z = |z|e^{i\theta}$ with $\theta \in \mathbb{R}$, and moreover $\theta$ is unique if one requires $\theta \in I$ with any fixed half-open interval $I$ of length $2\pi$.

**Definition 13.** We define the set-valued function, called the *argument*, by

$$\arg z = \big\{t \in \mathbb{R} : e^{it} = \frac{z}{|z|}\big\}, \qquad z \in \mathbb{C}^\times. \tag{26}$$

It is customary to write

$$\arg z = \theta + 2\pi n, \qquad n \in \mathbb{Z}, \tag{27}$$

for a fixed $\theta \in \mathbb{R}$ which depends on $z$, and say that $\arg z$ is a *multi-valued* function.

**Remark 14.** In the multi-valued formalism (27), it is understood that $\arg z$ returns *infinitely many values* $\theta + 2\pi n$, $n \in \mathbb{Z}$, all at once, while in the set-valued formalism (26), what $\arg z$ returns is a *single object*, that is the set of all the numbers $\theta + 2\pi n$ for $n \in \mathbb{Z}$. We will use these two formalisms interchangeably.

**Example 15.** Since $e^0 = 1$, we have

$$\arg 1 = 2\pi\mathbb{Z} = \{2\pi n : n \in \mathbb{Z}\}. \tag{28}$$

We have $(e^{\pi i})^2 = e^{2\pi i} = 1$ and $e^{\pi i} \neq 1$, hence $e^{\pi i} = -1$, implying that

$$\arg(-1) = \pi + 2\pi\mathbb{Z} = \{\pi + 2\pi n : n \in \mathbb{Z}\}. \tag{29}$$

Moreover, from $(e^{\frac{\pi i}{2}})^2 = e^{\pi i} = -1$ we infer that either $e^{\frac{\pi i}{2}} = i$ or $e^{\frac{\pi i}{2}} = -i$. Suppose that $e^{\frac{\pi i}{2}} = -i$. Then since $\operatorname{Im} e^{it} > 0$ for small $t > 0$ and $\operatorname{Im}(-i) < 0$, the intermediate value theorem implies that the real valued function $f(t) = \operatorname{Im} e^{it}$ has a zero at some $t$ satisfying $0 < t < \frac{\pi}{2}$. In other words, $e^{it} \in \mathbb{R}$ for some $0 < t < \frac{\pi}{2}$, and in light of the fact that $|e^{it}| = 1$, we have $e^{it} = \pm 1$ for some $0 < t < \frac{\pi}{2}$. In either case, we get $e^{2it} = 1$, which impossible, since $0 < 2t < \pi$, and $s = 2\pi$ is the smallest positive solution to $e^{is} = 1$. To conclude, we have

$$\arg i = \frac{\pi}{2} + 2\pi\mathbb{Z} = \big\{\frac{\pi}{2} + 2\pi n : n \in \mathbb{Z}\big\}. \tag{30}$$

**Exercise 16.** Compute $\arg(-i)$, $\arg(1+i)$, and $\arg(1-i)$.

**Remark 17.** If $\arg z = \theta + 2\pi n$, $n \in \mathbb{Z}$, we have

$$|z|e^{i\arg z} = |z|e^{i\theta + 2\pi i n} = |z|e^{i\theta}e^{2\pi i n} = ze^{2\pi i n} = z, \qquad n \in \mathbb{Z}. \tag{31}$$

