1. Suppose that $f(z) = \sum a_n(z-c)^n$ and $g(z) = \sum b_n(z-c)^n$ both converge in an open disk centered at c, and assume $b_0 \neq 0$. Show that

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} e_n (z-c)^n, \qquad \text{with} \qquad e_n = \frac{1}{b_0}\left(a_n - \sum_{k=0}^{n-1} b_{n-k} e_k\right),$$

where the power series converges in a disk $D_r(c)$ with some $r > 0$, and the empty sum in the definition of e_n when $n = 0$ is understood to be 0. By using this result, compute a first few terms of the Maclaurin series of $\sec z = \frac{1}{\cos z}$ and $\tan z$.

Solution: We will prove this a few steps. First we will show that f/g is a power series in a ball centered on c, and then we will show that it's coefficients must be given by the formula state above. To prove that f/g is a power series, we will prove a few lemmas

LEMMA 1: Let φ be power series in a ball centered on a, and let σ be power series in a ball centered on $\varphi(a)$, then $\sigma \circ \varphi$ is a power series in a ball centered on a.

Proof. The proof is simple, we let φ_n be the polynomial of degree n such that $\lim_{n\to\infty} \varphi_n = \varphi$ and similarly for σ_m. Then $\sigma_m \circ \varphi_n \to \sigma \circ \varphi_n \to \sigma \circ \varphi$ as $n, m \to \infty$. This proof is not the most rigorous but it gets the idea across. □

LEMMA 2: the function $f : \mathbb{C}^\times \to \mathbb{C}$ defined by $f(z) = 1/z$ has a power series representation around $z = c$ for any $c \neq 0$.

Proof. write $1/z$ as

$$\frac{1}{z} = \frac{1}{c-(z-c)} = \frac{1}{c}\frac{1}{1-\frac{z-c}{c}} = \sum_{n=0}^{\infty}\left(\frac{1}{c}\right)^n (z-c)^n \tag{1}$$

this equality is valid as long as $|(z-c)/c| < 1$. □

With these two lemmas, we can conclude that $h \equiv f/g$ is analytic on some ball centered on c, that is, there exists a power series representation for h:

$$h(z) = \sum_{n=0}^{\infty} e_n (z-c)^n \tag{2}$$

We can find the e_n by equating the coefficients of $gh = f$. Recall that the coefficients of gh can be calculated by

$$gh(z) = \sum_n \left(\sum_{i+j=n} b_i e_j\right)(z-c)^n \tag{3}$$

and thus equating the coefficients, we obtain

$$\boxed{e_n b_0 + \sum_{i=0}^{n-1} b_{n-i} e_i = a_n}$$

which is exactly what we wanted to show.

2. Sketch the following curves. *In the following exercises, we will let x, y stand for the real and imaginary parts of the complex number under consideration. The real number t will denote the curve parameter.*

   (a) The image of $\{z \in \mathbb{C} : \operatorname{Im} z = \operatorname{Re} z + 1\}$ under the mapping $z \mapsto z^2$.

   Solution: The original curve is $(t, t+1)$, and here $z \mapsto z^2$ can be expressed in component form by

$$(t, t+1) \to (-2t-1, 2t^2+t)$$

   Note that if $x = -2t-1$ and $y = 2t^2+t$, then $y = (x^2+x)/2$, and so the curve is an upwards facing parabola that passes through the points $z = -1$ and $z = 0$.

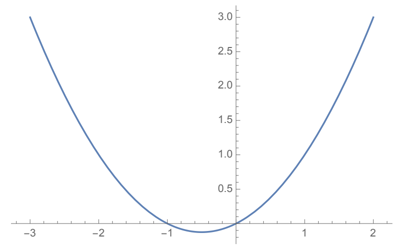

Figure 1: Figure for part (a)

   (b) The image of $\{z \in \mathbb{C} : \operatorname{Im} z = 1\}$ under the mapping $z \mapsto z^3$.

