COMPLEX DIFFERENTIABILITY

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Contents

1. The problem of extension

In complex analysis, we study a certain special class of functions of a complex variable, which has very strong analytical properties. This section introduces us to heuristic and historical reasons why we study this particular class.

Historically, complex functions arose from questions such as "What is e^z for a complex number z?" and "What is $\log i$?". The default understanding was that the values of e^z and $\log i$ are "out there", and we just need to "find" them. In our language, the question can be rephrased as follows.

> *Given a function $f(x)$ of a real variable, find an extension $F(z)$, that is in some sense natural.*

A complex function $F(z)$ is an *extension* of $f(x)$ if $F(t) = f(t)$ for real t. Note that the essence of this problem is not that we start with a real function, but the question of how we classify complex functions into two classes: The first class consists of functions that we consider "natural" or "nice", and the second class consists of all the rest. Extensions of real functions will provide us with a large supply of complex functions, but we want to study complex functions on their own, regardless of whether or not they are extensions of real functions. Hence the classification problem just mentioned is the question we are really after.

In order to get some insight on the extension problem, let us consider a real polynomial

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n. \tag{1}$$

Then everybody would agree that the most natural extension of it to the complex setting is

$$P(z) = a_0 + a_1 z + \ldots + a_n z^n. \tag{2}$$

In particular, when we say that a complex number z is a root of $p(x)$, what we have in mind is the statement $P(z) = 0$. However, $P(z)$ is not the only possible extension of $p(x)$, as, for example,

$$Q(z) = a_0 + a_1 \mathrm{Re} z + \ldots + a_n (\mathrm{Re} z)^n, \tag{3}$$

and

$$R(z) = \begin{cases} p(z) & \text{if } \operatorname{Im} z = 0, \\ 0 & \text{if } \operatorname{Im} z \neq 0, \end{cases} \tag{4}$$

are both extensions of the polynomial $p(x)$.

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Let us try to identify what makes the extension $P(z)$ special. In the definition (2) of $P(z)$, we are working with the variable z as a basic entity, whereas in (3) and (4), we break z apart into its real and imaginary parts, hence treating z as consisting of two real numbers. The basic philosophy of complex analysis is to treat the independent variable z as an elementary entity without any "internal structure." For polynomials, this simply means that we only allow addition and multiplication of complex numbers. For non-polynomial functions, we still need some clarifying to do.

As we know, in 1748, Euler used power series to extend the exponential and trigonometric functions to the complex setting. This is a generalization of how we extended $p(x)$ to $P(z)$, since polynomials are a special case of power series. We postpone a detailed study of power series to the subsequent chapter. The next big idea also came from Euler, around 1760. He was interested in evaluating the definite integral

$$\int_A^B f(x)\,\mathrm{d}x, \tag{5}$$

by extending f into the complex plane. Thus he assumed $F(z)$ was an extension of $f(x)$, which he wrote by using real and imaginary parts as $F(z) = M(x, y) + iN(x, y)$ with $z = x + iy$. Now let γ be a path in the complex plane joining the points A and B, and write

$$\int_\gamma F(z)\,\mathrm{d}z = \int_\gamma (M + iN)(\mathrm{d}x + i\mathrm{d}y) = \int_\gamma (M\mathrm{d}x - N\mathrm{d}y) + i\int_\gamma (N\mathrm{d}x + M\mathrm{d}y). \tag{6}$$

Since $F = M + iN$ is an extension of f, the integral (6) reduces to (5) if we take γ to be the real interval $[A, B]$. So the idea is to choose the functions M and N such that the integrals in the right hand side of (6) do not depend on the path γ. If this can be achieved, then we can hope to choose γ so that the integral (6) is easy to evaluate, which would give the value of the integral (5). The path independence requirement leads to the system

$$\frac{\partial N}{\partial x} = -\frac{\partial M}{\partial y}, \qquad \text{and} \qquad \frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}, \tag{7}$$

which are nowadays called the *Cauchy-Riemann equations*[1].

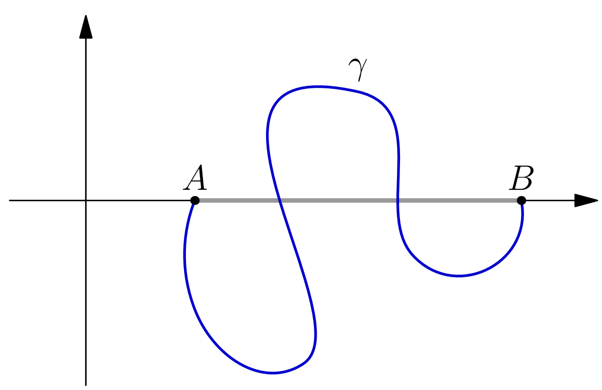

FIGURE 1. To reiterate, if we can extend the real function $f(x)$ into $F(z)$, in such a way that the real and imaginary parts of $F(z)$ satisfy the Cauchy-Riemann equations (7), then the integral (5) over the interval $[A, B]$ would be equal to the integral (6) along the curve γ, and the value of the latter would be independent of γ. This gives us a possibility to simplify the integral by choosing a suitable path γ.

[1]These equations were written down the first time in 1752 by Jean Le Rond d'Alembert (1717-1783) in a work not related to complex functions.

Example 1. The question arises if Euler's two procedures are consistent with each other. Let us check if the real and imaginary parts of the polynomial

$$P(z) = az^2 + bz + c, \tag{8}$$

satisfy the Cauchy-Riemann equations. Writing $z = x + iy$, we have

$$P(z) = a(x+iy)^2 + b(x+iy) + c = a(x^2 - y^2) + 2iaxy + bx + iby + c, \tag{9}$$

and so $P(z) = M(x, y) + iN(x, y)$ with

$$M(x, y) = a(x^2 - y^2) + bx + c, \qquad \text{and} \qquad N(x, y) = 2axy + by. \tag{10}$$

Computing the partial derivatives

$$\frac{\partial M}{\partial x} = 2ax + b, \qquad \frac{\partial M}{\partial y} = -2ay, \tag{11}$$

$$\frac{\partial N}{\partial x} = 2ay, \qquad \frac{\partial N}{\partial y} = 2ax + b, \tag{12}$$

shows that the Cauchy-Riemann equations are indeed satisfied.