Now recall that for $A, B \subset \mathbb{C}$, the *sum set* is defined as

$$A + B = \{a + b : a \in A,\, b \in B\}. \tag{32}$$

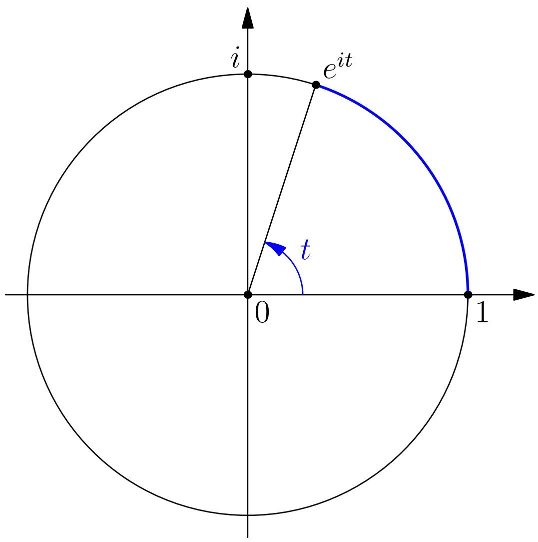
Then noting that

$$\arg w + \arg z = \{\tau + 2\pi n : n \in \mathbb{Z}\} + \{\theta + 2\pi n : n \in \mathbb{Z}\} = \{\tau + \theta + 2\pi n : n \in \mathbb{Z}\}, \tag{33}$$

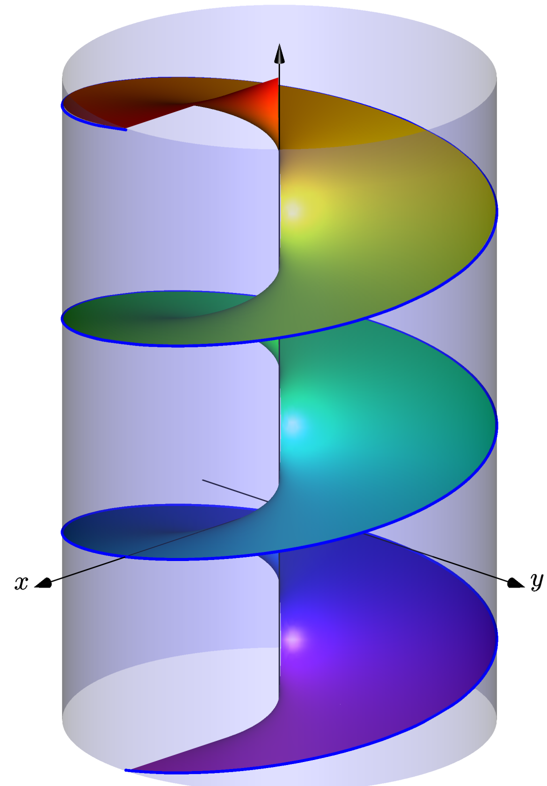
we can write

$$wz = |w|e^{i\arg w} \cdot |z|e^{i\arg z} = |w||z| \cdot e^{i(\arg w + \arg z)}. \tag{34}$$

In other words, when we multiply two complex numbers, the *magnitudes are multiplied* and the *arguments are added*.




(A) The map $t \mapsto e^{it}$ is an arclength parameterization of the unit circle $S^1$. The inverse of this parameterization is $\arg z$ for $z \in S^1$.




(B) The graph of $\arg z$. The blue helix represents the graph of $\arg z$ restricted to $|z| = 1$.

FIGURE 4. The parameterization $e^{it}$ of $S^1$, and the multi-valued function $\arg z$.

**Definition 18.** Let $F$ be a set-valued function, defined on some $A \subset \mathbb{C}$, and let $\Omega \subset A$. Then a single-valued function $f \in \mathscr{C}(\Omega)$ is called a *(continuous) branch of $F$ in $\Omega$*, if $f(z) \in F(z)$ for all $z \in \Omega$.

**Definition 19.** The *principal branch* of $\arg z$ is defined as

$$\operatorname{Arg} z = \theta, \qquad \text{where} \quad e^{i\theta} = \frac{z}{|z|} \quad \text{and} \quad -\pi < t \leq \pi. \tag{35}$$

**Example 20.** We have $\operatorname{Arg} 1 = 0$, $\operatorname{Arg}(-1) = \pi$, and $\operatorname{Arg} i = \dfrac{\pi}{2}$.

