1. *Starting with the complex number axioms, prove the following.*

(a) *If $zw = z$ then $w = 1$.*

Proof. Supposing that $z \neq 0$, then we have $zw = z$ and $z1 = z$ and thus $zw - z1 = z(w-1) = z - z = 0$. Now suppose that $z = a + bi$. If z were zero, then $z(a - bi) = a^2 + b^2 = 0$, and thus $a = b = 0$. Thus one or both of a, b is non zero. Now $z(w-1) = a(w-1) + ib(w-1) = 0$. Multiplying this last equation by $a - bi$, we obtain

$$(a^2 + b^2)(w - 1) = 0 \tag{1}$$

Finally since $a^2 + b^2 \neq 0$, and it is a real number, we can multiply by the inverse to obtain $w - 1 = 0$. □

(b) *If $zw = zu$ and $z \neq 0$ then $w = u$.*

Proof. In the last proof we essentially proved a lemma:

Lemma 1: If $zw = 0$ and $z \neq 0$ then $w = 0$

Using this lemma note that if $zw = zu$ then $z(w - u) = 0$ and since $z \neq 0$ we can conclude that $w - u = 0$ and thus $w = u$ (where we have added the inverse of $-u$ to both sides). □

(c) *$(wz)^{-1} = w^{-1}z^{-1}$ for $w, z \in \mathbb{C} \setminus \{0\}$.*

Proof. Assuming the fact that inverses exist (which is not in the axioms, but can by proved using the fact that there exist reals a, b such that $z = a + bi$), then:

$$(wz)(z^{-1}w^{-1}) = w1w^{-1} = ww^{-1} = 1 \tag{2}$$

Therefore $(wz)^{-1} = z^{-1}w^{-1}$ and by commutativity, $(wz)^{-1} = w^{-1}z^{-1}$. □

(d) *If $wz = 0$ then $w = 0$ or $z = 0$.*

Proof. If $z \neq 0$ then $w = 0$, by Lemma 1. This is logically equivalent to $z = 0$ or $w = 0$. □

2. *Prove the following.*

(a) *$|z| \geq 0$ for any $z \in \mathbb{C}$, and $|z| = 0$ if and only if $z = 0$.*

Proof. If $z = a + bi$ then by definition $|z| = \sqrt{(a+bi)(a-bi)} = \sqrt{a^2 + b^2}$. In an earlier problem we proved that $z = 0$ if and only if $a = b = 0$, and so if $z \neq 0$, $a^2 + b^2 > 0$ and thus $|z| > 0$ (also because we chose the positive branch of the square root). Now $z = 0$ is equivalent to $a^2 + b^2$ by the preceeding arguments, and so is also equivalent to $|z| = 0$ □

(b) *$z^{-1} = \frac{z}{z\bar{z}}$ and $|z^{-1}| = \frac{1}{|z|}$ for $z \neq 0$.*

Proof. I believe there is a slight error in this exercise, as

$$z\frac{\bar{z}}{\bar{z}z} = \frac{z\bar{z}}{z\bar{z}} = 1 \tag{3}$$

and thus $\bar{z}/\bar{z}z$ is the inverse of z. Here we are implicitly using the fact that $\bar{z}z \in \mathbb{R}$, or else we would not be able to discuss division. Now taking the modulus of z^{-1}, we obtain

$$\left|z^{-1}\right|^2 = \frac{1}{|z|^2}\bar{z} \times \frac{1}{|z|^2}\bar{\bar{z}} \tag{4}$$

Using the fact that $\bar{\bar{z}} = z$, we obtain

$$\left|z^{-1}\right|^2 = \frac{1}{|z|^2} \tag{5}$$

Which is easily seen to be equivalent to the desired result. ☐

(c) $||w| - |z|| \leq |w - z|$ *for* $w, z \in \mathbb{C}$.