Exercise 2. Show that the real and imaginary parts of a general polynomial with complex coefficients satisfy the Cauchy-Riemann equations.

Finally, the third approach was offered by Cauchy in his fundamental investigations. Implicit in his early writings, which he made explicit later, is the assumption that all complex functions $F(z)$ "worthy of their salt" are *complex differentiable*, in the sense that for each point z_0 in some region of $\mathbb{C}$, there is a complex number $\lambda \in \mathbb{C}$ such that

$$\frac{F(z) - F(z_0)}{z - z_0} \to \lambda \qquad \text{as} \quad z \to z_0. \tag{13}$$

The number λ is called the derivative of F at z_0, and we write $F'(z_0) = \lambda$. We will make it precise later, but for now, $z \to z_0$ can be understood to mean $|z - z_0| \to 0$.

Example 3. Let $F(z) = z^n$, and let $z_0 \in \mathbb{C}$ be fixed. Introducing $h = z - z_0$, we compute

$$\begin{aligned} F(z) = z^n = (z_0 + h)^n &= z_0^n + nz_0^{n-1}h + \frac{n(n-1)}{2} z_0^{n-2}h^2 + \ldots + h^n \\ &= F(z_0) + \left(nz_0^{n-1} + e(h)\right) h, \end{aligned} \tag{14}$$

where

$$e(h) = \frac{n(n-1)}{2} z_0^{n-2} h + \ldots + h^{n-1}, \tag{15}$$

and so in particular, $e(h) \to 0$ as $h \to 0$. This gives

$$\frac{F(z) - F(z_0)}{z - z_0} = nz_0^{n-1} + e(z - z_0) \to nz_0^{n-1} \qquad \text{as} \quad z \to z_0, \tag{16}$$

and thus $F'(z_0) = nz_0^{n-1}$.

To summarize, we have considered, at least formally, the following three approaches to classification of complex functions:

- We can ask if a function can be represented by power series.
- We can ask if the real and imaginary parts satisfy the Cauchy-Riemann equations.
- We can also ask if the function is complex differentiable.

It will turn out that all three approaches are equivalent, and they will lead to a very rich theory. From the next section on, we start our rigorous study.

2. Limits and continuity

Our first stop is the topology of complex numbers. Intuitively speaking, topology specifies when two complex numbers are infinitesimally close to each other.

Definition 4. A subset $\Omega \subset \mathbb{C}$ is called *open*, if for each $z \in \Omega$, there exists $\varepsilon > 0$ such that $D_\varepsilon(z) \subset \Omega$, where

$$D_\varepsilon(z) = \{w \in \mathbb{C} : |w - z| < \varepsilon\}. \tag{17}$$

The set of all open subsets of $\mathbb{C}$ is called the *topology of* $\mathbb{C}$.

Remark 5. We have the natural identification between $\mathbb{C}$ and $\mathbb{R}^2$ via the map $z \mapsto (\mathrm{Re} z, \mathrm{Im} z)$, and so the preceding definition can also be considered as a definition of open sets of $\mathbb{R}^2$. This is of course the default topology of $\mathbb{R}^2$, and from now on we will endow $\mathbb{R}^2$ with it.

Exercise 6. Show the following.

a) The unit disk $\mathbb{D} = D_1(0)$ is open.
b) The punctured plane $\mathbb{C} \setminus \{0\}$ is open.
c) The square $\{x + iy \in \mathbb{C} : 0 < x < 1,\ 0 < y < 1\}$ is open.
d) The square $\{x + iy \in \mathbb{C} : 0 < x \leq 1,\ 0 < y < 1\}$ is *not* open.

Exercise 7. Show that the following alternative definition of open sets leads to the same topology on $\mathbb{C}$. A subset $\Omega \subset \mathbb{C}$ is called *open*, if for each $z \in \Omega$, there exists a rectangle $R = \{x + iy : a < x < b, c < y < d\}$ containing z such that $R \subset \Omega$.

Definition 8. A sequence $\{z_n\} \subset \mathbb{C}$ of complex numbers is said to *converge to* $z \in \mathbb{C}$, if $|z_n - z| \to 0$ as $n \to \infty$, that is, if for any $\varepsilon > 0$, there exists N such that $z_n \in D_\varepsilon(z)$ whenever $n \geq N$. We write this fact as

$$\lim_{n\to\infty} z_n = z, \qquad \text{or} \qquad z_n \to z \quad \text{as} \quad n \to \infty. \tag{18}$$

Example 9. a) The sequence $\{z_n\}$ with $z_n = 1 + \frac{i}{n}$ converges to 1.
b) The sequence $\{i^n : n \in \mathbb{N}\}$ does *not* converge, i.e., it diverges.
c) The sequence $\{n^2 + i : n \in \mathbb{N}\}$ diverges.

Exercise 10. Show that $z_n \to z$ as $n \to \infty$ if and only if for any open set $U \ni z$ there exists N such that $z_n \in U$ for all $n \geq N$.

Lemma 11. *A sequence* $\{z_n\}$ *converges if and only if both* $\{\mathrm{Re} z_n\}$ *and* $\{\mathrm{Im} z_n\}$ *converge.*

Proof. Suppose that $z_n \to z$ as $n \to \infty$. Without loss of generality, let $z = 0$, so that $|z_n| \to 0$ as $n \to \infty$. Then, writing $z_n = x_n + iy_n$, we infer $|z_n|^2 = x_n^2 + y_n^2 \to 0$ as $n \to \infty$, which shows that $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$.