**Exercise 21.** Compute $\operatorname{Arg}(-i)$, $\operatorname{Arg}(1+i)$, and $\operatorname{Arg}(1-i)$.

## 4. LOGARITHMS

For $z = \rho e^{i\theta} \in \mathbb{C}^\times$ and $\zeta = \log \rho + i\theta$, it is clear that

$$\exp(\zeta) = \exp(\log \rho) \exp(i\theta) = \rho \exp(i\theta) = z. \tag{36}$$

Now, if $\exp(w) = z$, then $w = \zeta + 2\pi i n$ for some $n \in \mathbb{Z}$. Therefore, the solutions of the equation $\exp(\zeta) = z$ are precisely the numbers $\log \rho + i\theta + 2\pi i \mathbb{Z} = \log |z| + i \arg z$.

**Definition 22.** We define the set-valued *complex logarithm*, by

$$\log z = \exp^{-1}(z) = \{w \in \mathbb{C} : \exp w = z\}, \qquad z \in \mathbb{C}^\times. \tag{37}$$

It is customary to regard it as a multi-valued function, and simply write

$$\log z = \log |z| + i \arg z, \tag{38}$$

where "log" in the right hand side of course denotes the real logarithm.

**Remark 23.** The notation "log" is an *overloaded* notation:

- $\log x$ may mean the (single-valued) real logarithm, for $x \in \mathbb{R}$ with $x > 0$.
- $\log z$ is the multi-valued complex logarithm, as we have just defined.
- Any branch of the complex logarithm is often denoted by $\log z$ as well.

**Example 24.** We have

$$\begin{aligned} \log(-1) &= \pi i + 2\pi i n, \qquad n \in \mathbb{Z}, \\ \log i &= \frac{\pi i}{2} + 2\pi i n, \qquad n \in \mathbb{Z}, \end{aligned} \tag{39}$$

and

$$\overbrace{\log x}^{\text{complex log}} = \overbrace{\log x}^{\text{real log}} + 2\pi i n, \qquad n \in \mathbb{Z}, \tag{40}$$

for $x \in \mathbb{R}$ with $x > 0$.

**Definition 25.** The *principal branch* of $\log z$ is defined as

$$\operatorname{Log} z = \log|z| + i\operatorname{Arg} z, \qquad z \in \mathbb{C}^-, \tag{41}$$

where $\mathbb{C}^- = \mathbb{C} \setminus (-\infty, 0]$ is called the *slit-plane.*

**Example 26.** We have

$$\operatorname{Log}(-1) = \pi i, \qquad \operatorname{Log} i = \frac{\pi i}{2}, \tag{42}$$

and

$$\operatorname{Log} x = \log x, \tag{43}$$

for $x \in \mathbb{R}$ with $x > 0$, where "log" in the right hand side is the real logarithm.

**Remark 27.** (a) For $z \in \mathbb{C}^\times$, by construction, we have

$$\exp(\log z) = \exp(\log|z| + i\arg z) = |z|e^{\arg z} = z. \tag{44}$$

(b) However, the property $\log e^x = x$ of the real logarithm does not extend to the complex logarithm. What we have instead is

$$\begin{aligned} \log(\exp z) &= \log|\exp z| + i\arg\exp z = \log e^{\operatorname{Re} z} + i(\operatorname{Im} z + 2\pi n) \\ &= \operatorname{Re} z + i\operatorname{Im} z + 2\pi i n = z + 2\pi i n, \qquad n \in \mathbb{Z}. \end{aligned} \tag{45}$$

This is not surprising, since $\log z$ is now a multi-valued function.

(c) We can ask if the property $\log e^x = x$ holds for individual branches of logarithm. Let us consider the principal branch here as an example. First of all, $\operatorname{Log}(\exp z)$ is *not* defined if $\exp z \in \mathbb{R}$ and $\exp z < 0$, that is, if $\operatorname{Im} z = \pi + 2\pi n$ for some $n \in \mathbb{Z}$. So if $\pi + 2\pi(n-1) < \operatorname{Im} z < \pi + 2\pi n$ for some $n \in \mathbb{Z}$, then $\operatorname{Arg}\exp z = \operatorname{Im} z - 2\pi n$, and hence

$$\operatorname{Log}(\exp z) = \log|\exp z| + i\operatorname{Arg}\exp z = \log e^{\operatorname{Re} z} + i(\operatorname{Im} z - 2\pi n) = z - 2\pi i n. \tag{46}$$