Proof. Squaring the left hand side, we obtain

$$w\bar{w} + z\bar{z} - 2\,|w|\,|z| \tag{6}$$

Squaring the right hand side, we obtain

$$w\bar{w} + z\bar{z} - (w\bar{z} + z\bar{w}) \tag{7}$$

Taking the difference (7) by (6) we obtain

$$2\,|w|\,|z| - (w\bar{z} + z\bar{w}) \tag{8}$$

Note that $|w| = |\bar{w}|$ and $|\bar{w}|\,|z| = |\bar{w}z|$, and also note that $w\bar{z} = \bar{z\bar{w}}$, and so the equation (8) is equal to

$$2\,|\bar{w}z| - 2\bar{w}z \tag{9}$$

Where we have used that $2\text{Re}x = x + \bar{x}$. Now (9) is always greater than or equal to zero, because the real part is always less than or equal to the modulus (eg if $x = a+bi$ then $|x| = \sqrt{a^2 + b^2}$, while $z = a$). Now this proves that (6) is always greater than (5), which is clearly equivalent to the desired results ($a^2 \geq b^2 \implies a \geq b$). ☐

3. *Prove the following.*

(a) *The unit disk* $\mathbb{D} = D_1(0)$ *is open.*

Proof. It suffices to show that if $x \in D$, then there exists a ball centered on x which is fully contained in D. Let $|x| = r$, then we have $r < 1$, and so $\delta = 1 - r$ is a positive real number. Then $D_\delta(x)$ is fully contained in $D_1(0)$. To see why this is true, let $y \in D_\delta(x)$, then $|y - x| < \delta$, and by the triangle inequality

$$|1 - y| < |1 - x| + |x - y| < r + \delta < 1$$

Thus $D_\delta(x) \subset D_1(0)$. □

(b) *The punctured plane* $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ *is open.*

Proof. Let $\|x\| = r$, then $D_r(x) \subset \mathbb{C}^\times$. To see why this is true, let $y \in D_r(x)$, then

$$|y| > |x| - |y - x| > r - r > 0$$

Thus $y \neq 0$, and so $D_r(x) \subset \mathbb{C}^\times$. □

(c) *The square* $I^2 = \{x + iy \in \mathbb{C} : 0 < x < 1,\, 0 < y < 1\}$ *is open.*

Proof. Let $(a, b) \in I^2$, and let $\delta_1 = a$, $\delta_2 = 1 - a$, $\delta_3 = b$, and $\delta_4 = 1 - b$. It is clear these are all positive numbers less than 1. Let $\delta = \min \delta_1, \delta_2, \delta_3, \delta_4$, then $D_\delta((a, b)) \subset I^2$. To see why this is so, let $(x, y) \in D_\delta(a, b)$, then

$$|(x, y) - (a, b)| < \delta \implies |x - a| < \delta \text{ and } |y - b| < \delta$$

This allows us to conclude that

$$|x - a| < \delta \implies -a < -\delta < x - a < \delta \leq 1 - a$$

Which gives us $0 < x < 1$. An exactly analogous argument allows us to conclude that $0 < y < 1$. □

(d) *The square* $S = \{x + iy \in \mathbb{C} : 0 < x \leq 1,\, 0 < y < 1\}$ *is* not *open.*

Proof. Let $0 < \lambda < 1$, then we have $(1, \lambda) \in S$, and the ball of radius δ centered on $(1, \lambda)$ clearly contains the point $(1 + \delta/2, \lambda) \notin S$, and thus $B_\delta(1, \lambda)$ is not fully contained in S. Since this is true for arbitrary δ, we must conclude that S is not open. □

4. *Let* $\lim z_n = z$ *and* $\lim w_n = w$. *Show that the following hold. (You can assume that the corresponding results for real number sequences are given.)* Since we can assume that the corresponding results are given for real numbers, let us prove a simple lemma

Lemma 2: Let $\{ z_n \}$ be a complex valued sequence, then $\{ z_n \}$ converges in $\mathbb{C}$ if and only if the two real sequences $\{ \mathrm{Im}(z_n) \}$ and $\{ \mathrm{Re}(z_n) \}$ converge. Furthermore, if $\{ z_n \} \to z$ and $\{ \mathrm{Re}(z_n) \} \to a$ and $\{ \mathrm{Im}(z_n) \} \to b$, then $z = a + bi$.