In the converse direction, without loss of generality, let $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$. Then $x_n^2 + y_n^2 \to 0$, and so $|z_n| \to 0$ as $n \to \infty$. □

Exercise 12. Let $\lim z_n = z$ and $\lim w_n = w$. Show that the following hold.

a) $\lim(w_n \pm z_n) = w \pm z$ and $\lim(w_n z_n) = wz$.
b) $\lim \bar{z}_n = \bar{z}$ and $\lim |z_n| = |z|$.
c) If $z \neq 0$, then $z_n = 0$ for only finitely many indices n, and after the removal of those zero terms from the sequence $\{z_n\}$, we have $\lim \frac{1}{z_n} = \frac{1}{z}$.

Exercise 13. Let $\{z_n\}$ be a *Cauchy sequence*, in the sense that

$$|z_n - z_m| \to 0, \qquad \text{as} \quad \min\{n, m\} \to 0. \tag{19}$$

Show that there is $z \in \mathbb{C}$, to which $\{z_n\}$ converges.

Next, we define continuous functions as the ones that send convergent sequences to convergent sequences. This is sometimes called the *sequential criterion of continuity.*

Definition 14. Let $K \subset \mathbb{C}$ be a set. A function $f : K \to \mathbb{C}$ is called *continuous at* $w \in K$ if $f(z_n) \to f(w)$ as $n \to \infty$ for every sequence $\{z_n\} \subset K$ converging to w.

In the following lemma, we prove that our definition is equivalent to other common definitions of continuity. We only consider open sets as the domain for simplicity, although the argument can be modified to cover more general domains.

Lemma 15. *Let $f : \Omega \to \mathbb{C}$ be a function, with $\Omega \subset \mathbb{C}$ open, and let $w \in \Omega$. Then the following are equivalent*[2]*.*

a) f is continuous at w.
b) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $z \in D_\delta(w)$ implies $f(z) \in D_\varepsilon(f(w))$.
c) For any open set $V \subset \mathbb{C}$ containing the point $f(w)$, there exists an open set $U \subset \Omega$ containing w, such that $z \in U$ implies $f(z) \in V$.

Proof. Suppose that c) holds. Then for any $\varepsilon > 0$, applying Definition 14 with $V = D_\varepsilon(f(w))$, there exists an open set $U \subset \Omega$ containing w, such that $z \in U$ implies $f(z) \in D_\varepsilon(f(w))$. This open set U must contain a disk $D_\delta(w)$ with $\delta > 0$, which proves b).

Now suppose that b) holds, and let $\{z_n\}$ be a sequence with $\lim z_n = w$. Let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that $z \in D_\delta(w)$ implies $f(z) \in D_\varepsilon(f(w))$. In turn, there exists N such that $|z_n - w| < \delta$ for all $n \geq N$. Thus for $n \geq N$, we have $f(z_n) \in D_\varepsilon(f(w))$, showing that $\lim f(z_n) = f(w)$. This proves a).

Finally, assume a), and suppose that c) does *not* hold. Then there exists an open set $V \subset \mathbb{C}$ containing $f(w)$, such that the preimage $f^{-1}(V) = \{z \in \Omega : f(z) \in V\}$ does not contain any disk $D_\delta(w)$ with $\delta > 0$. By choosing $\delta = \frac{1}{n}$ with $n = 1, 2, \ldots$, we infer that there exists a sequence $\{z_n\} \subset \Omega$ satisfying $|z_n - w| < \frac{1}{n}$ and $f(z_n) \notin V$ for each n. Since $z_n \to w$, by assumption c) we have $f(z_n) \to f(w)$ as $n \to \infty$, and the latter implies (by definition of limit) that $f(z_n) \in V$ for all large n. This is impossible, and hence c) holds. ☐

Exercise 16. Show the following.

a) The polynomial $f(z) = z^n$ is continuous at every point of $\mathbb{C}$.
b) The rational function $f(z) = \frac{1}{z}$ is continuous at every point in $\mathbb{C} \setminus \{0\}$.

Definition 17. Given two functions $f, g : \Omega \to \mathbb{C}$, with $\Omega \subset \mathbb{C}$ open, we define their *sum*, *difference*, *product*, and *quotient* by

$$(f \pm g)(z) = f(z) \pm g(z), \qquad (fg)(z) = f(z)g(z), \qquad \text{and} \qquad \big(\frac{f}{g}\big)(z) = \frac{f(z)}{g(z)}, \tag{20}$$

for $z \in \Omega$, where for the quotient definition we assume that g does not vanish anywhere in Ω. Furthermore, we define the functions $\bar{f}$, $\mathrm{Re}f$, and $\mathrm{Im}f$ by

$$\bar{f}(z) = \overline{f(z)}, \qquad (\mathrm{Re}f)(z) = \mathrm{Re}f(z), \qquad (\mathrm{Im}f)(z) = \mathrm{Im}f(z), \qquad \text{for} \quad z \in \Omega. \tag{21}$$

Lemma 18. *Let $\Omega \subset \mathbb{C}$ be an open set, and let $f, g : \Omega \to \mathbb{C}$ be functions continuous at $w \in \Omega$. Then $f \pm g$ and fg are all continuous at w. Furthermore, suppose that $U \subset \mathbb{C}$ is an open set satisfying $g(\Omega) \subset U$, the latter meaning that $z \in \Omega$ implies $g(z) \in U$. Let $F : U \to \mathbb{C}$ be a function continuous at $g(w)$. Then the composition $F \circ g : \Omega \to \mathbb{C}$, defined by $(F \circ g)(z) = F(g(z))$, is continuous at w.*

[2]Note that in class we used c) as the definition of continuity.

