Problem 1. *Prove that*

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Proof. We note by the symmetry of $f(x) = \frac{1}{x}\sin x$, $2\int_0^\infty \frac{\sin x}{x} = \int_{-\infty}^{+\infty} \frac{\sin x}{x}$. We also know that $\int_{-R}^{R} \frac{\sin x}{x} dx = \operatorname{Im} \int_{-R}^{R} \frac{e^{ix}}{ix}$, and so we only have to calculate

$$\lim_{\epsilon \to 0, R \to \infty} \int_{l_-} \frac{e^{ix}}{x} dx + \int_{l_+} \frac{e^{ix}}{x} dx \tag{1}$$

where we are using the contours shown below. By Cauchy's theorem, we can conclude that

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \int_{c_\epsilon} \frac{e^{iz}}{z} dz + \int_{c_R} \frac{e^{z}}{z} dz \tag{2}$$

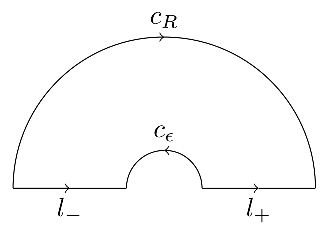


Figure 1: Contours used in solution

Now note that on $-c_R$ we have $z = R\exp(it) = R\cos t + iR\sin t$, thus

$$\int_{-C_R} \frac{\exp iz}{z} dz = \int_0^\pi \frac{\exp(iR\cos t)\exp(-R\sin t)}{R\exp(it)} iR\exp(it) dt$$

Then we obtain the estimate

$$\left| \int_{-C_R} \frac{\exp iz}{z} dz \right| = \left| \int_0^\pi \exp(iR\cos t)\exp(-R\sin t) i\, dt \right| \le \int_0^\pi \exp(-R\sin t) = 2\int_0^{\pi/2} \exp(-R\sin t) dt \tag{3}$$

This integral on the right vanishes, because $\sin t \ge 2t/\pi$ in the interval $[0, \pi/2]$ (proof: sin is concave), and thus

$$\int_0^{\pi/2} \exp(-R\sin t) dt \le \int_0^{\pi/2} \exp\left(-\frac{2Rt}{\pi}\right) dt = \frac{\pi}{2R}\left(1 - \exp\left(-\frac{R}{\pi}\right)\right)$$

and this clearly vanishes as $R \to \infty$. thus $\displaystyle\int_{C_R} \frac{\exp iz}{z} \to 0$ as $R \to \infty$. Now it remains to evaluate

$$\int_{C_\epsilon} \frac{\exp iz}{z} dz = \int_{C_\epsilon} \frac{1}{z} + f(z) dz \tag{4}$$

Where f is bounded (say, by M) as $z \to 0$. Then

$$\int_{C_\epsilon} \frac{1}{z} + f(z)dz \leq \int_{C_\epsilon} \frac{1}{z}dz + M\pi\epsilon \tag{5}$$

Thus $\int_{C\epsilon} \frac{1}{z} + f(z)dz \to \int_{C_\epsilon} \frac{1}{z}dz$ as $\epsilon \to 0$. This second integral is easy to evaluate, and is $i\pi$. Recalling that $\int_0^\infty \sin x/x = \operatorname{Im} \frac{1}{2} \int_{-\infty}^\infty e^{ix}/x\, dx$, we obtain the desired result

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

□

Problem 2. *Prove that*

$$\int_0^\infty \sin x^2\, dx = \int_0^\infty \cos x^2\, dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

Proof. Here we will use the fact that $\int_{-\infty}^\infty e^{-z^2} dz = \sqrt{\pi}$, which has a standard proof (using polar coordinates) which we will not reproduce here. By symmetry we can conclude that $\int_0^\infty e^{-z^2} dz = \sqrt{\pi}/2$. We will now integrate e^{-z^2} over the following contours: □

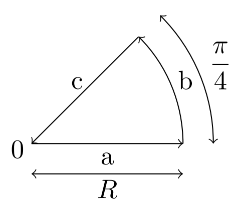


Figure 2: contours used in problem 2

Then we have

$$\lim_{R\to\infty} \int_a e^{-z^2} dz = \frac{\sqrt{\pi}}{2}$$

We also have that $\lim_{R\to\infty} \int_b e^{-z^2} dz = 0$. To see why this is so, we let $z = R \exp it$, $t \in [0, \pi/4]$, so that $z^2 = R^2 \exp 2it = R^2 \cos 2t + iR^2 \sin 2t$, and thus

$$\left| \int_b e^{-z^2} dz \right| = \left| \int_0^{\pi/4} \exp\left(-R^2 \cos 2t\right) \exp\left(-iR^2 \sin 2t\right) iR \exp\left(it\right) dt \right| \leq R \int_0^{\pi/4} \exp\left(-R^2 \cos 2t\right) dt$$