Proof. For brevity, let $z_n = a_n + ib_n$, where $a_n, b_n \in \mathbb{R}$. First suppose that z_n converges to $z = a + ib$, then given any ϵ there exists an N such that $n > N$ implies that

$$|z_n - z| \leq \epsilon \implies |(a_n - a) + (b_n - b)i| \leq \epsilon \tag{10}$$

Now since $|a_n - a + (b_n - b)i| = \sqrt{(a_n - a)^2 + (b_n - b)^2}$, we can conclude from (10) that for $n > N$ we have both

$$|a_n - a| \leq \epsilon \qquad \text{and} \qquad |b_n - b| \leq \epsilon \tag{11}$$

Thus $a_n \to a$ and $b_n \to b$.

Conversely, suppose that $z_n = a_n + ib_n$ and that $a_n \to a$ and $b_n \to b$. Then there exists an N such that $n > N$ implies

$$|a_n - a| \leq \frac{\epsilon}{2} \qquad \text{and} \qquad |b_n - b| \leq \frac{\epsilon}{2} \tag{12}$$

And so for $n > N$

$$|z_n - (a + ib)| = |a_n - a + i(b_n - b)| \leq |a_n - a| + |b_n - b| \leq \epsilon \tag{13}$$

Where we have used the triangle inequality. This proves that $z_n \to (a + ib)$. □

(a) $\lim(w_n \pm z_n) = w \pm z$ *and* $\lim(w_n z_n) = wz$.

Proof. By the lemma, it suffices to prove that the real and imaginary parts of $w_n \pm z_n$ converge to the real and imaginary parts of $w \pm z$. Let $w_n = a_n + ib_n$ and let $z_n = c_n + id_n$. By the fact that $w_n \to w = a + ib$ and $z_n \to z = c + id$, we can conclude that $a_n \to a$, $b_n \to b$, $c_n \to c$ and $d_n \to d$. Now

$$w_n \pm z_n = (a_n \pm c_n) + i(b_n \pm d_n) \tag{14}$$

And since $a_n \pm c_n \to a \pm c$ and $b_n \pm d_n \to b \pm c$, because we assume the corresponding results hold in $\mathbb{R}$, we can conclude that

$$w_n \pm z_n \to (a \pm c) + i(b \pm d) = w \pm z \tag{15}$$

In a similar fashion

$$w_n z_n = a_n c_n - b_n d_n + i(a_n d_n + b_n c_n) \tag{16}$$

And using the fact that that $a_n c_n - b_n d_n \to ac - bd$ and $a_n d_n + b_n c_n \to ad + bc$, because these are both sequences in $\mathbb{R}$, we conclude that

$$w_n z_n \to ac - bd + i(ad + bc) = wz \tag{17}$$

□

(b) $\lim \bar{z}_n = \bar{z}$ *and* $\lim |z_n| = |z|$.

Proof.

$$\bar{z_n} = a_n - b_n i \to a - bi = \bar{z} \tag{18}$$

Where we have used the fact that $z_n \to z$ to conclude that both $a_n \to a$ and $b_n \to b$, and then used the lemma to conclude that $a_n - b_n i \to a - bi$. We will prove that $\lim |z_n| = |z|$ without recourse to real and imaginary parts, to shake things up a bit. Given $z_n \to z$ we can conclude that for $\epsilon > 0$ there exists an N

such that $n > N$ implies

$$|z_n - z| \leq \epsilon$$

Using the "inverse" triangle inequality, we can conclude that for $n > N$

$$||z_n| - |z|| \leq |z_n - z| \leq \epsilon$$

Which proves that $|z_n| \to |z|$. □

(c) *If $z \neq 0$, then $z_n = 0$ for only finitely many indices n, and after the removal of those zero terms from the sequence $\{z_n\}$, we have $\lim \frac{1}{z_n} = \frac{1}{z}$.*

Proof. Let N be large enough so that for $n > N$ $z_n \neq 0$. Suppose that $n > N$, then

$$\frac{1}{z_n} = \frac{a_n - ib_n}{a_n^2 + b_n^2} = \frac{a_n}{a_n^2 + b_n^2} - i\frac{b_n}{a_n^2 + b_n^2} \tag{19}$$