Proof. The results are immediate from the definition of continuity. For instance, let us prove that fg is continuous at w. Thus let $\{z_n\} \subset \Omega$ be an arbitrary sequence converging to w. Then $f(z_n) \to f(w)$ and $g(z_n) \to g(w)$ as $n \to \infty$, and Exercise 12 gives $f(z_n)g(z_n) \to f(w)g(w)$ as $n \to \infty$. Hence fg is continuous at w. □

Exercise 19. Show that if $f : \Omega \to \mathbb{C}$ is continuous at $w \in \Omega$, then the functions $\bar{f}$, Ref, Imf, and $\frac{1}{f}$ are continuous at w, where in the case of $\frac{1}{f}$ we assume that $f(w) \neq 0$.

Definition 20. A function $f : \Omega \to \mathbb{C}$ is called *continuous in* Ω, if f is continuous at each point of Ω. The set of all continuous functions in Ω is denote by $\mathscr{C}(\Omega)$.

Lemma 21. *A function $f : \Omega \to \mathbb{C}$ is continuous in Ω if and only if for any open set $V \subset \mathbb{C}$, its preimage $f^{-1}(V) = \{z \in \Omega : f(z) \in V\}$ is open.*

Proof. Let f be continuous in Ω, and suppose that there exists an open set $V \subset \mathbb{C}$ such that $f^{-1}(V)$ is *not* open. The latter means that there is $w \in f^{-1}(V)$ with the property that $D_\delta(w) \not\subset f^{-1}(V)$ for any $\delta > 0$. In other words, $f^{-1}(V)$ cannot contain any open set that contains w. This contradicts the assumption that f is continuous at each point of Ω.

Now assume that f is *not* continuous at some point, say $w \in \Omega$. Then there would exist an open set $V \subset \mathbb{C}$ such that $f^{-1}(V)$ does not contain any nontrivial disk entered at w, which would mean that $f^{-1}(V)$ is not open. To conclude, if f was not continuous in Ω, there would exist an open set whose preimage is not open. □

Exercise 22. Show that if $f, g \in \mathscr{C}(\Omega)$, then $f \pm g, fg, \bar{f}, \text{Re}f, \text{Im}f \in \mathscr{C}(\Omega)$.

3. Complex differentiability

With the notions of limits and continuity at hand, we can now make Cauchy's concept of complex derivative precise.

Definition 23. A function $f : \Omega \to \mathbb{C}$, with $\Omega \subset \mathbb{C}$ open, is called *complex differentiable at* $z_0 \in \Omega$, if there is a function $g : \Omega \to \mathbb{C}$, which is continuous at z_0, such that

$$f(z) = f(z_0) + g(z)(z - z_0), \qquad z \in \Omega. \tag{22}$$

We call the value $g(z_0)$ the *derivative of f at z_0*, and write

$$f'(z_0) \equiv \frac{\mathrm{d}f}{\mathrm{d}z}(z_0) := g(z_0). \tag{23}$$

It immediate from (22) that if f is complex differentiable at z_0 then f is continuous at z_0. The following lemma gives a *sequential criterion* of complex differentiability.

Lemma 24. *Let $\Omega \subset \mathbb{C}$ be an open set. A function $f : \Omega \to \mathbb{C}$ is complex differentiable at $z_0 \in \Omega$ if and only if there exists a number $\lambda \in \mathbb{C}$ such that*

$$\frac{f(z_n) - f(z_0)}{z_n - z_0} \to \lambda \qquad \text{as} \quad n \to \infty, \tag{24}$$

for every sequence $\{z_n\} \subset \Omega$ converging to z_0.

Proof. Let f be complex differentiable at z_0. Then by (22), we have

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0} \qquad \text{for} \quad z \in \Omega \setminus \{z_0\}. \tag{25}$$

Since g is continuous at z_0, for any sequence $\{z_n\} \subset \Omega$ converging to z_0, we have

$$g(z_n) = \frac{f(z_n) - f(z_0)}{z_n - z_0} \to \lambda := g(z_0) \qquad \text{as} \quad n \to \infty. \tag{26}$$

This establishes the "only if" part of the lemma.

Now suppose that there exists $\lambda \in \mathbb{C}$ such that (24) holds for every sequence $\{z_n\} \subset \Omega$ converging to z_0. Then we define a function $g : \Omega \to \mathbb{C}$ by

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0} \quad \text{for} \quad z \in \Omega \setminus \{z_0\}, \qquad \text{and} \qquad g(z_0) = \lambda. \tag{27}$$

This function is continuous at z_0, and by construction satisfies (22). □

The following exercise shows that the limit λ in (24) is guaranteed *not* to depend on the sequence $\{z_n\}$, as long as a limit exists for every sequence $\{z_n\} \subset \Omega$ converging to z_0.

Exercise 25. Let $\Omega \subset \mathbb{C}$ be open, and let $g : \Omega \to \mathbb{C}$ be a function. Let $w \in \mathbb{C}$, and suppose that for every sequence $\{z_n\} \subset \Omega$ converging to w, there exists $\lambda \in \mathbb{C}$ such that $g(z_n) \to \lambda$ as $n \to \infty$. Then show that there exists $\lambda \in \mathbb{C}$ such that $g(z_n) \to \lambda$ as $n \to \infty$, for every sequence $\{z_n\} \subset \Omega$ converging to w.

Let us look at some explicit examples of complex differentiation.