Since $\cos(2t) \geq 1 - 4t/\pi$, this integral on the right is less than

$$R \int_0^{\pi/4} \exp\left(-R^2\right) \exp\left(4R^2 t\pi\right) dt = R\pi \exp\left(-R^2\right) \frac{\exp\left(R^2\right) - 1}{4R^2} \leq \frac{1 - \exp\left(-R^2\right)}{R}$$

and this clearly vanishes as $R \to \infty$. Finally, we evaluate along $-c$, where we have $z = t \exp(\pi/4)$ $t \in [0, \infty]$, which gives us

$$\int_{-c} \exp\left(-z^2\right) dz = \int_0^\infty \exp\left(-it^2\right) \exp\left(\pi/4\right) dt = \int_0^\infty \left(\cos t^2 - i \sin t^2\right) \frac{1+i}{\sqrt{2}} dt \tag{6}$$

Performing more algebraic manipulations, we obtain

$$\int_{-c} \exp\left(-z^2\right) dz = \frac{1}{\sqrt{2}} \int_0^\infty \left(\cos t^2 + \sin t^2\right) + i\left(\cos t^2 - \sin t^2\right) dt$$

By cauchy's theorem, we can conclude that $\int_{-c} e^{-z^2} dz = \int_a e^{-z^2} = \sqrt{\pi}/2 + 0i$, and thus

$$\underbrace{\int_0^\infty \cos t^2 dt = \int_0^\infty \sin t^2 dt}_{\text{imaginary part}} \text{ and } \int_0^\infty \cos t^2 dt = \frac{\sqrt{2\pi}}{4}$$

which is what we wanted to show.

Problem 3. *Evaluate the integrals*

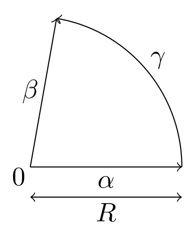
$$\int_0^\infty \exp\left(-ax\right) \sin bx \, dx \qquad \textit{and} \qquad \int_0^\infty \exp\left(-ax\right) \cos bx \, dx$$

where a and b are positive real constants.

Solution: These are clearly the real and imaginary part of the integral

$$\int_0^\infty \exp\left((-a+ib)z\right) dz$$

to solve this problem, we will use the following contours



Here the curve β is the ray aligned with $a+ib$. First it is clear that $\lim_{R\to\infty} \int_\alpha \exp\left((-a+ib)z\right) dz$ is the desired integral. We also have

$$\lim_{R\to\infty} \int_\beta \exp\left((-a+ib)z\right) dz = \int_0^\infty \exp\left((-a+ib)(a+ib)t\right)(a+ib) = (a+ib) \int_0^\infty \exp\left(-(a^2+b^2)t\right) = \frac{a+ib}{a^2+b^2}$$

It remains to show that $\int_\gamma \exp\left((-a+ib)z\right) dz$ vanishes as R goes to infinity. To prove this, let γ be parametrized by $R\exp\left(it\right)$, where $t \in [0, \theta]$, where $\theta < \pi/2$; this follows from the fact that a, b are positive. Then we have

$$\int_\gamma \exp\left((-a+ib)z\right)dz = \int_0^\theta \exp\left(-R(a\cos t + b\sin t) + iR(b\cos t - a\sin t)\right)Ri\exp\left(it\right)dt$$

Taking absolute values, we can conclude that this integral vanishes if the integral

$$\int_0^{\pi/2} R\exp\left(-R(a\cos t + b\sin t)\right)dt$$

vanishes. This is true because the minimum value of $a\cos t + b\sin t$ in $(0,\pi/2)$, let's call it m, is clearly positive and so since $R\exp\left(-Rm\right) \to 0$ as $R \to \infty$, we have the desired result. Thus

$$\int_0^\infty \exp\left(-ax\right)\sin bx\,dx = \frac{b}{a^2+b^2} \qquad \text{and} \qquad \int_0^\infty \exp\left(-ax\right)\cos bx\,dx = \frac{a}{a^2+b^2} \tag{7}$$