This means that

$$\operatorname{Log}(\exp z) = z, \qquad \text{if} \quad -\pi < \operatorname{Im} z < \pi. \tag{47}$$

(d) For $w, z \in \mathbb{C}^\times$, we have

$$\begin{aligned} \log(wz) &= \log|wz| + i\arg(wz) = \log|w| + \log|z| + i(\arg w + \arg z) \\ &= \log w + \log z, \end{aligned} \tag{48}$$

where the equality should be understood as an equality between sets.

(e) Note that $\operatorname{Arg}(wz) = \operatorname{Arg} w + \operatorname{Arg} z$ is not true in general. For example, $\operatorname{Arg} 1 = 0$, but $\operatorname{Arg}(-1) + \operatorname{Arg}(-1) = 2\pi$. If $\operatorname{Arg}(wz) = \operatorname{Arg} w + \operatorname{Arg} z$ for some particular numbers $w$ and $z$, then we would have

$$\operatorname{Log}(wz) = \operatorname{Log} w + \operatorname{Log} z. \tag{49}$$

A sufficient condition for $\operatorname{Arg}(wz) = \operatorname{Arg} w + \operatorname{Arg} z$ to hold would be $\operatorname{Arg} w, \operatorname{Arg} z \in (-\frac{\pi}{2}, \frac{\pi}{2})$, that is, $\operatorname{Re} w > 0$ and $\operatorname{Re} z > 0$.

**Theorem 28.** *Let $\ell \in \mathscr{C}(D_r(a))$, with $D_r(a) \subset \mathbb{C}^\times$. Then the following are equivalent.*

*(a) $\ell$ is a branch of logarithm in $D_r(a)$.*
*(b) $\exp(\ell(a)) = a$ and $\ell \in \mathscr{O}(D_r(a))$ with*

$$\ell'(z) = \frac{1}{z} \qquad \text{for} \quad z \in D_r(a). \tag{50}$$

*(c) There is some $n \in \mathbb{Z}$ and $\alpha \in \mathbb{C}$ with $\exp \alpha = a$, such that*

$$\ell(z) = \lambda(\tfrac{z}{a}) + \alpha + 2\pi i n \qquad \text{for} \quad z \in D_r(a), \tag{51}$$

*where $\lambda$ is the logarithmic series introduced in Theorem 3.*

*Proof.* Suppose that $\ell$ is a branch of logarithm in $D_r(a)$, and let $\lambda_a(z) = \lambda(\frac{z}{a}) + \alpha$ with some $\alpha \in \mathbb{C}$ satisfying $\exp \alpha = a$. Then by Theorem 3(b) the function $\lambda_a$ is a branch of logarithm in $D_r(a)$, and so we have

$$\exp(\ell(z)) = z = \exp(\lambda_a(z)), \qquad z \in D_r(a), \tag{52}$$

which implies that

$$\exp(\ell(z) - \lambda_a(z)) = 1, \qquad z \in D_r(a), \tag{53}$$

or

$$\ell(z) - \lambda_a(z) = 2\pi i g(z), \qquad z \in D_r(a), \tag{54}$$

for some function $g : D_r(a) \to \mathbb{Z}$. Since $\ell - \lambda_a$ is continuous, so is $g$, and hence $g = \text{const}$ in $D_r(a)$. This means that

$$\ell(z) = \lambda_a(z) + 2\pi i n, \qquad z \in D_r(a), \tag{55}$$

for some constant $n \in \mathbb{Z}$, establishing the implication (a) $\Longrightarrow$ (c).