Example 26. a) Consider $f(z) = z^2$, and its differentiability at some point $z_0 \in \mathbb{C}$. Introducing $h = z - z_0$, we compute

$$f(z) = z^2 = (z_0 + h)^2 = z_0^2 + 2z_0 h + h^2 = f(z_0) + (z_0 + z)(z - z_0), \tag{28}$$

and hence $f(z) = f(z_0) + g(z)(z - z_0)$ with $g(z) = z_0 + z$. The function $g(z) = z_0 + z$ is clearly continuous at z_0, which means that f is complex differentiable at z_0, with

$$f'(z_0) = g(z_0) = (z_0 + z)\big|_{z=z_0} = 2z_0. \tag{29}$$

Note that z_0 should be considered as a parameter that is fixed.

b) Let us consider $f(z) = \bar{z}$. Let $z_0 \in \mathbb{C}$, and take $z_n = z_0 + h_n$, where $\{h_n\} \subset \mathbb{R}$ is a real sequence converging to 0. Then we have

$$f(z_n) - f(z_0) = \bar{z}_n - \bar{z}_0 = \bar{z}_0 + \bar{h}_n - \bar{z}_0 = h_n, \qquad \text{and} \qquad z_n - z_0 = h_n, \tag{30}$$

which implies that

$$\frac{f(z_n) - f(z_0)}{z_n - z_0} = \frac{h_n}{h_n} = 1. \tag{31}$$

Now take $w_n = z_0 + ih_n$, where $\{h_n\} \subset \mathbb{R}$ is a real sequence converging to 0. Then we have

$$f(w_n) - f(z_0) = \bar{w}_n - \bar{z}_0 = \bar{z}_0 + \overline{ih}_n - \bar{z}_0 = -ih_n, \qquad \text{and} \qquad w_n - z_0 = ih_n, \tag{32}$$

which implies that

$$\frac{f(w_n) - f(z_0)}{w_n - z_0} = \frac{-ih_n}{ih_n} = -1. \tag{33}$$

The conclusion is that $f(z) = \bar{z}$ is *not* complex differentiable at any point in $\mathbb{C}$, as we have two sequences, both converging to z_0, but giving different limits as in (31) and (33).

c) Let $f(z) = \frac{1}{z}$, and let $z_0 \in \mathbb{C}$ be a nonzero complex number. Then we have

$$f(z) - f(z_0) = \frac{1}{z} - \frac{1}{z_0} = \frac{z_0 - z}{z z_0}, \tag{34}$$

so that

$$f(z) = f(z_0) + g(z)(z - z_0), \qquad \text{where} \quad g(z) = -\frac{1}{z z_0}. \tag{35}$$

Since $z_0 \neq 0$, the function $g(z)$ is continuous at $z = z_0$, and hence $f(z) = \frac{1}{z}$ is complex differentiable at $z = z_0$, with

$$f'(z_0) = g(z_0) = -\frac{1}{z_0^2}. \tag{36}$$

Exercise 27. a) Let $f(z) = z^n$ where $n \geq 1$ is an integer. Determine if f is complex differentiable, and if it is, compute the derivative.

b) Show that $f(z) = \mathrm{Re} z$ is *not* complex differentiable at any point in $\mathbb{C}$.

In the following remark, we will see that complex differentiability implies the Cauchy-Riemann equations, meaning that Euler's second approach is included in Cauchy's complex differentiation approach.

Remark 28. We have seen, by way of the example $f(z) = \bar{z}$, that complex differentiability is a very strong condition. Here we want to shed a bit more light on this observation. Let $\Omega \subset \mathbb{C}$ be an open set, and let $f : \Omega \to \mathbb{C}$ be a complex valued function. Introducing $\tilde{u} = \mathrm{Re} f$ and $\tilde{v} = \mathrm{Im} f$, we can write

$$f(z) = \tilde{u}(z) + i\tilde{v}(z), \qquad z \in \Omega. \tag{37}$$

Equivalently, we have

$$f(x + iy) = \tilde{u}(x + iy) + i\tilde{v}(x + iy), \tag{38}$$

for all $(x, y) \in \mathbb{R}^2$ satisfying $x+iy \in \Omega$. Now we introduce the real functions $u(x, y) = \tilde{u}(x+iy)$ and $v(x, y) = \tilde{v}(x + iy)$, and turn the preceding formula into

$$f(x + iy) = u(x, y) + iv(x, y), \tag{39}$$

for all $(x, y) \in \mathbb{R}^2$ satisfying $x + iy \in \Omega$. Under the (natural) identification between the pair $(x, y) \in \mathbb{R}^2$ and the complex number $x + iy \in \mathbb{C}$, the functions u and v are of course identical to the functions $\tilde{u}$ and $\tilde{v}$, respectively. Then Ω can be considered as a subset of the plane $\mathbb{R}^2$, and we can finally write

$$f(x + iy) = u(x, y) + iv(x, y), \qquad (x, y) \in \Omega. \tag{40}$$

So far, $f : \Omega \to \mathbb{C}$ was an arbitrary function. Now let us assume that f is complex differentiable at $z_0 \in \Omega$, and let $z_0 = x_0 + iy_0$ with $(x_0, y_0) \in \mathbb{R}^2$.

As in the example, we first take $z_n = z_0 + h_n$, where $\{h_n\} \subset \mathbb{R}$ is an arbitrary real sequence converging to 0. Then we have

$$\begin{aligned} \frac{f(z_n) - f(z_0)}{z_n - z_0} &= \frac{u(x_0 + h_n, y_0) + iv(x_0 + h_n, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h_n} \\ &= \frac{u(x_0 + h_n, y_0) - u(x_0, y_0)}{h_n} + i\frac{v(x_0 + h_n, y_0) - v(x_0, y_0)}{h_n}, \end{aligned} \tag{41}$$

and since the left hand side converges to $f'(z_0)$, the real and imaginary parts of the right hand side must converge. Moreover, the sequence $\{h_n\}$ is an arbitrary real sequence converging to 0, so we conclude that the x-derivatives of u and v must exist at (x_0, y_0), and

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0). \tag{42}$$

Next, we take $z_n = z_0 + ih_n$, where $\{h_n\} \subset \mathbb{R}$ is as before. Then we have

$$\begin{aligned} \frac{f(z_n) - f(z_0)}{z_n - z_0} &= \frac{u(x_0, y_0 + h_n) + iv(x_0, y_0 + h_n) - u(x_0, y_0) - iv(x_0, y_0)}{ih_n} \\ &= \frac{v(x_0, y_0 + h_n) - v(x_0, y_0)}{h_n} - i\frac{u(x_0, y_0 + h_n) - u(x_0, y_0)}{h_n}, \end{aligned} \tag{43}$$

which implies that the y-derivatives of u and v exist at (x_0, y_0), and that

$$f'(z_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0). \tag{44}$$

Now by comparing (42) and (44), we infer the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \qquad \text{at} \quad (x_0, y_0). \tag{45}$$

To conclude, the real and imaginary parts of a complex differentiable function must satisfy the Cauchy-Riemann equations. We see that this strong condition is related to the fact that a sequence of complex numbers can converge to a point from many different directions.