   Solution: Here we can write the original set as the image of the curve $(t, 1)$, and then $z \mapsto z^3$ becomes

$$(t, 1) \mapsto (t^3 - 3t, 3t^2 - 1) \tag{4}$$

   For very negative t this is in the -/+ quadrant and for very positive t it is in the +/+ quadrant. The interesting part of the curve is near $t = 0$, where for $t > 0$ it is initially heading in the (-1,0) direction, but changes it's mind at $t = 1$ and starts heading in the $(+1, 0)$ direction. The curve is symetric with respect to the transformation $(x, y) \to (-x, y)$.

   (c) The image of the circle $\partial D_r = \{z \in \mathbb{C} : |z| = r\}$ under the mapping $z \mapsto \exp z$, for $r = \pi$ and for $r = \frac{3}{2}\pi$.

   Solution: The original curve can be parametrized by $r \exp(it)$, and thus the image is $\exp(r \exp(it)) = (\exp(r\cos t)\cos(r \sin t), \exp(r\cos t)\sin(r\sin t))$, which has a radial length $R = \exp(r \cos t)$ and an angle $r \sin t$, that is $(\log R)^2 + \theta^2 = r^2$, and thus $R = \exp\left(\pm\sqrt{r^2 - \theta^2}\right)$, where $\theta \in (-r, r)$. These curves look like

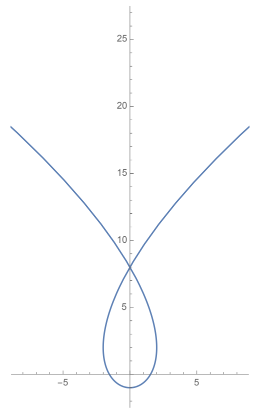

Figure 2: Part (b)

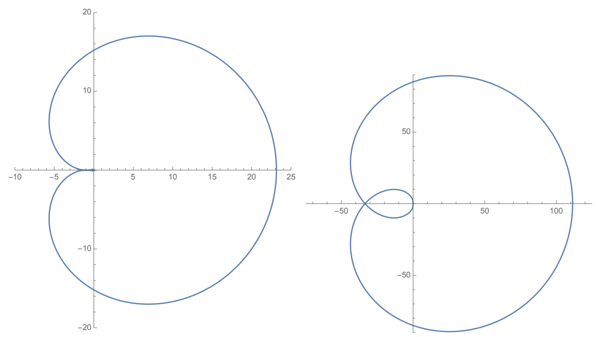

Figure 3: Part (c)

(d) The image of $\{z \in \mathbb{C} : \operatorname{Im} z = 1\}$ under the multi-valued mapping $z \mapsto \sqrt[3]{z}$. Identify the part of the curve that corresponds to the principal branch of $z \mapsto \sqrt[3]{z}$.

Solution: Here we can write the set as the image of the curve $(t, 1)$, $t \in \mathbb{R}$, and thus since $\sqrt[3]{z} = \exp\left(\frac{1}{3}\log z\right)$, with $\log(z) = \log_{\mathbb{R}}|z| + i\operatorname{br}\arg(z) + i2\pi n$, with

$$\operatorname{br}\arg(t,1) = \begin{cases} \arctan\dfrac{1}{t} & t > 0 \\ \dfrac{\pi}{2} & t = 0 \\ \pi - \arctan\left|\dfrac{1}{t}\right| & t < 0 \end{cases}$$

And so we obtain

$$\sqrt[3]{(t,1)} = (1+t^2)^{1/6}\exp\left(i\frac{\operatorname{br}\arg(1,t)}{3} + i\frac{2\pi k}{3}\right) \qquad k = 0, 1, 2 \tag{5}$$

This gives us three branches, which look like the curves shown below in Figure 4.