The implication (c) $\Longrightarrow$ (b) is immediate from Theorem 3. To prove (b) $\Longrightarrow$ (a), we consider the function

$$f(z) = z \exp(-\ell(z)), \tag{56}$$

and compute

$$f'(z) = \exp(-\ell(z)) - z \exp(-\ell(z))\ell'(z) = \exp(-\ell(z)) - z \exp(-\ell(z)) \cdot \frac{1}{z} = 0, \tag{57}$$

which implies that $f = \text{const}$ in $D_r(a)$. This means that $f(z) = 1$ for $z \in D_r(a)$, because

$$f(a) = a \exp(-\ell(a)) = \frac{a}{\exp \ell(a)} = 1, \tag{58}$$

and thus

$$\exp \ell(z) = \frac{1}{\exp(-\ell(z))} = \frac{z}{f(z)} = \frac{z}{f(a)} = z \qquad \text{for} \quad z \in D_r(a), \tag{59}$$

completing the proof. □

## 5. Powers

For $x, s \in \mathbb{R}$ with $x > 0$, we can define the *real positive power* $x^s$ by $x^s = e^{s \log x}$, where $\log x$ is the real logarithm. This can be extended to the complex setting as follows.

**Definition 29.** For $\sigma \in \mathbb{C}$, we define the multi-valued *complex power* $z \mapsto z^\sigma$ by

$$z^\sigma = \exp(\sigma \log z), \qquad z \in \mathbb{C}^\times. \tag{60}$$

The *principal value* of the power $z^\sigma$ is defined as

$$\text{p.v.}\, z^\sigma = \exp(\sigma \text{Log}\, z), \qquad z \in \mathbb{C}^-. \tag{61}$$

**Remark 30.** The notation $z^\sigma$ is even more overloaded than "log":

- $x^s$ may mean the (single-valued) real power, for $x, s \in \mathbb{R}$ with $x > 0$.
- $z^\sigma$ is the multi-valued complex power, as we have just defined.
- Any branch of the complex power is often denoted by $z^\sigma$ as well.
- The notation "p.v." in the principal value we have introduced is not completely standard, and therefore often omitted, resulting in the notation $z^\sigma$ also for p.v. $z^\sigma$.

**Example 31.** We have

$$\begin{aligned} 1^\sigma &= \exp(2\pi\sigma i n), \qquad n \in \mathbb{Z}, \\ \text{p.v.}\, 1^\sigma &= \exp(\sigma \cdot 0) = 1, \\ \text{p.v.}\, i^i &= \exp(i \cdot \tfrac{\pi i}{2}) = e^{-\frac{\pi}{2}}. \end{aligned} \tag{62}$$

**Remark 32.** The following simple properties can be derived.

- p.v. $x^s = \exp(s \text{Log}\, x) = x^s$ for $x, s \in \mathbb{R}$ with $x > 0$, where $x^s$ in the right hand side is the real positive power.
- $z^0 = \exp(0) = 1$.
- $z^1 = \exp(\log z) = z$.
- $z^{-1} = \exp(-\log z) = \frac{1}{z}$.
- $z^n = \exp(n \log z) = \underbrace{\exp(\log z) \cdots \exp(\log z)}_{n \text{ times}} = \underbrace{z \cdot z \cdots z}_{n \text{ times}}$.

**Exercise 33** (Abel 1826)**.** Show that

$$\text{p.v.}\, z^{s+it} = |z|^s e^{-t\text{Arg}z} \left(\cos\left(s\text{Arg}z + t \log|z|\right) + i \sin\left(s\text{Arg}z + t\log|z|\right)\right), \tag{63}$$

for $z \in \mathbb{C}^-$ and $s, t \in \mathbb{R}$.

**Remark 34** (Powers with real exponents)**.** For $z = |z|e^{i\theta} \in \mathbb{C}^\times$ and $s \in \mathbb{R}$, we have

$$z^s = \exp(s \log|z|) \exp(is \arg z) = |z|^s e^{is\theta} e^{2\pi sik}, \qquad k \in \mathbb{Z}, \tag{64}$$

where $|z|^s$ is the real positive power. If $s$ is *irrational*, then there is no $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$ that satisfy $2\pi sk = 2\pi n$ except $n = k = 0$, and hence $z^s$ has infinitely many distinct values. On the other hand, if $s$ is *rational*, and say, $s = \frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ minimal, then

$$e^{2\pi sik} = e^{2\pi si(mq+r)} = e^{2\pi ipm} e^{2\pi sir} = e^{2\pi sir}, \tag{65}$$

where $k = mq + r$ with $m \in \mathbb{Z}$ and $0 \le r < q$, implying that $z^s$ has exactly $q$ distinct values.