At this point, a natural question is if complex differentiability is a too strong condition, i.e., if there would be not enough complex differentiable functions to generate any interesting theory. We have seen that $f(z) = z^n$ and $f(z) = \frac{1}{z}$ are complex differentiable, and will see later various assurances that the class of complex differentiable functions is large enough. Another question is if complex differentiability is equivalent to the Cauchy-Riemann equations. We will see in the next section that complex differentiability implies a bit more than the Cauchy-Riemann equations, and provided this extra condition holds, they are indeed equivalent.

The usual differentiation rules work also for complex derivatives.

Theorem 29. *Let $\Omega \subset \mathbb{C}$ be an open set, and suppose that $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ are complex differentiable at $z_0 \in \Omega$. Then $f \pm g$ and fg are all complex differentiable at z_0, and their derivatives are given by*

$$(f \pm g)'(z_0) = f'(z_0) \pm g'(z_0), \qquad \text{and} \qquad (fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \tag{46}$$

Furthermore, let $U \subset \mathbb{C}$ be open, with $g(\Omega) \subset U$, and let $F : U \to \mathbb{C}$ be complex differentiable at $g(z_0)$. Then the composition $F \circ g : \Omega \to \mathbb{C}$ is complex differentiable at z_0, and

$$(F \circ g)'(z_0) = F'(g(z_0))g'(z_0). \tag{47}$$

Proof. Let us prove the chain rule (47). Since F is differentiable at $g(z_0)$, by definition, there is a function $\tilde{F} : U \to \mathbb{C}$, continuous at $g(z_0)$, and with $F'(g(z_0)) = \tilde{F}(g(z_0))$, such that

$$F(w) = F(g(z_0)) + \tilde{F}(w)(w - g(z_0)), \qquad w \in U. \tag{48}$$

Similarly, there is a function $\tilde{g} : \Omega \to \mathbb{C}$, continuous at z_0, and with $g'(z_0) = \tilde{g}(z_0)$, such that

$$g(z) = g(z_0) + \tilde{g}(z)(z - z_0), \qquad z \in \Omega. \tag{49}$$

Plugging $w = g(z)$ into (48), we get

$$F(g(z)) = F(g(z_0)) + \tilde{F}(g(z))(g(z) - g(z_0)) = F(g(z_0)) + \tilde{F}(g(z))\tilde{g}(z)(z - z_0), \tag{50}$$

where in the last step we have used (49). By Lemma 18 the function $z \mapsto \tilde{F}(g(z))\tilde{g}(z)$ is continuous at z_0, which confirms that $F \circ g$ is complex differentiable at z_0, with

$$(F \circ g)'(z_0) = \tilde{F}(g(z_0))\tilde{g}(z_0) = F'(g(z_0))g'(z_0). \tag{51}$$

The sum and product rules can be proven similarly. □

Corollary 30. *Let $\Omega \subset \mathbb{C}$ be an open set, and let $g : \Omega \to \mathbb{C}$ be complex differentiable at $z_0 \in \Omega$, with $g(z_0) \neq 0$. Then with $F(w) = \frac{1}{w}$, we have*

$$\Big(\frac{1}{g}\Big)'(z_0) = (F \circ g)'(z_0) = -\frac{g'(z_0)}{[g(z_0)]^2}. \tag{52}$$

Exercise 31. a) Compute the derivative of $f(z) = z^{-n} := \frac{1}{z^n}$, where $n \geq 1$ is an integer.

b) Derive a formula for $\big(\frac{f}{g}\big)'$.

The following might be the most important definition in complex analysis, as complex analysis can be thought of as the study of holomorphic functions.

Definition 32. A function $f : \Omega \to \mathbb{C}$, with $\Omega \subset \mathbb{C}$ open, is said to be *holomorphic in* Ω, if f is complex differentiable at each point of Ω. The set of all holomorphic functions in Ω is denoted by $\mathscr{O}(\Omega)$.

Remark 33. Obviously, holomorphic functions are continuous, that is, $\mathscr{O}(\Omega) \subset \mathscr{C}(\Omega)$.

Exercise 34. Let $\Omega \subset \mathbb{C}$ be an open set, and let $f, g \in \mathscr{O}(\Omega)$. Prove the following.

a) We have $f \pm g \in \mathscr{O}(\Omega)$ and $fg \in \mathscr{O}(\Omega)$, with

$$(f \pm g)' = f' \pm g', \qquad \text{and} \qquad (fg)' = f'g + fg'. \tag{53}$$

b) Let $U \subset \mathbb{C}$ be open, with $g(\Omega) \subset U$, and let $F \in \mathscr{O}(U)$. Then $F \circ g \in \mathscr{O}(\Omega)$, and

$$(F \circ g)' = (F' \circ g)g'. \tag{54}$$

c) Suppose that g does not vanish anywhere in Ω. Then we have $\frac{1}{g} \in \mathscr{O}(\Omega)$ with

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}, \qquad \text{and} \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}. \tag{55}$$

4. Real differentiability and the Cauchy-Riemann equations

In this section, we will have a closer look at the relation between complex differentiability and the Cauchy-Riemann equations. Before doing so we introduce a convenient notation.