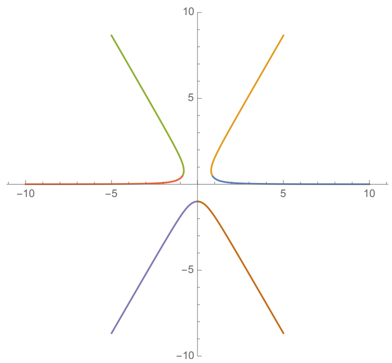

Figure 4: Part (d)

(e) The image of $\{z \in \mathbb{C} : \operatorname{Im} z = 1\}$ under the multi-valued mapping $z \mapsto \log z$. Identify the part of the curve that corresponds to the principal branch $z \mapsto \operatorname{Log} z$.

Solution: Here we have $(t, 1) \to \log_{\mathbb{R}} \sqrt{1+t^2} + i \operatorname{br}\arg(1,t) + i2\pi n$, $n \in \mathbb{Z}$, with $\operatorname{br}\arg$ as defined above. These branches look like those curves shown below, in figure 5.

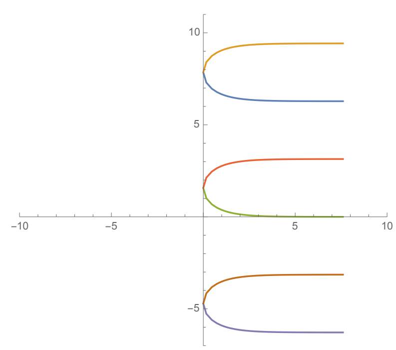

Figure 5: Part (e)

3. Find as many mistakes as you can in the following reasonings.

(a) $-1 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1) \cdot (-1)} = \sqrt{1} = 1.$

Solution: The first equality is okay. The second equality is not okay, because i is only one of the two values of $\sqrt{-1}$, in other words, if we replace i by $\sqrt{-1}$, we are assuming a specific branch of the square root function. The third inequality is true in general only as a statement about the square root *set* valued function. In the last equality we choose a particular branch of the square root function. The problem arises because 1 and -1 are both in $\sqrt{1}$.

(b) We have $e^{2\pi i} = 1$, and hence $e^{1+2\pi i} = e$. This means that

$$e = (e^{1+2\pi i})^{1+2\pi i} = e^{(1+2\pi i)(1+2\pi i)} = e^{1-4\pi^2+4\pi i} = e^{1-4\pi^2},$$

or $e^{-4\pi^2} = 1$.

Solution: It helps to use the exp notation. First note that $(e^{1+2\pi i})^{(1+2\pi i)} = \exp((1+2\pi i)(\log\exp 1 + 2\pi i))$. Now we see what is wrong, $\log\exp \neq$ identity, but rather $\log\exp =$ identity $+2\pi i n$, $n \in \mathbb{Z}$. Thus

$$\exp((1+2\pi i)(\log\exp 1 + 2\pi i)) = \exp((1+2\pi i)(1+2\pi i n)),\ n \in \mathbb{Z} = \exp(1-4\pi^2 n + 2\pi i n) = \exp(1-4\pi^2 n). \tag{6}$$

The problem is we are picking different elements in the *set* $\exp(1+2\pi i)^{1+2\pi i}$.

4. Prove the following.

(a) For the principal branch of the power function, we have

$$z^{s+it} = |z|^s e^{-t\mathrm{Arg}z}\left(\cos\left(s\mathrm{Arg}z + t\log|z|\right) + i\sin\left(s\mathrm{Arg}z + t\log|z|\right)\right),$$

where s and t are real numbers.

Proof. For the principal branch of the power function, we have

$$z^{s+it} = \exp((s+it)\mathrm{Log}(z))$$

Where here $\mathrm{Log}(z) = \log_{\mathbb{R}}|z| + i\mathrm{Arg}(z)$. Substituting this into the above and expanding, we obtain

$$z^{s+it} = \exp(s\log_{\mathbb{R}}|z| - t\mathrm{Arg}(z) + i(t\log_{\mathbb{R}}|z| + s\mathrm{Arg}(z)))$$