**Remark 35** (Roots)**.** For $z = |z|e^{i\theta} \in \mathbb{C}^\times$ and $n \in \mathbb{N}$, we define the *$n$-th root of $z$* to be

$$\sqrt[n]{z} = z^{\frac{1}{n}} = \sqrt[n]{|z|}\, e^{i\theta/n} e^{2\pi ik/n}, \qquad k \in \mathbb{Z}, \tag{66}$$

where $\sqrt[n]{|z|}$ is the real positive $n$-th root. It is easily checked that $\sqrt[n]{z}$ has exactly $n$ distinct values, corresponding to, e.g., $k = 0, 1, \ldots, n-1$. We have

$$(\sqrt[n]{z})^n = (\sqrt[n]{|z|})^n (e^{i\theta/n})^n (e^{2\pi ik/n})^n = |z|e^{i\theta} e^{2\pi ik} = z, \tag{67}$$

so (66) with any fixed $k$ is a right inverse of the function $z \mapsto z^n$. We claim that all right inverses are obtained in this manner, in the sense that $\sqrt[n]{z} = \{w \in \mathbb{C} : w^n = z\}$. To prove this claim, let $w = \rho e^{i\tau}$, and require

$$w^n = \rho^n e^{i\tau n} = |z| e^{i\theta}. \tag{68}$$

Since $|w^n| = \rho^n$, we infer that $\rho^n = |z|$, and hence $\rho = \sqrt[n]{|z|}$. Furthermore, $e^{i\tau n} = e^{i\theta}$ is equivalent to the condition that $\tau n = \theta + 2\pi i k$ for some $k \in \mathbb{Z}$, or

$$w = \rho e^{i\theta/n} e^{2\pi i k/n}, \qquad k \in \mathbb{Z}, \tag{69}$$

establishing the claim.

## 6. Circular functions

**Definition 36.** We define the (circular) *cosine* and the (circular) *sine*, respectively, by

$$\cos z = \frac{\exp(iz) + \exp(-iz)}{2} \qquad \text{and} \qquad \sin z = \frac{\exp(iz) - \exp(-iz)}{2i}, \tag{70}$$

for $z \in \mathbb{C}$.

**Remark 37.** For $t \in \mathbb{R}$, we have

$$\cos t = \frac{\exp(it) + \exp(-it)}{2} = \operatorname{Re} e^{it}, \tag{71}$$

and

$$\sin t = \frac{\exp(it) - \exp(-it)}{2i} = \operatorname{Im} e^{it}, \tag{72}$$

which, in light of the fact that $t \mapsto e^{it}$ is an arclength parameterization of the unit circle $S^1$, shows that Definition 36 extends the usual geometric definition of $\cos t$ and $\sin t$ to complex values of $t$, cf. Figure 5.

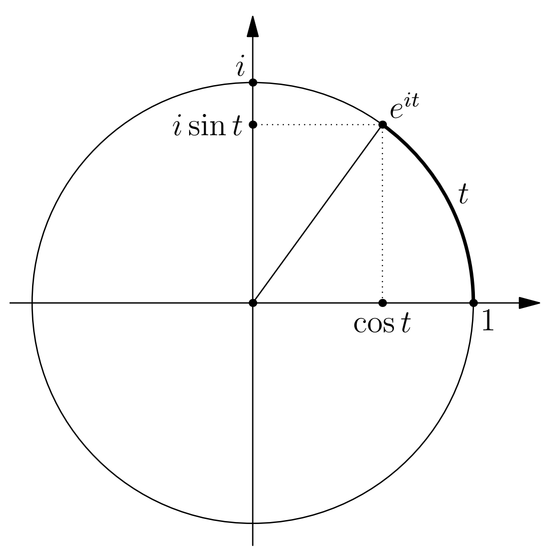



Figure 5. The functions $\cos t$ and $\sin t$ for real values of $t$.

