VECTOR CALCULUS

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Abstract. Integration and differentiation of vector fields in 2, 3 and 4-dimensions. Duality.

Contents

1. Line integrals

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a continuously differentiable function, representing a curve L in $\mathbb{R}^n$. Suppose that $u : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function. Then $f(t) = u(\gamma(t))$ is a function in $[a, b]$, and by the fundamental theorem of calculus, we have

$$u(\gamma(b)) - u(\gamma(a)) = f(b) - f(a) = \int_a^b f'(t)\mathrm{d}t = \int_a^b Du(\gamma(t))\gamma'(t)\mathrm{d}t. \tag{1}$$

This quantity does not depend on the curve L, let alone the parametrization γ, as long as the endpoints $\gamma(a)$ and $\gamma(b)$ stay fixed. We are going to interpret the integral in the right hand side as the integral of Du over the curve L, and attempt to generalize it to a class of objects broader than the derivatives of scalar functions. We make the following observations.

- The integral must depend not only on the curve L as a subset of $\mathbb{R}^n$, but also on a directionality property of the curve, since switching the endpoints $\gamma(a)$ and $\gamma(b)$ would flip the sign of (1).
- The derivative Du is a row vector at each point $x \in \mathbb{R}^n$. Thus it might be possible to generalize (1) from integration of Du to that of $F : \mathbb{R}^n \to \mathbb{R}^n$.

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Let $F:\mathbb{R}^n\to\mathbb{R}^n$ be smooth, and define

$$\int_\gamma F\cdot d\gamma=\int_a^b F(\gamma(t))\cdot\gamma'(t)\mathrm{d}t. \tag{2}$$

The right hand side makes sense, because for each $t\in[a,b]$, we have $F(\gamma(t))\in\mathbb{R}^n$ and $\gamma'(t)\in\mathbb{R}^n$, and so $F(\gamma(t))\cdot\gamma'(t)\in\mathbb{R}$. Objects such as F are called *vector fields*.

Now let us check if the integral (2) depends on the parametrization γ. Suppose that $\phi:[c,d]\to[a,b]$ is a continuously differentiable function, with $\phi([c,d])=[a,b]$, $\phi(c)=a$ and $\phi(d)=b$. Then by applying the change of variables formula

$$\int_{\phi(c)}^{\phi(d)} f=\int_c^d (f\circ\phi)\phi', \tag{3}$$

to $f(t)=F(\gamma(t))\cdot\gamma'(t)$, we have

$$\int_a^b F(\gamma(t))\cdot\gamma'(t)\mathrm{d}t=\int_c^d F(\gamma(\phi(s)))\cdot\gamma'(\phi(s))\phi'(s)\mathrm{d}s=\int_c^d F(\eta(s))\cdot\eta'(s)\mathrm{d}s, \tag{4}$$

where $\eta(s)=\gamma(\phi(s))$ is the new parametrization. Hence the integral does not depend on the parametrization, as long as the endpoints of the curve are kept fixed. On the other hand, if $\phi(c)=b$ and $\phi(d)=a$, then we have

$$\int_a^b F(\gamma(t))\cdot\gamma'(t)\mathrm{d}t=-\int_b^a F(\gamma(t))\cdot\gamma'(t)\mathrm{d}t=-\int_c^d F(\eta(s))\cdot\eta'(s)\mathrm{d}s, \tag{5}$$

which tells us that if the endpoints of the curve get switched under reparametrization, then the sign of the integral flips. Therefore, the integral (2) depends only on those aspects of the parametrization γ that specify a certain "directionality" property of the underlying curve. This "directionality" property is called orientation.

