MATH 249 ASSIGNMENT 1

DUE WEDNESDAY FEBRUARY 18

1. Prove the following.
   (a) If $zw = z$ then $w = 1$.
   (b) If $zw = zu$ and $z \neq 0$ then $w = u$.
   (c) $(wz)^{-1} = w^{-1}z^{-1}$ for $w, z \in \mathbb{C} \setminus \{0\}$.
   (d) If $wz = 0$ then $w = 0$ or $z = 0$.
   You are allowed to use the complex number axioms as stated in the notes, *and* the properties that are proved in the notes.
2. Prove the following.
   (a) $|z| \geq 0$ for any $z \in \mathbb{C}$, and $|z| = 0$ if and only if $z = 0$.
   (b) $z^{-1} = \frac{\bar{z}}{z\bar{z}}$ and $|z^{-1}| = \frac{1}{|z|}$ for $z \neq 0$.
   (c) $||w| - |z|| \leq |w - z|$ for $w, z \in \mathbb{C}$.
3. Prove the following.
   (a) The unit disk $\mathbb{D} = D_1(0)$ is open.
   (b) The punctured plane $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ is open.
   (c) The square $\{x + iy \in \mathbb{C} : 0 < x < 1,\ 0 < y < 1\}$ is open.
   (d) The square $\{x + iy \in \mathbb{C} : 0 < x \leq 1,\ 0 < y < 1\}$ is *not* open.
4. Let $\lim z_n = z$ and $\lim w_n = w$. Show that the following hold. (You can assume that the corresponding results for real number sequences are given.)
   (a) $\lim(w_n \pm z_n) = w \pm z$ and $\lim(w_n z_n) = wz$.
   (b) $\lim \bar{z}_n = \bar{z}$ and $\lim |z_n| = |z|$.
   (c) If $z \neq 0$, then $z_n = 0$ for only finitely many indices n, and after the removal of those zero terms from the sequence $\{z_n\}$, we have $\lim \frac{1}{z_n} = \frac{1}{z}$.
5. Let $\{z_n\}$ be a *Cauchy sequence*, in the sense that
   $$|z_n - z_m| \to 0, \qquad \text{as} \quad \min\{n, m\} \to \infty. \tag{1}$$
   Show that there is $z \in \mathbb{C}$, to which $\{z_n\}$ converges. (Assume that the corresponding result for real number sequences is given.)
6. (a) Let $f(z) = z^n$ where $n \geq 1$ is an integer. Determine if f is complex differentiable, and if it is, compute the derivative. Show also that f is continuous in $\mathbb{C}$.
   (b) Compute the derivative of $f(z) = z^{-n} := \frac{1}{z^n}$ in $\mathbb{C} \setminus \{0\}$, where $n \geq 1$ is an integer. Show also that f is continuous in $\mathbb{C} \setminus \{0\}$.
   (c) Show that $f(z) = \mathrm{Re} z$ is *not* complex differentiable at any point in $\mathbb{C}$.
7. Let $\Omega \subset \mathbb{C}$ be an open set, and let $f, g : \Omega \to \mathbb{C}$ be complex differentiable at $w \in \Omega$.

Date: Winter 2015.

(a) Show that $f \pm g$ and fg are complex differentiable at w, with

$$(f \pm g)'(w) = f'(w) \pm g'(w), \qquad \text{and} \qquad (fg)'(w) = f'(w)g(w) + f(w)g'(w). \tag{2}$$

(b) Show that if $g(w) \neq 0$ then $\frac{1}{g}$ complex differentiable at w, with

$$\Big(\frac{1}{g}\Big)'(w) = -\frac{g'(w)}{[g(w)]^2}, \qquad \text{and} \qquad \Big(\frac{f}{g}\Big)'(w) = \frac{f'(w)g(w) - f(w)g'(w)}{[g(w)]^2}. \tag{3}$$

(c) Consider the rational function

$$f(z) = \frac{a_0 + a_1 z + \ldots + a_n z^n}{b_0 + b_1 z + \ldots + b_m z^m}, \tag{4}$$

where $a_0, \ldots, a_n$ and $b_0, \ldots, b_m$ are complex numbers, and show that f is complex differentiable wherever the denominator is nonzero.

8. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^x},$$

where x is a *real* variable. Determine all possible intervals in which the series converges absolutely uniformly. Show that there is a continuous function f defined on $(1, \infty)$, to which the series converges locally uniformly, but *not* uniformly, in $(1, \infty)$.

9. (a) Let $R > 0$ and $S > 0$ be the convergence radii of the power series $f(z) = \sum a_n (z-c)^n$ and $g(z) = \sum b_n (z-c)^n$, respectively. Show that

$$f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{j+k=n} a_j b_k \right) (z-c)^n,$$

where the convergence radius of the power series is at least $\min\{R, S\}$.

(b) Derive the Maclaurin series of $\sin^2 z$ and $\cos^2 z$, and in each case, explicitly compute a first few coefficients.

Homework policy

You are welcome to consult each other provided (1) you list all people and sources who aided you, or whom you aided and (2) you write-up the solutions independently, in your own language. If you seek help from other people, you should be seeking general advice, not specific solutions, and must disclose this help. This applies especially to internet fora such as `MathStackExchange`.

Similarly, if you consult books and papers outside your notes, you should be looking for better understanding of or different points of view on the material, not solutions to the problems.