Definition 35. Let $\{z_n\}$ and $\{w_n\}$ be sequences of complex numbers. Then the notation

$$z_n = o(w_n) \qquad \text{as} \quad n \to \infty, \tag{56}$$

means that

$$\lim_{n\to\infty} \frac{|z_n|}{|w_n|} = 0. \tag{57}$$

We also write

$$z_n = s_n + o(w_n) \qquad \text{as} \quad n \to \infty, \tag{58}$$

to mean

$$z_n - s_n = o(w_n) \qquad \text{as} \quad n \to \infty. \tag{59}$$

Definition 36. Let $f : K \to \mathbb{C}$ with $K \subset \mathbb{C}$ and let $w \in K$. Then the notation

$$f(z) = o(g(z)) \qquad \text{as} \quad z \to w, \tag{60}$$

where $g : U \to \mathbb{C}$ is some function defined on an open set $U \subset \mathbb{C}$ with $w \in U$, means that

$$f(z_n) = o(g(z_n)) \qquad \text{as} \quad n \to \infty, \tag{61}$$

for every sequence $\{z_n\} \subset U \cap K$ converging to w. We also write

$$f(z) = F(z) + o(g(z)) \qquad \text{as} \quad z \to w, \tag{62}$$

to mean

$$f(z) - F(z) = o(g(z)) \qquad \text{as} \quad z \to w. \tag{63}$$

Let $f : \Omega \to \mathbb{C}$ be a function where $\Omega \subset \mathbb{C}$ is open. With this notation, we can write the definitions of continuity and complex differentiability as follows.

- f continuous at $w \in \Omega$ iff

$$f(z) = f(w) + o(1) \qquad \text{as} \quad z \to w. \tag{64}$$

- f is complex differentiable at $w \in \Omega$ iff

$$f(z) = f(w) + \lambda(z - w) + o(z - w) \qquad \text{as} \quad z \to w, \tag{65}$$

  for some $\lambda \in \mathbb{C}$. If such λ exists, we write $f'(w) = \lambda$.

Let us write

$$f(x+iy) = u(x,y) + iv(x,y), \qquad x+iy \in \Omega, \tag{66}$$

as in Remark 28. Then we consider Ω as a subset of $\mathbb{R}^2$, and define the vector-valued function $F : \Omega \to \mathbb{R}^2$ by

$$F(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}, \qquad (x,y) \in \Omega. \tag{67}$$

The functions f and F can and will be considered identical, but in this section we are going to make a distinction between them. If f is complex differentiable at $x+iy \in \Omega$, then (65) can be rewritten in terms of F as

$$F((x,y)+h) = F(x,y) + Ah + o(|h|) \qquad \text{as} \quad \mathbb{R}^2 \ni h \to 0, \tag{68}$$

where $A \in \mathbb{C}\mathbb{R}$ is the matrix representing the multiplication by $f'(x+iy)$, i.e.,

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad \text{where} \qquad (a,b) \in \mathbb{R}^2 \quad \text{and} \quad f'(x+iy) = a + bi. \tag{69}$$

Here the meaning of the notation $o(|h|)$ in (68) is what it should be, i.e., (68) means

$$\frac{|F((x,y)+h) - F(x,y) - Ah|}{|h|} \to 0 \qquad \text{as} \quad h \to 0, \tag{70}$$

where $|s|$ denotes the (Euclidean) norm of the vector $s \in \mathbb{R}^2$.

Now, the condition (68) or (70) is precisely what it means for the function F to be Fréchet differentiable at (x,y) with its derivative (or the Jacobian) $DF(x,y)$ equal to A.

Definition 37. Let $\Omega \in \mathbb{R}^n$ be an open set, let $F : \Omega \to \mathbb{R}^m$, and let $p \in \Omega$. If there exists a matrix $A \in \mathbb{R}^{m\times n}$ such that

$$F(p+h) = F(p) + Ah + o(|h|) \qquad \text{as} \quad \mathbb{R}^n \ni h \to 0, \tag{71}$$

then we say that F is *Fréchet differentiable at* $p \in \Omega$, and write $DF(p) = A$. Fréchet differentiability is also referred to as *real differentiability*, or simply *differentiability*.

We have proved the following theorem.

Theorem 38. *If $f \in \mathscr{O}(\Omega)$, then $F : \Omega \to \mathbb{R}^2$, as above, is Fréchet differentiable at each point of Ω, and the Jacobian $DF(x,y)$ is the matrix representing the multiplication by the complex number $f'(x+iy)$, for each $(x,y) \in \Omega$.*

Remark 39. Suppose that $F : \Omega \to \mathbb{R}^2$ is Fréchet differentiable at $(x,y) \in \Omega$. Then writing $F = \begin{pmatrix} u \\ v \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in components, and taking a sequence $h_n = \begin{pmatrix} t_n \\ 0 \end{pmatrix} \in \mathbb{R}^2$ with $\mathbb{R} \ni t_n \to 0$ in the definition (71), we get

$$\begin{aligned} u(x+t_n,y) &= u(x,y) + at_n + o(|t_n|), \\ v(x+t_n,y) &= v(x,y) + ct_n + o(|t_n|). \end{aligned} \tag{72}$$

Since $\{t_n\}$ is an arbitrary real sequence converging to 0, this implies the existence of the partial derivatives $\frac{\partial u}{\partial x}(x,y)$ and $\frac{\partial v}{\partial x}(x,y)$, as well as the equalities $\frac{\partial u}{\partial x}(x,y) = a$ and $\frac{\partial v}{\partial x}(x,y) = c$. On the other hand, if we take a sequence $h_n = \begin{pmatrix} 0 \\ t_n \end{pmatrix} \in \mathbb{R}^2$ with $t_n \to 0$, we get

$$\begin{aligned} u(x,y+t_n) &= u(x,y) + bt_n + o(|t_n|), \\ v(x,y+t_n) &= v(x,y) + dt_n + o(|t_n|), \end{aligned} \tag{73}$$

which implies that the partial derivatives $\frac{\partial u}{\partial y}(x,y)$ and $\frac{\partial v}{\partial y}(x,y)$ exist and are equal to b and d, respectively. To conclude, we have

$$DF(x,y) = \begin{pmatrix} \frac{\partial u}{\partial x}(x,y) & \frac{\partial u}{\partial y}(x,y) \\ \frac{\partial v}{\partial x}(x,y) & \frac{\partial v}{\partial y}(x,y) \end{pmatrix}. \tag{74}$$

If $DF(x,y)$ represents the multiplication by a complex number, as in the preceding theorem, then we must have $a = d$ and $b = -c$, which gives the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{at} \qquad (x,y). \tag{75}$$

It turns out that the converse of the preceding theorem is also true.