And expanding this using the fact that $|z|^s = \exp s\log_{\mathbb{R}}|z|$ and $\exp i\theta = \cos\theta + i\sin\theta$, we obtain

$$z^{s+it} = |z|^s \exp(-t\mathrm{Arg}(z))(\cos(t\log_{\mathbb{R}}|z| + s\mathrm{Arg}(z)) + i\sin(t\log_{\mathbb{R}}|z| + s\mathrm{Arg}(z))$$

□

(b) Let $\Omega \subset \mathbb{C}$ be an open set, and let $f \in \mathscr{O}(\Omega)$ be a *holomorphic branch of the n-th root* in the sense that $[f(z)]^n = z$ for $z \in \Omega$ ($n \in \mathbb{N}$). Suppose also that $\log \in \mathscr{O}(\Omega)$ is a branch of logarithm in the set Ω. Then we have $f(z) = \exp(\frac{1}{n}\log z)\exp(\frac{2\pi i k}{n})$ for all $z \in \Omega$ and for some $k \in \{0, 1, \ldots, n-1\}$.

Proof. Recall that in class we proved that the n-th root *set-valued* function satisfies

$$\sqrt[n]{z} = \underbrace{\sqrt[n]{|z|}}_{\mathbb{R}}\exp\left(\frac{\arg(z)}{n}\right), \qquad 0 \text{ if } z = 0 \tag{7}$$

We also know that $\frac{1}{n}\log z = \frac{1}{n}\log_{\mathbb{R}}|z| + \frac{1}{n}\arg z$, and thus

$$\exp\left(\frac{1}{n}\log z\right) = \sqrt[n]{z} \tag{8}$$

Where the equality holds as set-valued functions. Now since $\operatorname{br}\log$ is a branch of the logarithm, we can write $\log z = \operatorname{br}\log + 2\pi k i$, $k \in \mathbb{Z}$, and thus we can conclude that

$$\sqrt[n]{z} = \exp\left(\frac{1}{n}\operatorname{br}\log z\right)\exp\left(\frac{2\pi i k}{n}\right),\ k \in \mathbb{Z} \tag{9}$$

This is still a multivalued function. By the periodicity of exp, we can suppose that $k \in \mathbb{Z}/n\mathbb{Z}$. Since f is a branch of the above $\sqrt[n]{}$ function, we can conclude that

$$f(z) = \exp\left(\frac{1}{n}\operatorname{br}\log z\right)\exp\left(\frac{2\pi i k(z)}{n}\right) \tag{10}$$

where $k : \mathbb{C} \to \mathbb{Z}/n\mathbb{Z}$.

$$\exp\left(\frac{2\pi i}{n}k(z)\right) = \frac{f(z)}{\exp\left(\frac{1}{n}\operatorname{br}\log z\right)} \tag{11}$$

and since the right side is a *continuous* function of z, we must conclude that the left side is also a continuous function, but since $k \in \mathbb{Z}/n\mathbb{Z}$, if k is discontinuous anywhere, then the function on the left will also be discontinous, and so we must conclude that k is a constant function on Ω. Finally, we have

$$f(z) = \exp\left(\frac{1}{n}\operatorname{br}\log z\right)\exp\left(\frac{2\pi i k}{n}\right) \tag{12}$$

for some $k \in \mathbb{Z}/n\mathbb{Z}$, constant. □

(c) In the setting of (b), such a function f cannot exist if $n \geq 2$ and if $0 \in \Omega$.

Proof. Suppose there did exist such a f. Then clearly since $f(0)^n = 0$, we must have $f(0) = 0$. Then note that since

$$[f(z)]^n = z \tag{13}$$

for all z in some open set Ω, we must have $nf(z)^{n-1}f'(z) = 1$, and in particular

$$f(z)^{n-1}f'(0) = \frac{1}{n} \tag{14}$$

Since $n \geq 2$, we have $f(0)^{n-1} = 0$, but this clearly contradicts (14), and so we must conclude that f is *not* holomorphic on Ω. □

5. Prove the following.

(a) $\sin z = 0$ if and only if $z = \pi n$ for some $n \in \mathbb{Z}$.