Problem 4 (Schwarz reflection principle)**.** *Let $\Omega \subset \mathbb{H}$ be an open set and let $\Sigma = \{\, z \in \partial\Omega : \operatorname{Im} z = 0 \,\}$ be a nonempty open subset of the real axis. Suppose that f is holomorphic in Ω, continuous on $\Omega \cup \Sigma$, and takes real values on Σ. Let $\tilde{\Omega} = \Omega \cup \Sigma \cup \overline{\Omega}$, where here $\overline{\Omega}$ denotes the image of Ω under the mapping $z \mapsto \overline{z}$. Define the function $F \in C(\tilde{\Omega})$ by*

$$F(z) = \begin{cases} f(z) & z \in \Omega \cup \Sigma \\ \overline{f(\overline{z})} & z \in \overline{\Omega} \end{cases}$$

Prove that F is holomorphic in $\tilde{\Omega}$.

Proof. The only thing to prove is that F is differentiable at every point z in it's domain. We will prove this in cases

Case 1. $z \in \Omega$, then by the openness of Ω there exists a neighborhood $N \subset \Omega$ around z such that $F = f$ on N. Since f is holomorphic on N, F is holomorphic on N.

Case 2. $z \in \overline{\Omega}$. It is clear that $z \mapsto \overline{z}$ is an open mapping, and so there exists a neighborhood $N \subset \overline{\Omega}$ which contains z and thus $\overline{N} \subset \Omega$ contains $\overline{z}$. Pick h so small so that $z + h \in N$, and consider the difference

$$\frac{F(z+h) - F(z)}{h} = \overline{\left(\frac{f(\overline{z}+\overline{h}) - f(\overline{z})}{\overline{h}}\right)}$$

By the holomorphicity of f in Ω, we can conclude that the right side converges to $f'(\overline{z})$ as $h \to 0$, that is

$$\lim_{h\to 0}\frac{F(z+h) - F(z)}{h} = \overline{\lim_{h\to 0}\left(\frac{f(\overline{z}+\overline{h}) - f(\overline{z})}{\overline{h}}\right)} = \overline{f'(\overline{z})}$$

This proves that F is holomorphic in $\overline{\Omega}$. Furthermore, since $f(z) \in \mathbb{R}$ for $z \in \Sigma$, we can conclude that for $u \in \overline{\Omega}$ we have $\lim_{u\to z} F(u) = \lim_{u\to z} \overline{f(\overline{u})} = f(z)$, and so F extends continuously to Σ from both Ω and $\overline{\Omega}$.

Case 3. Now consider the case where $z \in \Sigma$. Now I claim that for every point z in Σ there exists a ball B which is centered on z such that $B \cap \mathbb{H} \subset \Omega$. To prove this claim, note that $\mathbb{R}$ and $\operatorname{bdry} \Omega$ are closed, and so $\Sigma \equiv \operatorname{bdry} \Omega \cap \mathbb{R}$ is closed. But since Σ is also open by hypothesis, we must conclude that $\Sigma = \mathbb{R}$ or $\Sigma = \emptyset$, since $\mathbb{R}$ is connected. However, this seems a bit too restrictive, and so we won't use it in our proof. Rather, we shall prove a weaker claim that F is holomorphic on the interior of Σ with respect to $\mathbb{R}$, which we shall denote by Σ^o (e.g. if $\Sigma = [0, 1]$, then Σ^o is $(0, 1)$, not $\emptyset$). Then Σ^o is an open subset of $\operatorname{bdry} \Omega$ (with the relative topology), and so for each point z of Σ^o there exists an open ball B around z which contains none of $\operatorname{bdry} \Omega - \Sigma^o$. Now consider $B \cap \mathbb{H}$. By the fact that $z \in \Sigma$, we can conclude that $B \cap \mathbb{H}$ intersects Ω, but if $B \cap \mathbb{H}$ also intersects the exterior of Ω, then we can write

$$B \cap \mathbb{H} = \Omega \cap (B \cap \mathbb{H}) \cup \operatorname{exterior} \Omega \cap (B \cap \mathbb{H})$$

Thus writing $B \cap \mathbb{H}$ as the union of two non-empty disjoint open sets, contradicting the fact that $B \cap \mathbb{H}$ is connected. This proves that $B \cap \mathbb{H} \subset \Omega$. With all that, we can conclude that $\overline{B \cap \mathbb{H}} = B \cap \overline{H} \subset \bar{\Omega}$ and $B \cap \mathbb{R} \subset \Sigma^o$, and so B is completely contained in $\tilde{\Omega}$.

