MATH 249 ASSIGNMENT 3

DUE WEDNESDAY APRIL 1

Note: For hints on the first 3 problems, see page 64 of Stein and Shakarchi.

1. Prove that
$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$
2. Prove that
$$\int_0^\infty \sin x^2\,\mathrm{d}x = \int_0^\infty \cos x^2\,\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$
3. Evaluate the integrals
$$\int_0^\infty e^{-ax}\sin bx\,\mathrm{d}x, \qquad \text{and} \qquad \int_0^\infty e^{-ax}\cos bx\,\mathrm{d}x,$$
where $a > 0$ and $b > 0$ are constants.
4. Let $\Omega \subset \mathbb{H} \equiv \{\operatorname{Im} z > 0\}$ be an open set, and let $\Sigma = \{z \in \partial\Omega : \operatorname{Im} z = 0\}$ be a nonempty open subset of the real axis $\{\operatorname{Im} z = 0\}$. Suppose that f is holomorphic in Ω, continuous in $\Omega \cup \Sigma$, and takes real values on Σ. Let
$$\tilde{\Omega} = \Omega \cup \Sigma \cup \{\bar{z} : z \in \Omega\}.$$
Define the function $F \in C(\tilde{\Omega})$ by $F = f$ in $\Omega \cup \Sigma$ and $F(\bar{z}) = \overline{f(z)}$ for $z \in \Omega$. Show that F is holomorphic in $\tilde{\Omega}$. This result is known as the *Schwarz reflection principle*.
5. Show that an entire analytic function with bounded real part must be constant.
6. Let $f \in \mathscr{O}(\mathbb{C})$ and suppose that $|f(z)| \leq M(1 + \sqrt{|z|})$ for all $z \in \mathbb{C}$, with some constant $M > 0$. Show that f is constant.
7. Let f be an entire function satisfying $|f(z)| \to \infty$ as $|z| \to \infty$. Show that $f : \mathbb{C} \to \mathbb{C}$ is surjective. Derive the fundamental theorem of algebra as a corollary.

Date: Winter 2015.