Remark 1.1. Intuitively, and in practice, an *oriented curve* is a curve given by some concrete parametrization γ, with the understanding that one can freely replace it by any other parametrization $\eta=\gamma\circ\phi$, as long as $\phi'>0$. To define it precisely, we need some preparation. Let $L\subset\mathbb{R}^n$ be a curve, admitting a parametrization $\gamma:[a,b]\to\mathbb{R}^n$ which is a continuously differentiable function with $\gamma'\neq 0$ in (a,b). Suppose that P is the set of all such parametrizations of L. Then for any two parametrizations $\gamma_1:[a_1,b_1]\to\mathbb{R}^n$ and $\gamma_2:[a_2,b_2]\to\mathbb{R}^n$ from P, there exists a continuously differentiable function $\phi:[a_2,b_2]\to[a_1,b_1]$ with $\phi'\neq 0$ in (a_2,b_2), such that $\gamma_2=\gamma_1\circ\phi$. This gives a way to decompose P into two mutually disjoint classes P_1 and P_2: If $\phi'>0$, then γ_1 and γ_2 are in the same class, and if $\phi'<0$, then γ_1 and γ_2 are in different classes. The curve L, together with a choice of P_1 or P_2, is called an *oriented curve*. So the classes P_1 and P_2 are the possible *orientations* of the curve L. As mentioned before, in practice, we specify an orientated curve simply by giving a concrete parametrization.

Example 1.2. Let $\gamma:[0,\pi]\to\mathbb{R}^2$ be given by $\gamma(t)=(\cos t,\sin t)$, and let $\eta:[0,\sqrt{\pi}]\to\mathbb{R}^2$ be given by $\eta(s)=(\cos s^2,\sin s^2)$. Then we have $\eta=\gamma\circ\phi$ with $\phi(s)=s^2$ for $s\in[0,\sqrt{\pi}]$, and since $\phi'>0$ in $(0,\sqrt{\pi})$, these two parametrizations define the same oriented curve. On the other hand, $\xi(\tau)=(-\cos\tau,\sin\tau)$, $\tau\in[0,\pi]$, gives the same curve as γ and η, but ξ is in the orientation opposite to that of γ and η, and thus as an *oriented* curve, ξ is different than γ.

Definition 1.3. Let $\Omega\subset\mathbb{R}^n$ be an open set. Then a *vector field* in Ω is a continuously differentiable function $F:\Omega\to\mathbb{R}^n$.

For convenience, we restate the definition of integration of vector fields over oriented curves.

Definition 1.4. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let F be a vector field in Ω. Let $\gamma : [a, b] \to \Omega$ be an oriented curve. Then we define the *line integral* of F over γ as

$$\int_\gamma F \cdot d\gamma = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t. \tag{6}$$

If γ is a closed curve, that is, if $\gamma(a) = \gamma(b)$, then this integral is also called the *circulation* of F over γ.

We have already established that the integral in the right hand side does not depend on the parametrization γ, as long as the orientation is kept fixed.

Remark 1.5. The following notations are sometimes used for circulations:

$$\oint_\gamma F \cdot d\gamma, \qquad \ointctrclockwise_\gamma F \cdot d\gamma, \qquad \ointclockwise_\gamma F \cdot d\gamma. \tag{7}$$

Note that the second and third notations have arrows specifying orientations, and that they make sense only in two dimensions.

Example 1.6. (a) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $F(x, y) = (-y, x)$, and let $\gamma(t) = (\cos t, \sin t)$, $t \in [0, \pi]$. Then we have

$$\int_\gamma F \cdot d\gamma = \int_0^\pi \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \mathrm{d}t = \int_0^\pi \mathrm{d}t = \pi. \tag{8}$$

(b) Let F be as before, and let η be the unit circle, oriented *counter-clockwise*, and let γ be the unit circle, with *no orientation* specified. Then we have

$$\int_\eta F \cdot d\eta = \oint_\eta F \cdot d\eta = \ointctrclockwise_\gamma F \cdot d\gamma = \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \mathrm{d}t = 2\pi. \tag{9}$$