**Remark 38.** The following properties can easily be derived from Definition 36.

(a) $\exp(iz) = \cos z + i \sin z$ for $z \in \mathbb{C}$ (Euler 1748).
(b) $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$ for $z \in \mathbb{C}$.
(c) $\cos^2 z + \sin^2 z = 1$ for $z \in \mathbb{C}$.

(d) $(\cos z)' = -\sin z$ and $(\sin z)' = \cos z$ for $z \in \mathbb{C}$.
(e) $\cos(w + z) = \cos w \cos z - \sin w \sin z$ for $w, z \in \mathbb{C}$.
(f) $\sin(w + z) = \sin w \cos z + \cos w \sin z$ for $w, z \in \mathbb{C}$.

**Theorem 39.** *The maps* $\cos : \mathbb{C} \to \mathbb{C}$ *and* $\sin : \mathbb{C} \to \mathbb{C}$ *are surjective.*

*Proof.* Given $w \in \mathbb{C}$, consider the equation $\cos z = w$, that is,

$$\exp(iz) + \exp(-iz) = 2w. \tag{73}$$

With $u = \exp(iz)$, this is equivalent to $u + \frac{1}{u} = 2w$, or

$$u^2 - 2wu + 1 = 0. \tag{74}$$

We see that $u = 0$ is not a solution. By writing

$$(u - w)^2 = u^2 - 2wu + w^2 = w^2 - 1, \tag{75}$$

we find the solution $u$ in terms of the multi-valued square root as

$$u = w + \sqrt{w^2 - 1}. \tag{76}$$

Since $u \neq 0$, we can solve $u = \exp(iz)$ as $z = -i\log u$. Note that to each $w \in \mathbb{C}$, in general there corresponds two distinct values of $u$, and to each value of $u$, there corresponds infinitely many distinct values of $z$. This can be written as

$$z = \cos^{-1}(w) = -i\log(w + \sqrt{w^2 - 1}), \tag{77}$$

where the composition of multi-valued functions is interpreted in the obvious way.

The equation $\sin z = w$ can similarly be solved by

$$z = \sin^{-1}(w) = -i\log(w + \sqrt{w^2 + 1}), \tag{78}$$

completing the proof. □

**Exercise 40.** Prove the following.
(a) $\sin z = 0$ if and only if $z = \pi n$ for some $n \in \mathbb{Z}$.
(b) $\cos z = 0$ if and only if $z = \frac{\pi}{2} + \pi n$ for some $n \in \mathbb{Z}$.
(c) The periods of sin are precisely the numbers $2\pi n$, $n \in \mathbb{Z}$.
(d) The periods of cos are precisely the numbers $2\pi n$, $n \in \mathbb{Z}$.

**Exercise 41.** Show that $\cos z = \cos w$ if and only if either $z + w = 2\pi n$ for some $n \in \mathbb{Z}$, or $z - w = 2\pi n$ for some $n \in \mathbb{Z}$.

**Exercise 42.** Let $\cot : \mathbb{C} \setminus \pi\mathbb{Z} \to \mathbb{C}$ and $\tan : \mathbb{C} \setminus (\frac{\pi}{2} + \pi\mathbb{Z}) \to \mathbb{C}$ be defined by

$$\cot z = \frac{\cos z}{\sin z}, \qquad \text{and} \qquad \tan z = \frac{\sin z}{\cos z},$$

respectively. Prove the following.
(a) The functions cot and tan are holomorphic in their respective domains, with

$$(\cot z)' = -\frac{1}{\sin^2 z}, \qquad \text{and} \qquad (\tan z)' = \frac{1}{\cos^2 z}.$$

(b) The periods of cot and tan are the numbers $\pi n$, $n \in \mathbb{Z}$.
(c) With

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \dots, \qquad |z| < 1,$$

we have

$$(\arctan z)' = \frac{1}{1 + z^2} \qquad \text{and} \qquad \arctan(\tan z) = z \qquad \text{for} \quad |z| < 1.$$