Theorem 40. *Let $F : \Omega \to \mathbb{R}^2$ be Fréchet differentiable at each point of Ω, and let the components $u, v : \Omega \to \mathbb{R}$ of $F = \begin{pmatrix} u \\ v \end{pmatrix}$ satisfy the Cauchy-Riemann equations in Ω. Then the function $f : \Omega \to \mathbb{C}$ defined by $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic in Ω, with*

$$f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad \text{in} \quad \Omega. \tag{76}$$

Proof. By definition, for $(x,y) \in \Omega$, we have

$$F((x,y)+h) = F(x,y) + Ah + o(|h|) \qquad \text{as} \quad \mathbb{R}^2 \ni h \to 0, \tag{77}$$

with

$$A = \begin{pmatrix} \frac{\partial u}{\partial x}(x,y) & \frac{\partial u}{\partial y}(x,y) \\ \frac{\partial v}{\partial x}(x,y) & \frac{\partial v}{\partial y}(x,y) \end{pmatrix}. \tag{78}$$

Because of the Cauchy-Riemann equations, the matrix A represents the multiplication by the complex number $\lambda = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)$, and so (77) can be rewritten as

$$f(z+h) = f(z) + \lambda h + o(|h|) \qquad \text{as} \quad \mathbb{C} \ni h \to 0, \tag{79}$$

where $z = x + iy$. This shows that f is complex differentiable at z. □

The following theorem provides a simple criterion for Fréchet differentiability.

Theorem 41. *Suppose that the partial derivatives of $u : \Omega \to \mathbb{R}$ exist and are continuous in Ω. Then for each $p \in \Omega$, there exists $k \in \mathbb{R}^2$ such that*

$$u(p+h) = u(p) + k \cdot h + o(|h|) \qquad \text{as} \quad \mathbb{R}^2 \ni h \to 0, \tag{80}$$

that is, u is Fréchet differentiable in Ω.

Proof. Without loss of generality, we assume $0 \in \Omega$ and will only consider differentiability at the point $p = 0$. Take a sequence $\{h_n\} \subset \mathbb{R}^2$ with $h_n = (x_n, y_n) \to 0$ as $n \to \infty$. Then by the mean value theorem, for each n, there exists $\xi_n \in [-|x_n|, |x_n|]$ such that

$$u(x_n, 0) - u(0,0) = x_n \frac{\partial u}{\partial x}(\xi_n, 0), \tag{81}$$

and similarly, there exists $\eta_n \in [-|y_n|, |y_n|]$ such that

$$u(x_n, y_n) - u(x_n, 0) = y_n \frac{\partial u}{\partial y}(x_n, \eta_n). \tag{82}$$

Summing the two equalities we infer

$$u(x_n, y_n) - u(0,0) = x_n \frac{\partial u}{\partial x}(\xi_n, 0) + y_n \frac{\partial u}{\partial y}(x_n, \eta_n). \tag{83}$$

Since the partial derivatives are continuous, and $|\xi_n| \leq |x_n|$ and $|\eta_n| \leq |y_n|$, we have

$$\frac{\partial u}{\partial x}(\xi_n, 0) \to \frac{\partial u}{\partial x}(0,0) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x_n, \eta_n) \to \frac{\partial u}{\partial y}(0,0) \qquad \text{as} \quad n \to \infty. \tag{84}$$

This means that with $k = (\frac{\partial u}{\partial x}(0,0), \frac{\partial u}{\partial y}(0,0))$, we have

$$\begin{aligned} \frac{u(x_n, y_n) - u(0,0) - k \cdot h_n}{\sqrt{x_n^2 + y_n^2}} &= \frac{x_n(\frac{\partial u}{\partial x}(\xi_n, 0) - \frac{\partial u}{\partial x}(\xi_n, 0))}{\sqrt{x_n^2 + y_n^2}} + \frac{y_n(\frac{\partial u}{\partial y}(x_n, \eta_n) - \frac{\partial u}{\partial y}(0,0))}{\sqrt{x_n^2 + y_n^2}} \\ &\to 0 \qquad \text{as} \quad n \to \infty, \end{aligned} \tag{85}$$

showing that u is Fréchet differentiable at $(0,0)$ with $Du(0,0) = k$. □

It is clear that if each component of $F = (u, v) : \Omega \to \mathbb{R}^2$ is Fréchet differentiable then F itself is Fréchet differentiable. Combined with Theorem 40 and Theorem 41, this observation implies the following sufficient condition on complex differentiability of f in terms of its real and imaginary parts.

Corollary 42. *Let $u, v : \Omega \to \mathbb{R}$ be two functions whose partial derivatives exist and continuous in Ω. In addition, assume that u and v satisfy the Cauchy-Riemann equations in Ω. Then the complex function $f = u + iv$ is holomorphic in Ω.*

Example 43. Consider the complex function $f(x + iy) = e^x \cos y + ie^x \sin y$, or equivalently, the vector-valued function $F(x, y) = \begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix}$. The Jacobian of F at (x, y) is equal to

$$J(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}, \tag{86}$$

which is clearly continuous in $\mathbb{R}^2$. We also see that the components of F satisfy the Cauchy-Riemann equations. Thus we have $f \in \mathscr{O}(\mathbb{C})$.