(c) Let ξ be the unit circle, oriented *clockwise*, and let γ be as before. Then we have

$$\int_\xi F \cdot d\xi = \oint_\xi F \cdot d\xi = \ointclockwise_\gamma F \cdot d\gamma = -\ointctrclockwise_\gamma F \cdot d\gamma = -2\pi. \tag{10}$$

(d) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $F(x, y, z) = (-y, x - z, z)$, and let $\gamma(t) = (\cos t, \sin t, \sin t)$, $t \in [0, 2\pi]$. Then we have

$$\int_\gamma F \cdot d\gamma = \oint_\gamma F \cdot d\gamma = \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t - \sin t \\ \sin t \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} -\sin t \\ \cos t \\ \cos t \end{pmatrix} \mathrm{d}t = 2\pi. \tag{11}$$

Remark 1.7. Another notation for the line integral (6) is

$$\int_\gamma F_1(x)dx_1 + \ldots + F_n(x)dx_n = \int_\gamma F \cdot d\gamma. \tag{12}$$

For example, with $\gamma(t) = (t, 2t)$, $t \in [0, 1]$, we have

$$\int_\gamma x\mathrm{d}x + (x + y)\mathrm{d}y = \int_0^1 (t + (t + 2t) \cdot 2)\mathrm{d}t = \int_0^1 7t\,\mathrm{d}t = \frac{7}{2}. \tag{13}$$

2. Conservative fields: Scalar potential

A convenient way to generate vector fields is to differentiate scalar functions. That is, if $\Omega \subset \mathbb{R}^n$ is an open set, and $\phi : \Omega \to \mathbb{R}$ is a smooth function, then

$$\nabla\phi(x) = (\partial_1\phi(x), \ldots, \partial_n\phi(x)), \tag{14}$$

is a vector field in Ω. This is called the *gradient* of ϕ. For example, if $\phi(x, y) = x^2 + \sin y$, then $\nabla\phi(x, y) = (2x, \cos y) \in \mathbb{R}^2$ for $(x, y) \in \mathbb{R}^2$. The notation

$$\text{grad}\phi = \nabla\phi, \tag{15}$$

is also used for the gradient.

Definition 2.1. If $F = \nabla\phi$ in $\Omega \subset \mathbb{R}^n$ for some scalar function ϕ, then we say that F is a *conservative vector field* in Ω, and that ϕ is a *scalar potential* of F in Ω.

Let $F = \nabla\phi$ and let $\gamma : [a, b] \to \Omega$ be an oriented curve. Then we have

$$\frac{d\phi(\gamma(t))}{dt} = \nabla\phi(\gamma(t)) \cdot \gamma'(t) = F(\gamma(t)) \cdot \gamma'(t), \tag{16}$$

and so

$$\int_\gamma F \cdot d\gamma = \int_a^b \frac{d\phi(\gamma(t))}{dt} = \phi(\gamma(b)) - \phi(\gamma(a)), \tag{17}$$

leading to the following result.

Proposition 2.2. *The line integral of a conservative field over a curve depends only on the endpoints of the curve. In particular, the circulations of a conservative field vanish.*

Example 2.3. (a) The vector field $F(x, y) = (2x, 4y)$ is conservative in $\mathbb{R}^2$, since $F = \nabla\phi$ with $\phi(x, y) = x^2 + 2y^2$. So for any curve γ connecting $P = (0, 0)$ to $Q = (2, 1)$, we have

$$\int_\gamma F \cdot d\gamma = \phi(Q) - \phi(P) = 6. \tag{18}$$

(b) Let $F(x, y) = (y - 2x, 2y + x)$, and let us try to decide if F is conservative. First, observe that if $F = \nabla\phi$, then

$$\int_\gamma F \cdot d\gamma = \phi(x, y) - \phi(0, 0), \tag{19}$$

where γ is any curve connecting the origin $(0, 0)$ to the point (x, y). We turn this around, and *construct* a candidate potential ϕ such that $\phi(0, 0) = 0$ by the formula

$$\phi(x, y) = \int_\gamma F \cdot d\gamma. \tag{20}$$

We take the straight line segment $\gamma(t) = (xt, yt)$, $t \in [0, 1]$. Thus

$$\phi(x, y) = \int_0^1 \begin{pmatrix} yt - 2xt \\ 2yt + xt \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} x \\ y \end{pmatrix} \mathrm{d}t = \int_0^1 (-2x^2 + 2xy + 2y^2)t dt = y^2 - x^2 + xy. \tag{21}$$

