MATH 249 ASSIGNMENT 2

DUE WEDNESDAY MARCH 18

1. Suppose that $f(z) = \sum a_n(z-c)^n$ and $g(z) = \sum b_n(z-c)^n$ both converge in an open disk centered at c, and assume $b_0 \neq 0$. Show that
$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} e_n (z-c)^n, \qquad \text{with} \qquad e_n = \frac{1}{b_0}\left(a_n - \sum_{k=0}^{n-1} b_{n-k} e_k \right),$$
where the power series converges in a disk $D_r(c)$ with some $r > 0$, and the empty sum in the definition of e_n when $n = 0$ is understood to be 0. By using this result, compute a first few terms of the Maclaurin series of $\sec z = \frac{1}{\cos z}$ and $\tan z$.
2. Sketch the following curves.
   (a) The image of $\{z \in \mathbb{C} : \operatorname{Im} z = \operatorname{Re} z + 1\}$ under the mapping $z \mapsto z^2$.
   (b) The image of $\{z \in \mathbb{C} : \operatorname{Im} z = 1\}$ under the mapping $z \mapsto z^3$.
   (c) The image of the circle $\partial D_r = \{z \in \mathbb{C} : |z| = r\}$ under the mapping $z \mapsto \exp z$, for $r = \pi$ and for $r = \frac{3}{2}\pi$.
   (d) The image of $\{z \in \mathbb{C} : \operatorname{Im} z = 1\}$ under the multi-valued mapping $z \mapsto \sqrt[3]{z}$. Identify the part of the curve that corresponds to the principal branch of $z \mapsto \sqrt[3]{z}$.
   (e) The image of $\{z \in \mathbb{C} : \operatorname{Im} z = 1\}$ under the multi-valued mapping $z \mapsto \log z$. Identify the part of the curve that corresponds to the principal branch $z \mapsto \operatorname{Log} z$.
3. Find as many mistakes as you can in the following reasonings.
   (a) $-1 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1)\cdot(-1)} = \sqrt{1} = 1$.
   (b) We have $e^{2\pi i} = 1$, and hence $e^{1+2\pi i} = e$. This means that
   $$e = \left(e^{1+2\pi i}\right)^{1+2\pi i} = e^{(1+2\pi i)(1+2\pi i)} = e^{1-4\pi^2+4\pi i} = e^{1-4\pi^2},$$
   or $e^{-4\pi^2} = 1$.
4. Prove the following.
   (a) For the principal branch of the power function, we have
   $$z^{s+it} = |z|^s e^{-t\operatorname{Arg} z}\left(\cos\left(s\operatorname{Arg} z + t\log|z|\right) + i \sin\left(s\operatorname{Arg} z + t\log|z|\right)\right),$$
   where s and t are real numbers.
   (b) Let $\Omega \subset \mathbb{C}$ be an open set, and let $f \in \mathscr{O}(\Omega)$ be a *holomorphic branch of the n-th root* in the sense that $[f(z)]^n = z$ for $z \in \Omega$ ($n \in \mathbb{N}$). Suppose also that $\log \in \mathscr{O}(\Omega)$ is a branch of logarithm in the set Ω. Then we have $f(z) = \exp(\frac{1}{n}\log z)\exp(\frac{2\pi i k}{n})$ for all $z \in \Omega$ and for some $k \in \{0, 1, \ldots, n-1\}$.
   (c) In the setting of (b), such a function f cannot exist if $n \geq 2$ and if $0 \in \Omega$.

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5. Prove the following.
   (a) $\sin z = 0$ if and only if $z = \pi n$ for some $n \in \mathbb{Z}$.
   (b) $\cos z = 0$ if and only if $z = \frac{\pi}{2} + \pi n$ for some $n \in \mathbb{Z}$.
   (c) The periods of sin are precisely the numbers $2\pi n$, $n \in \mathbb{Z}$.
   (d) The periods of cos are precisely the numbers $2\pi n$, $n \in \mathbb{Z}$.
   (e) $\cos z = \cos w$ if and only if either $z + w = 2\pi n$ for some $n \in \mathbb{Z}$, or $z - w = 2\pi n$ for some $n \in \mathbb{Z}$.
   (f) A statement analogous to (e) for sin.
6. In this exercise, we will construct an inverse function $\arccos : \Omega \to \mathbb{C}$ to the cosine, with the domain $\Omega = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z = 0,\ |z| \geq 1\}$.
   (a) Show that $z \mapsto e^{iz}$ maps the strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < \pi\}$ bijectively onto the upper half plane $\mathbb{H} = \{\operatorname{Im} z > 0\}$.
   (b) Construct a branch $f \in \mathscr{O}(\Omega)$ of $z \mapsto \sqrt{z^2 - 1}$ satisfying $f(0) = i$. *Hint*: Construct a branch of $\sqrt{z-1}$ in $\mathbb{C} \setminus [1, \infty)$, and a branch of $\sqrt{z+1}$ in $\mathbb{C} \setminus (-\infty, -1]$, by relying on appropriate branches of logarithms.
   (c) Show that $z \mapsto \frac{1}{2}(z + z^{-1})$ maps $\mathbb{H}$ bijectively onto Ω.
   (d) Show that cos maps S bijectively onto Ω, with the inverse $\arccos : \Omega \to S$ given by

$$\arccos z = -i \operatorname{Log}(z + \sqrt{z^2 - 1}),$$

   where $\sqrt{z^2 - 1}$ denotes the branch f constructed in (b).