Now it is clear that if $n > N$ then $a_n^2 + b_n^2 > 0$, and so we can use the corresponding result from sequences in $\mathbb{R}$ to conclude that if $c_n \to c$ and $d_n \to d$ then $c_n/d_n \to c/d$, as long as $d \neq 0$, to conclude that

$$\frac{a_n}{a_n^2 + b_n^2} \to \frac{a}{a^2 + b^2} \qquad \text{and} \qquad \frac{b_n}{a_n^2 + b_n^2} \to \frac{b}{a^2 + b^2} \tag{20}$$

Applying the lemma gives us

$$\frac{1}{z_n} \to \frac{a - bi}{a^2 + b^2} = \frac{1}{z} \tag{21}$$

□

5. *Let $\{z_n\}$ be a* Cauchy sequence*, in the sense that*

$$|z_n - z_m| \to 0, \qquad \text{as} \quad \min\{n, m\} \to \infty. \tag{22}$$

Show that there is $z \in \mathbb{C}$, to which $\{z_n\}$ converges. (Assume that the corresponding result for real number sequences is given.)

Proof. Assuming the corresponding result for real number sequences, the proof of this fact becomes completely trivial. It is obvious that if $z_n = a_n + ib_n$ is a cauchy sequence then a_n and b_n are also cauchy sequences. If this is not obvious, it suffices to remark that $|a_n - a_m| \leq |z_n - z_m|$. Assuming that the corresponding result holds for real numbers, we obtain $a_n \to a$ and $b_n \to b$ for some real numbers a and b. Then applying the lemma we proved in the last section, we conclude that $z_n \to a + ib$. □

6. Prove the following

(a) *Let $f(z) = z^n$ where $n \geq 1$ is an integer. Determine if f is complex differentiable, and if it is, compute the derivative. Show also that f is continuous in $\mathbb{C}$.*

Proof. Since $\mathbb{C}$ is a ring, we can use the binomial formula

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i = a^n + na^{n-1}b + b^2 g(z,b)$$

Where $g(z,b)$ is continuous at $b = 0$, and $g(z,0)$ is finite. Applying this to the difference quotient:

$$\frac{(z+h)^n - z^n}{h} = nz^{n-1} + hg(z,h)$$

Now taking the limit $h \to 0$, the last term clearly vanishes, and we conclude that $f(z) = z^n$ is differentiable everywhere and

$$f'(z) = nz^{n-1}$$

Since f is complex differentiable everywhere, it must be continuous everywhere. If f is complex differentiable, then for given any ϵ there exists a δ such that $|h| \leq \delta$ implies

$$|f(z+h) - f(z)| \leq (|f'(z)| + \epsilon)\,|h|$$

And since the right side clearly goes to zero as $h \to 0$, we conclude that $|f(z+h) - f(z)| \to 0$, which is equivalent to f continuous at z. □

(b) *Compute the derivative of* $f(z) = z^{-n} := \frac{1}{z^n}$ *in* $\mathbb{C} \setminus \{0\}$, *where* $n \geq 1$ *is an integer. Show also that* f *is continuous in* $\mathbb{C} \setminus \{0\}$.

Proof. Consider the difference quotient

$$\frac{1/(z+h)^n - 1/(z)^n}{h} = \frac{1}{h}\left(\frac{z^n - (z+h)^n}{(z+h)^n z^n}\right)$$

Applying the binomial formula again, we obtain

$$\frac{1}{h}\left(\frac{z^n - (z+h)^n}{(z+h)^n z^n}\right) = -\frac{nz^{n-1} + g(z,h)h}{(z+h)^n z^n}$$

Now taking the limit as $h \to 0$, it is clear that for $z \neq 0$ we have

$$-\frac{nz^{n-1} + g(z,h)h}{(z+h)^n z^n} \to -n\frac{1}{z^{n+1}}$$

and thus $f(z) = 1/z^n$ is differentiable away from the origin with

$$f'(z) = -\frac{n}{z^{n+1}}$$

As in the last exercise, the complex differentiability of f on $\mathbb{C}/\{0\}$ implies that f is continuous on $\mathbb{C}/\{0\}$. □

(c) *Show that* $f(z) = \text{Re}z$ *is* not *complex differentiable at any point in* $\mathbb{C}$.