Finally, the computation

$$\nabla\phi(x, y) = (y - 2x, x + 2y), \tag{22}$$

shows that ϕ is indeed a potential for F. Note that such a procedure is guaranteed to yield a proper potential only if F is conservative to begin with.

(c) Let $F(x,y) = (xy, 0)$, and let γ be the oriented square given by the schematic $(0,0) \to (0,1) \to (1,1) \to (1,0) \to (0,0)$. The integral over the first side $(0,0) \to (0,1)$ vanishes, because $x = 0$ there. Similarly, the integral over the fourth side $(1,0) \to (0,0)$ vanishes. Moreover, the third side $(1,1) \to (1,0)$ is orthogonal to F, so the line integral vanishes. Thus, the circulation over γ is equal to the integral over the second side $(0,1) \to (1,1)$:

$$\int_\gamma F \cdot d\gamma = \int_0^1 x dx = \frac{1}{2}. \tag{23}$$

As the circulation is nonzero, F is definitely *not* a conservative field in any region that contains the square γ.

In fact, the following result is true.

Proposition 2.4. *A vector field F is conservative in an open set $\Omega \subset \mathbb{R}^n$ if and only if the circulation of F over any closed curve contained in Ω is zero.*

Proof. Without loss of generality, let us assume that Ω is path-connected, i.e., that any pair of points in Ω can be joined by a piecewise smooth path (If not, we simply consider path-connected pieces of Ω one by one). Fix a point $P \in \Omega$, and for $Q \in \Omega$, define

$$\phi(Q) = \int_\gamma F \cdot d\gamma, \tag{24}$$

where γ is any curve connecting P to Q. The value $\phi(Q)$ does not depend on γ, because the circulation of F over any closed curve vanishes. We claim that ϕ is a potential for F.

For simplicity, let us work in 2 dimensions. Suppose that $Q = (x,y) \in \Omega$ is given, and let $D \subset \Omega$ be an open disk centred at Q. Pick a point $(a,b) \in D$ such that $a < x$ and $b < y$. Then we have

$$\phi(x,y) = \phi(a,b) + \int_a^x F_x(s,b)ds + \int_b^y F_y(x,t)dt, \tag{25}$$

which, upon differentiating with respect to y, yields

$$\partial_y \phi(x,y) = F_y(x,y). \tag{26}$$

Similarly, we have

$$\phi(x,y) = \phi(a,b) + \int_b^y F_y(x,s)ds + \int_a^x F_x(t,y)dt, \tag{27}$$

giving $\partial_x \phi(x,y) = F_x(x,y)$. The proof is complete. □

3. Irrotational fields: Poincaré's lemma

Given a vector field F, how do we check if it is conservative? We start in 2 dimensions. Suppose that F is conservative, that is, let $F = (F_x, F_y) = (\partial_x \phi, \partial_y \phi)$, where F_x and F_y are the components of F. Then we have

$$\partial_y F_x = \partial_y \partial_x \phi = \partial_x \partial_y \phi = \partial_x F_y. \tag{28}$$

So if F is conservative, then $\partial_x F_y - \partial_y F_x = 0$.

Definition 3.1. In 2 dimensions, the *curl* of F is defined to be

$$\mathrm{curl} F = \partial_x F_y - \partial_y F_x. \tag{29}$$

The following alternative notations are also used:

$$\nabla \times F = \nabla \wedge F = \mathrm{rot} F = \mathrm{curl} F. \tag{30}$$

Definition 3.2. If $\mathrm{curl} F = 0$ in Ω, then the vector field F is called *irrotational*, *longitudinal* or *curl-free in* Ω.