Proof. Recall the definitions of sin and cos

$$\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i} \qquad \cos(z) = \frac{\exp(iz) + \exp(-iz)}{2} \tag{15}$$

Now it is clear that if $\sin z = 0$ then $\exp(iz) = \exp(-iz)$, and thus $\exp(i2z) = 1$, and thus, as we proved in class, we must have $2z = 2\pi n$, for some $n \in \mathbb{Z}$, and thus $z = \pi n$ for some $n \in \mathbb{Z}$ ☐

(b) $\cos z = 0$ if and only if $z = \frac{\pi}{2} + \pi n$ for some $n \in \mathbb{Z}$.

Proof. This is proved in almost the same exact way as the previous problem. We use the fact that $\exp(i\pi) = -1$, which is true because $\exp(i\pi)\exp(i\pi) = 1$ while $\pi \neq 2\pi n$ for any $n \in \mathbb{Z}$. Then $\cos z = 0$ if and only if $\exp(iz) = -\exp(-iz)$, that is $\exp(2iz) = -1$, and thus $2z = \pi + 2\pi n$, $n \in \mathbb{Z}$, and thus $z = \pi/2 + \pi n$, $n \in \mathbb{Z}$. ☐

(c) The periods of sin are precisely the numbers $2\pi n$, $n \in \mathbb{Z}$.

Proof. We will use the fact (proved in class) that the periods of exp are precisely the numbers $2\pi n$, $n \in \mathbb{Z}$. Now suppose that $\sin(z + h) = \sin(z)$ for all z, then we can conclude that

$$\exp(iz)(\exp(ih) - 1) - \exp(-iz)(\exp(-ih) - 1) = 0 \tag{16}$$

Since this is true for all z, we can pick $z = 0$, whereby we conclude that $\exp(ih) = \exp(-ih)$, which implies $h = \pi n$. Since $\exp(\pi n) = (-1)^n$, we obtain

$$\exp(iz)((-1)^n - 1) - \exp(-iz)((-1)^n - 1) = 0 \tag{17}$$

and thus for n odd we conclude that $\sin(z) = 0$ for all z, a contradiction thanks to the last problem. For n even we have equality. Thus $\sin(z + h) = \sin(z)$ for all z implies that h cant be anything but $h = 2\pi n$, and this h *is* a period of sin because $\exp(z + 2\pi n) = \exp(z)$ for all z. ☐

(d) The periods of cos are precisely the numbers $2\pi n$, $n \in \mathbb{Z}$.

Proof. Using a similar argument to last proof, we conclude that $\cos(z + h) = \cos(z)$ for all z if and only if h satisfies

$$\exp(iz)(\exp(ih) - 1) + \exp(-iz)(\exp(-ih) - 1) = 0 \tag{18}$$

for all z. Using $z = w + \pi/2$ (which is a bijective map), we conclude that

$$i\exp(iw)(\exp(ih) - 1) - i\exp(-iw)(\exp(-ih) - 1) = 0 \tag{19}$$

which is exactly equivalent to $\sin(w + h) = \sin(w)$, and thus h must be of the form $2\pi n$. Again, if $h = 2\pi n$, then clearly $\cos(z + h) = \cos(z)$, because cos is a simple sum of exp. ☐

(e) $\cos z = \cos w$ if and only if either $z + w = 2\pi n$ for some $n \in \mathbb{Z}$, or $z - w = 2\pi n$ for some $n \in \mathbb{Z}$.

Proof. To prove this, we will prove a useful identity. We recall the addition formula for cos

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

(This is a trivial consequence of the definition of cos in terms of exp), and thus

$$\cos(a+b) - \cos(a-b) = -2\sin(a)\sin(b)$$

where we have used the fact that $\sin(-b) = -\sin(b)$ and $\cos(-b) = \cos(b)$ (another trivial consequence of the definitions). Then, letting $u = a+b$ and $v = a-b$, we obtain

$$\cos(u) - \cos(v) = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Now we can use part (a) to conclude that

$$\cos(u) = \cos(v) \iff \frac{u+v}{2} = n\pi \text{ or } \frac{u-v}{2} = n\pi$$

which is exactly what we wanted to show. □

(f) A statement analogous to (e) for sin.