Proof. To prove this, note that

$$\frac{\text{Re}(z+h) - \text{Re}(z)}{h} = \frac{\text{Re}h}{h}$$

If we approach $h = 0$ along the line $h = t$, where $t \in \mathbb{R}$, then the limit above is identically equal to 1. If however, we approach $h = 0$ along the line $h = it$, where $t \in \mathbb{R}$, then the limit above is identically 0. Therefore the limit does not exist, and so $\text{Re} : \mathbb{C} \to \mathbb{C}$ is not complex differentiable. □

7. *Let* $\Omega \subset \mathbb{C}$ *be an open set, and let* $f, g : \Omega \to \mathbb{C}$ *be complex differentiable at* $w \in \Omega$.

(a) *Show that* $f \pm g$ *and* fg *are complex differentiable at* w, *with*

$$(f \pm g)'(w) = f'(w) \pm g'(w), \qquad \text{and} \qquad (fg)'(w) = f'(w)g(w) + f(w)g'(w). \tag{23}$$

Proof. We note that if f is differentiable at $z \in \mathbb{C}$, then there exists a number $\lambda \in C$ such that

$$\lim_{h \to 0} \frac{|f(z+h) - f(z) - \lambda h|}{|h|} = 0 \tag{24}$$

Now to prove that $f + g$ is differentiable at w, note that

$$0 \leq \frac{|(f+g)(w+h) - (f+g)(w) - (f'(w) + g'(w))h|}{|h|} \leq \frac{|f(w+h) - f(w) - f'(w)h|}{|h|} + \frac{|g(w+h) - g(w) - g'(w)h|}{|h|} \tag{25}$$

And since this right side clearly tends to 0 as $h \to 0$, we can conclude that $f + g$ is differentiable at w with $(f + g)'(w) = f'(w) + g'(w)$. To prove the second claim, Note that

$$\begin{aligned} &\frac{|f(w+h)g(w+h) - f(w)g(w) - f'(w)g(w)h - f(w)g'(w)h|}{|h|} \\ =&\frac{|f(w+h)g(w+h) - f(w)g(w+h) + f(w)g(w+h) - f(w)g(w) - f'(w)g(w)h - f(w)g'(w)h|}{|h|} \\ \leq&\frac{|f(w+h)g(w+h) - f(w)g(w+h) - f'(w)g(w)h|}{|h|} + \frac{|f(w)|\,|g(w+h) - g(w) - g'(w)h|}{|h|} \end{aligned} \tag{26}$$

Taking the limit as $h \to 0$ makes the second term (the one on the right) vanish, and so it remains to prove that the first term vanishes. We can write the first term as

$$\begin{aligned} &\frac{|f(w+h)g(w+h) - f(w)g(w+h) - f'(w)g(w+h)h + f'(w)g(w+h)h - f'(w)g(w)h|}{|h|} \\ \leq&\frac{|f(w+h) - f(w) - f'(w)h|\,|g(w+h)|}{|h|} + |f'(w)|\,|g(w+h) - g(w)| \end{aligned} \tag{27}$$

Now by the differentiability of f, the first term vanishes when we take the limit $h \to 0$, and by the continuity of g the second term vanishes when we take the limit $h \to 0$. □

(b) *Show that if $g(w) \neq 0$ then $\frac{1}{g}$ complex differentiable at w, with*

$$\left(\frac{1}{g}\right)'(w) = -\frac{g'(w)}{[g(w)]^2}, \qquad \text{and} \qquad \left(\frac{f}{g}\right)'(w) = \frac{f'(w)g(w) - f(w)g'(w)}{[g(w)]^2}. \tag{28}$$

Proof. Probably the easiest way to prove these is to prove the chain rule.

