MATH 249 ASSIGNMENT 4

NOT TO BE HANDED IN

1. Compute the integrals $\int_{C_r} z^n dz$ for $n \in \mathbb{Z}$, where $C_r = \partial D_r$ and $r > 0$. Recall that C_r has the counter-clockwise orientation.
2. Evaluate the integrals
$$\int_0^{+\infty} \frac{x^m dx}{1 + x^n}, \qquad \text{and} \qquad \int_0^{\pi} \frac{d\theta}{a + \sin^2 \theta},$$
where m, n are integers satisfying $1 \leq m \leq n - 2$, and $a > 0$ is real.
3. Calculate the residues of $\tan \pi z$ and $\cot \pi z$ at their poles.
4. Show that the sum of the residues of a rational function (together with the residue at ∞) is equal to zero. As part of this exercise you need to introduce a natural definition for the residue at ∞.
5. Let f be holomorphic and bounded in a punctured neighbourhood of 0. Show that
$$g(z) = \begin{cases} z^2 f(z) & \text{for } z \neq 0, \\ 0 & \text{for } z = 0, \end{cases}$$
is holomorphic in a neighbourhood of 0.
6. Let $c \in \mathbb{C}$ be an isolated essential singular point of f. Prove that for any given $\alpha \in \mathbb{R}$ and an arbitrarily small $r > 0$, there exists $z \in D_r(c) \setminus \{c\}$ such that $\Re f(z) = \alpha$.
7. Let $\gamma : [a, b] \to A$ be a piecewise differentiable curve, where $A \subset \mathbb{C}$ is an open set, and let $g : A \times \Omega \to \mathbb{C}$ be a continuous function of two complex variables, where $\Omega \subset \mathbb{C}$ is also an open set. Assume that for any fixed $w \in A$, the function $z \mapsto g(w, z)$ is holomorphic in Ω. Then prove that
$$f(z) = \int_\gamma g(w, z) \, \mathrm{d}w,$$
is holomorphic in Ω.

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