Now we will show that F is holomorphic at z (which is the centre of B). To do, we will use *Morera's* theorem, and prove that the integral of F over every triangle T in B vanishes. The proof is quite simple: if T does not interesect the real axis, then we are done, for then the triangle is contained entirely in either Ω or $\bar{\Omega}$, where F is holomorphic. If the triangle does intersect the real axis, we can decompose the triangle as in the following picture

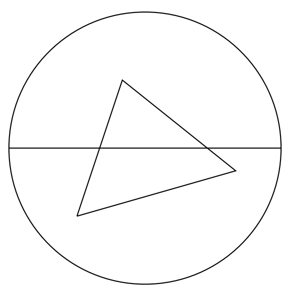


Figure 3: Triangle intersecting $\mathbb{R}$

Since this is the integral of F over two polygonal paths contained in the closure of Ω and $\bar{\Omega}$ respectively, which are the limits of triangles strictly contained in Ω and $\bar{\Omega}$, we can conclude *by continuity* that the integral of F vanishes over the entire triangle. This proves that F is holomorphic on B.

□

Problem 5. *Show that an entire analytic function with bounded real part must be constant.*

Proof. We will use the fact that if f is entire and f is bounded then f is constant. This was proved in class, and is known as Liouville's theorem.

Suppose that f has bounded real part, then consider $\exp \circ f$. By the fact that $|\exp(f(z))| = \exp(\operatorname{Re}(f(z)))$, we can conclude that that $\exp \circ f$ is totally bounded. Thus $\exp \circ f$ is constant, and so clearly f must also be constant. $\square$

Problem 6. *Let f be entire and suppose that $|f(z)| \leq M(1 + \sqrt{|z|})$ for all $z \in \mathbb{C}$, with some constant $M > 0$. Show that f is constant.*

Proof. We will use that fact that if f is entire then

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{C_R} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

for any circle C_R (with radius R) containing z. Then we can conclude that

$$\left| f^{(k)}(z) \right| \leq \frac{k!}{2\pi} \int_{C_R} \left| \frac{f(\zeta)}{(\zeta - z)^{k+1}} \right| d\zeta \leq M(1 + \sqrt{R}) \frac{k!}{R^k}$$

Sending $R \to \infty$, we see that $f^{(k)}(z)$ vanishes for all z and for all $k \geq 1$. In particular $f'(z) = 0$ for all z and thus f is constant on $\mathbb{C}$. $\square$

Problem 7. *Let f be an entire function satisfying $|f(z)| \to \infty$ as $|z| \to \infty$. Show that $f : \mathbb{C} \to \mathbb{C}$ is surjective. Derive the fundamental theorem of algebra as a corollary.*

Proof. First note that if f satisfies the hypotheses of the theorem, then so does $f - \zeta$ for any $\zeta \in \mathbb{C}$. Then there exists a disk D so large that $|f - \zeta| \geq 1$ outside this disk, and in particular, $\frac{1}{f - \zeta} \leq 1$ outside of D. Now since $f - \zeta$ is clearly *not* constant, we can conclude that $1/(f - \zeta)$ is also not constant, and thus $1/(f - \zeta)$ is unbounded. By the fact that $1/(f - \zeta) \leq 1$ outside of D, we must conclude that $1/(f - \zeta)$ is unbounded on the inside of D. But this implies that $1/(f - \zeta)$ has some discontinuities in D, which can only occur if there exists some z such that $f(z) = \zeta$. This proves surjectivity.

Now to prove the fundamental theorem of algebra as corollary, let f be a polynomial of degree n. Then $|f(z)| \geq a\,|z|^n - b\,|z|^{n-1}$ for some $a, b \in \mathbb{R}$ (this follows from $f(z) = a_n z^n + \cdots + a_0$, and the triangle inequality). But for large $|z|$, $a\,|z|^n - b\,|z|^{n-1}$ clearly gets arbitarily large, and thus $|f(z)| \to \infty$ as $|z| \to \infty$. We can therefore use the above theorem to prove that there exists a z such that $f(z) = 0$. $\square$