THEOREM 3 (Chain Rule): Let $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be differentiable at z and $f(z)$ respectively. Then $g \circ f : \mathbb{C} \to \mathbb{C}$ is differentiable at z, with $(g \circ f)'(z) = g'(f(z))f'(z)$

Proof. Let's set up the difference:

$$\phi(h) = g(f(z+h)) - g(f(z)) - g'(f(z))f'(z)h \tag{29}$$

And let $f(z) = y$ and let $k = f(z+h) - f(z)$, then

$$\phi(h) = g(y+k) - g(y) - g'(y)k + g'(y)(f(z+h) - f(z)) - g'(y)f'(z)h \tag{30}$$

Then we can write $|\phi(h)| \,/\, |h|$ as

$$\frac{|\phi(h)|}{|h|} \leq \frac{|g(y+k) - g(y) - g'(y)k|}{|k|}\frac{|k|}{|h|} + |g'(y)|\,\frac{|f(z+h) - f(z) - f'(z)h|}{|h|} \tag{31}$$

Now by the differentiability of f at z, this second term vanishes when $h \to 0$, and so it remains to prove that the first term vanishes. We note that by continuity of f, $\lim_{h\to 0} k = 0$, and we also note that by the complex differentiability of f $|k| \,/\, |h| = |k/h| \to |f'(z)|$, and so this factor is bounded. Then the entire first term vanishes because the first factor tends to zero as $k \to 0$ which occurs when $h \to 0$. This finishes the proof. ☐

Now we can simply apply the chain rule. We already showed that if $\gamma(z) = 1/z$, then γ is differentiable on $\mathbb{C}/\{0\}$ with $\gamma'(z) = -1/z^2$, and so the function $h : \mathbb{C} \to \mathbb{C}$ defined by

$$h(z) = \frac{1}{g(z)} = \gamma \circ g(z) \tag{32}$$

is differentiable whenever g is differentiable and $g(z) \neq 0$. It's derivative is given by the chain rule, and is

$$h'(z) = \gamma'(g(z))g'(z) = -\frac{g'(z)}{g(z)^2} \tag{33}$$

Similarly, the function $q(z) = f(z)/g(z)$ is just the function $f(z)h(z)$, where $h(z)$ is defined as above, and so we can use exercise 7(a) to conclude that $q(z)$ is differentiable whenever f and g are, and when g is non-zero. Then it's derivative is given by

$$q'(z) = f'(z)h(z) + f(z)h'(z) = \frac{f'(z)}{g(z)} - \frac{f(z)g'(z)}{g(z^2)} = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2} \tag{34}$$

□

(c) *Consider the rational function*

$$f(z) = \frac{a_0 + a_1 z + \ldots + a_n z^n}{b_0 + b_1 z + \ldots + b_m z^m}, \tag{35}$$

where $a_0, \ldots, a_n$ *and* $b_0, \ldots, b_m$ *are complex numbers, and show that* f *is complex differentiable wherever the denominator is nonzero.*

Proof. Applying the previous problem, we note that $f(z) = n(z)/d(z)$, and so f is differentiable whenever $n(z)$ and $d(z)$ are differentiable and whenever $d(z) \neq 0$. Now we shall prove that $n(z)$ and $d(z)$ are differentiable everywhere. By 7(a), it suffices to prove that each term in the sum is differentiable. An arbitrary term in the sum can be written as λz^k, where $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. Note that if we define $\Lambda : \mathbb{C} \to \mathbb{C}$ by $\Lambda(x) = \lambda x$, then Λ is differentiable with $\Lambda' = \lambda$. This is true because $\lim_{h\to 0}(\lambda(x+h) - \lambda(x))/h$ is clearly λ. Since the function $p(z) = \lambda z^k$ can be written as the composition $p = \Lambda \circ q(z)$, where $q(z) = z^k$, we can conclude by the chain rule that $p(z)$ is differentiable, with derivative $p'(z) = \lambda k z^{k-1}$. Thus each term in $d(z)$ and $n(z)$ is differentiable everywhere, and thus $d(z)$ and $n(z)$ are differentiable everywhere. Whenever $d(z)$ is non-zero, the quotient n/d will also be differentiable, by 7(b). □

8. *Consider the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^x},$$

where x *is a* real *variable. Determine all possible intervals in which the series converges absolutely uniformly. Show that there is a continuous function* f *defined on* $(1, \infty)$, *to which the series converges locally uniformly, but* not *uniformly, in* $(1, \infty)$.

Theorem 4: The series

$$\sum_{n=1}^{\infty} \frac{1}{n^x},$$

converges on $x \in (1, \infty)$.