Proof. Following the same arguments as part (e), we obtain the identity

$$\sin(a+b) - \sin(a-b) = 2\cos(a)\sin(b)$$

and thus with $u = a+b$ and $v = a-b$, we obtain

$$\sin(u) - \sin(v) = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

and thus $\sin(u) = \sin(v)$ if and only if

$$\frac{u+v}{2} = \frac{\pi}{2} + \pi n \text{ or } \frac{u-v}{2} = n\pi$$

and so the analogous statement for sin is

$$\sin(u) = \sin(v) \iff u+v = \pi + 2\pi n \qquad \text{or} \qquad u - v = 2\pi n$$

□

6. In this exercise, we will construct an inverse function $\arccos : \Omega \to \mathbb{C}$ to the cosine, with the domain $\Omega = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z = 0, |z| \geq 1\}$.

   (a) Show that $z \mapsto e^{iz}$ maps the strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < \pi\}$ bijectively onto the upper half plane $\mathbb{H} = \{\operatorname{Im} z > 0\}$.

Proof. Let $\Sigma = \{ z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi \}$, then we have that $\exp : \Sigma \to \mathbb{H}$ bijectively. To see why this must be so, we note that the imaginary part of z determines it's angle, and only those angles in $(0, \pi)$ get mapped to the upper half plane, and since exp is bijective on $\mathbb{R} \times [0, 2\pi)$ to $\mathbb{C}^\times$, we must have a bijection between Σ and $\mathbb{H}$. Now we note that $i : S \to \Sigma$ bijectively, for if $z = x + iy \in S$, then $x \in (0, \pi)$, and thus $iz = -y + ix$, and so $iz \in \Sigma$, and vice-versa. Thus $\exp \circ i$ takes S to $\mathbb{H}$ bijectively. □

(b) Construct a branch $f \in \mathscr{O}(\Omega)$ of $z \mapsto \sqrt{z^2 - 1}$ satisfying $f(0) = i$. *Hint*: Construct a branch of $\sqrt{z-1}$ in $\mathbb{C} \setminus [1, \infty)$, and a branch of $\sqrt{z+1}$ in $\mathbb{C} \setminus (-\infty, -1]$, by relying on appropriate branches of logarithms.

Proof. Let $\operatorname{br}_\pi \arg$ be the holomorphic branch of the argument function defined on $C - \mathbb{R}^+$, such that $\operatorname{br}_\pi \arg(-1) = \pi$. Similarly let $\operatorname{br}_0 \arg$ be the holomorphic branch of the argument function defined on $C - \mathbb{R}^-$ such that $br_0 \arg(1) = 0$. Then define g by

$$g(z) = \sqrt{|z-1|} \exp\left(\frac{\operatorname{br}_\pi \arg(1-z)}{2} \right) \qquad 0 \text{ if } z = 1 \tag{20}$$

Note that g is defined on $\mathbb{C} - [1, \infty)$, is holomorphic, and satisfies $g(z)^2 = z - 1$, thus $g(z)$ is a holomorphic branch of $\sqrt{z-1}$. Similarly, define h by

$$h(z) = \sqrt{|1+z|} \exp\left(\frac{\operatorname{br}_0 \arg(1+z)}{2} \right) \qquad 0 \text{ if } z = -1$$

Then h is defined on $\mathbb{C} - (-\infty, 1]$, is holomorphic, and satisfies $h(z)^2 = z + 1$, thus $h(z)$ is a holomorphic branch of $\sqrt{z+1}$. Finally $f = g \cdot h$ is a holomorphic branch of $\sqrt{z^2 - 1}$, defined on Ω. □

(c) Show that $z \mapsto \frac{1}{2}(z + z^{-1})$ maps $\mathbb{H}$ bijectively onto Ω.