Proof. Note that we can write the series as $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^x$. First note that the sequence is monotonically increasing, and so it suffices to prove that the sequence of partial sums is upper-bounded, for it will then converge to it's supremum. We will use a trick that works for many sequences whose coefficients are positive numbers that are monotonically decreasing. Let

$$\sum_{n=1}^{\infty} a_n \tag{36}$$

We can suggestively write this out

$$(a_1) + (a_2 + a_3) + (a_4 + \cdots + a_7) + (a_8 + \cdots + a_{15}) + (a_{16} + \cdots + a_{31}) + \cdots \tag{37}$$

Using the fact that the terms are monotonically decreasing, we can conclude that this sum is less than the sum

$$a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \cdots \tag{38}$$

and so if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, then so will $\sum_{n=1}^{\infty} a_n$, because their partial sums are both monotonically increasing sequences. Furthermore, we can slightly modify this argument to prove that (37) is greater than

$$a_1 + 2a_3 + 4a_7 + 8a_{15} + 16a_{31} + \cdots \tag{39}$$

Which in turn is greater than $a_2 + 2a_4 + 4a_8 + \cdots$. But this last series is just

$$\sum_{k=0}^{\infty} 2^k a_{2^{k+1}} = \frac{1}{2}\left(\sum_{k=1}^{\infty} 2^k a_{2^k}\right) \tag{40}$$

Which will diverge if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ diverges, and so if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ diverges, so will $\sum_{n=1}^{\infty} a_n$.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^x \tag{41}$$

We have $\sum_{k=0}^{\infty} 2^k a_{2^k}$ equal to

$$\sum_{k=0}^{\infty} 2^k \left(\frac{1}{2^k}\right)^x = \sum_{k=0}^{\infty} \left(\left(\frac{1}{2}\right)^{x-1}\right)^k \tag{42}$$

Since this is a geometric series, it converges if and only if $x > 1$. Therefore $\sum_{n=1}^{\infty}(1/n)^x$ converges if and only if $x > 1$. This completes the proof. □

Theorem 5: Defining $f_n(x) = \sum_{j=1}^{n}(1/j)^x$, we have the sequence of functions f_n converging locally uniformly in $(1, \infty)$ to a continuous function f defined on $(1, \infty)$.

Proof. By the previous theorem, f_n converges point-wise in $(1, \infty)$ to a function f defined by $f(x) = \sum_{j=1}^{\infty}(1/j)^x$. To prove that f_n converges locally uniformly in $(1, \infty)$, given any point $a \in (1, \infty)$, there exists a $\delta > 0$ such that $a \in (1 + \delta, \infty)$. Let us agree to call $I_\delta = (1 + \delta, \infty)$. Claim: f_n is a cauchy sequence of functions on $(1 + \delta, \infty)$. To prove this, note that

$$|f_n(x) - f_m(x)| < |f_n(1 + \delta) - f_m(1 + \delta)| \tag{43}$$

for all $x \in I_\delta$, and since the sequence defined by $f_n(1 + \delta)$ converges, it is cauchy, and thus given any $\epsilon > 0$ there exists an N such that $n, m > N$ implies that $|f_n(1 + \delta) - f_m(1 + \delta)| < \epsilon$. Then (43) implies that $|f_n(x) - f_m(x)| < \epsilon$ for every $n, m > N$, which is equivalent to f_n being a cauchy convergent sequence of functions. Now if we can prove that f_n is continuous, we will be able to conclude that f is continuous, for the limit of a cauchy convergent sequence of continuous functions is a continuous function. To prove that f_n is continuous at each x, we note that it suffices to prove that α^x is continuous for any $\alpha \geq 1$. To prove this, note that $\alpha^{x+h} - \alpha^x = \alpha^x(\alpha^h - 1)$. Now we know that $\lim_{h \to 0} \alpha^h = 1$, and so we have $\lim_{h \to 0} \alpha^{x+h} - \alpha^x = 0$, proving that α^x is a continuous function of x. This proves that f_n is a continuous function of x, for all x, and thus since $f = \lim_{n\to\infty} f_n$ locally uniformly, we can conclude that f is continuous. $\square$

9. Prove the following

(a) *Let $R > 0$ and $S > 0$ be the convergence radii of the power series $f(z) = \sum a_n(z-c)^n$ and $g(z) = \sum b_n(z-c)^n$, respectively. Show that*

$$f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{j+k=n} a_j b_k \right) (z-c)^n,$$

where the convergence radius of the power series is at least $\min\{R, S\}$.

Proof.

LEMMA 6: let $h(z) = \sum_{n=0}^{\infty} s_n |z-c|^n$. Let R be the radius of convergence of h, defined by

$$R = \sup\{\, r > 0 \mid a_n r^n \leq M \text{ for all } n \text{ for some } M > 0 \,\} \tag{44}$$

Then $h(z)$ absolutely uniformly coverges for each z such that $|z-c| < R$, and conversely, if $h(z)$ converges for some z with $|z-c| = \rho$, then $R \geq \rho$.

Proof. These results were proved in class, and are basic consequences of "Abel's observation", which essentially applies the M-test to power series and geometric series. $\square$

Let $|z-c| = r < \min\{R, S\}$, then both of the series $f(z)$ and $g(z)$ converge absolutely uniformly, by the lemma. Then we can conclude that the double sum

$$f(z)g(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i r^i b_j r^j$$

converges absolutely whether we sum along j and then i, i and then j, or along some rearrangement $\sigma : \mathbb{N}^2 \to \mathbb{N}$. This result was proved in class. Let us pick the rearrangement

$$\{\, (0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \cdots \} \tag{45}$$

This rearrangement order the elements (i, j) by their sum $i + j$. Then we are guaranteed that the series

$$k(z) = \sum_{n=0}^{\infty} \underbrace{\left(\sum_{i+j=n} a_i b_j \right)}_{p_n} r^{i+j} = \sum_{n=0}^{\infty} p_n r^n \tag{46}$$

converges absolutely uniformly. Now by the result in class we can conclude that $\lim k(z) = \lim f(z)g(z)$ (limit of the series), and thus $k(z) = f(z)g(z)$. Since this is true for all $|z-c| \leq \min\{R, S\}$, we can conclude that the radius of convergence of k is at least $\min\{R, S\}$. This proves the result. $\square$

(b) *Derive the Maclaurin series of $\sin^2 z$ and $\cos^2 z$, and in each case, explicitly compute a first few coefficients.*

Proof. We will use a trick here. First note that since $\sin z$ and $\cos z$ have power series representations that converge everywhere (proof, use the ratio test), we can conclude that $\sin^2 z$ and $\cos^2 z$ have power series representations that converge everywhere, by the previous exercise. Now note the standard trigonometric identities

$$2\sin^2 z = 1 - \cos(2z) \qquad 2\cos^2 z = 1 + \cos(2z)$$

We know from basic calculus class that we can write the maclaurin series for $\cos(2z)$ as

$$\cos(2z) = \sum_{i=0}^{\infty} (-1)^n \frac{(2z)^{2n}}{(2n)!}$$

Since we know that $\sin^2 z$ is equal to it's maclaurin series everywhere, we can write

$$\sin^2 z = \sum_{n=0}^{\infty} a_n z^n$$

and thus use our trigonometric identity to conclude that

$$\sum_{n=0}^{\infty} 2a_n z^n = 1 - \sum_{i=0}^{\infty} (-1)^n \frac{(2z)^{2n}}{(2n)!}$$

Now since these power series are equal everywhere in $\mathbb{C}$, we can conclude that the coefficient of each power of z is equal. This gives us a system of equations

$$2a_0 + 1 = 1 \qquad 2a_{2n} + (-1)^n \frac{2^{2n}}{(2n)!} = 0 \qquad 2a_{2n+1} = 0$$

Where $n > 1$. These can be solved immediately to give

$$\sin^2(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n-1}}{(2n)!} z^{2n}$$

Using a similar argument, we obtain

$$\cos^2(z) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n-1}}{(2n)!} z^{2n}$$

We can write out the first few terms:

$$\begin{aligned} \sin^2 z &\approx z^2 - \frac{z^4}{3} + \frac{2z^6}{45} - \frac{z^8}{315} + \frac{2z^{10}}{14175} \\ \cos^2 z &\approx 1 - z^2 + \frac{z^4}{3} - \frac{2z^6}{45} + \frac{z^8}{315} - \frac{2z^{10}}{14175} \end{aligned} \tag{47}$$

□