Proof. First we will prove injectivity. If $z + z^{-1} = u + u^{-1}$, then multiplying both sides by zu, we obtain $(uz - 1)(z - u) = 0$. Since z, u are in the upper half plane, we can conclude that $uz \notin \mathbb{R}$, because that would imply that $\operatorname{Arg}(u) + \operatorname{Arg}(z) = 2\pi$, which contradicts the fact that both $\operatorname{Arg}(u)$ and $\operatorname{Arg}(z)$ are in $(0, \pi)$. Thus $z - u = 0$, and we have injectivity.

It is clear that for any $z \in \mathbb{H}$, $\frac{1}{2}(z + z^{-1}) \in \Omega$, because if $|z| = 1$, then $\left|z^{-1}\right| = 1$ and thus $\left|z + z^{-1}\right| \leq 1$, by the triangle inequality. If $|z| > 1$, then $\left|z^{-1}\right| = \lambda < 1$, and so we can write $z = \lambda \exp(i\operatorname{Arg}(z)) + (1 - \lambda) \exp(i\operatorname{Arg}(z))$, and then $z + z^{-1} = 2\operatorname{Re}\exp(-\operatorname{Arg}(z))\lambda + (1 - \lambda) \exp(i\operatorname{Arg}(z)) \in H$. A similar argument shows that if $|z| < 1$, then $z + z^{-1}$ is in the lower half plane. This proves that $(z + z^{-1})/2 \in H$ for all $z \in \mathbb{H}$.

To prove surjectivity, we wish to solve the equation $z + z^{-1} = 2w$, for any $w \in \Omega$. This is equivalent to solving the polynomial equation $z^2 - 2zw + 1 = (z - w)^2 + (w^2 - 1)$. This has roots

$$z \in w + \sqrt{w^2 - 1} \tag{21}$$

Now if $w \in \mathbb{H}$, then one of the two elements in $\sqrt{w^2-1}$ must also be in $\mathbb{H}$, and thus $z \in \mathbb{H}$. Therefore $\frac{1}{2}(z+z^{-1})$ is surjective onto $\mathbb{H}$, but since $-z^{-1} \in \mathbb{H}$ for $z \in \mathbb{H}$, and $(-z^{-1})+(-z^{-1})^{-1} = -(z+z^{-1})$, we have $\frac{1}{2}(z+z^{-1})$ surjective onto $-\mathbb{H}$ (the lower half plane). Finally, if $w \in (-1,1)$, then $\sqrt{w^2-1}$ contains $\pm\lambda i$ for some nonzero λ, and thus $w+\sqrt{w^2-1}$ contains an element in $\mathbb{H}$. This proves the claim $\square$

(d) Show that cos maps S bijectively onto Ω, with the inverse $\arccos : \Omega \to S$ given by

$$\arccos z = -i \operatorname{Log}(z + \sqrt{z^2-1}),$$

where $\sqrt{z^2-1}$ denotes the branch f constructed in (b).

Proof. Let $\varphi(z) = \frac{1}{2}(z+z^{-1})$, then $\cos = \varphi \circ \exp \circ i$. Since $\exp \circ i : S \to \mathbb{H}$ and $\varphi : H \to \Omega$ are bijections, (with the principal branch $-i \circ \operatorname{Log} : \mathbb{H} \to S$) we can conclude that cos takes S to Ω bijectively. Now recall that we found that the inverse of φ is implicitly defined by the equation

$$z \in \varphi(z) + \sqrt{\varphi(z)^2-1}$$

And so using the branch f for $\sqrt{z^2-1}$ we constructed earlier, we can argue that $\varphi^{-1} = z + f(z)$. Then since $\cos = \varphi \circ \exp \circ i$, we have $\arccos = -i \circ \operatorname{Log} \circ \varphi^{-1}$, or

$$\arccos(z) = -i\operatorname{Log}(z+\sqrt{z^2-1})$$

Which is what we want to show. $\square$