We may summarize the preceding discussion as follows.

Proposition 3.3. *Conservative fields are irrotational.*

In the converse direction, we have the following.

Theorem 3.4 (Poincaré's lemma)**.** *If F is irrotational in Ω, where $\Omega \subset \mathbb{R}^2$ is open and convex, then F is conservative in Ω.*

A proof will be given in the next section.

Example 3.5. Let $F(x,y) = (y-x, y+x)$. It is irrotational in $\mathbb{R}^2$, since

$$\mathrm{curl} F(x,y) = \partial_x(y+x) - \partial_y(y-x) = 0, \tag{31}$$

and hence conservative in $\mathbb{R}^2$ by Poincaré's lemma. We construct a scalar potential for F by the formula

$$\phi(x,y) = \int_\gamma F \cdot d\gamma. \tag{32}$$

Since F is conservative, γ can be any path connecting $(0,0)$ to (x,y). We take the straight line segment $\gamma(t) = (xt, yt)$, $t \in [0,1]$. Thus

$$\phi(x,y) = \int_0^1 \begin{pmatrix} yt - xt \\ yt + xt \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} x \\ y \end{pmatrix} \mathrm{d}t = \int_0^1 (-x^2 + 2xy + y^2) t dt = xy + \frac{y^2 - x^2}{2}. \tag{33}$$

We can verify that

$$\nabla\phi(x,y) = (y-x, x+y). \tag{34}$$

Example 3.6. Is the condition of convexity in Poincaré's lemma important? As the first example below shows, a non-conservative vector field can be curl-free in a general domain, demonstrating that *some* assumption on the domain is necessary.

(a) The vector field

$$F(x,y) = \big(- \frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \big). \tag{35}$$

is curl-free in $\mathbb{R}^2 \setminus \{(0,0)\}$, since

$$\mathrm{curl} F(x,y) = \frac{x^2 + y^2 - x \cdot 2x}{(x^2+y^2)^2} + \frac{x^2 + y^2 - y \cdot 2y}{(x^2+y^2)^2} = 0. \tag{36}$$

However, its circulation over the circle $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$, is nonzero:

$$\oint_\gamma F \cdot d\gamma = \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \mathrm{d}t = 2\pi. \tag{37}$$

Poincaré's lemma is consistent with this, since $\mathbb{R}^2 \setminus \{(0,0)\}$ is not a convex set.

(b) However, the existence of such peculiar vector fields does not mean that the domain is not able to support conservative vector fields. For example, the vector field

$$F(x,y) = \big(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \big), \tag{38}$$

is conservative in $\mathbb{R}^2 \setminus \{(0,0)\}$, as

$$\phi(x,y) = \ln \sqrt{x^2 + y^2}, \tag{39}$$

is a potential for F in $\mathbb{R}^2 \setminus \{(0,0)\}$.

4. Green's theorem

In the preceding section, we have derived a criterion to tell if a vector field is conservative in terms of whether or not the quantity $\mathrm{curl}F = \partial_x F_y - \partial_y F_x$ is identically zero. Hence this quantity measures the failure of a vector field to be conservative.

Remark 4.1. Let $F = (F_x, F_y)$ be a 2-dimensional vector field, and let Q be the square region with the corners at (x, y), $(x+h, y)$, $(x+h, y+h)$, and $(x, y+h)$, where $h > 0$ is small. Denote by γ the boundary of Q with the counter-clockwise orientation. We approximate the line integrals over the sides as

$$\begin{aligned} &\int_{(x,y)}^{(x+h,y)} F \cdot d\gamma \approx F_x(x,y)h, \qquad \int_{(x+h,y)}^{(x+h,y+h)} F \cdot d\gamma \approx F_y(x+h,y)h, \\ &\int_{(x+h,y+h)}^{(x,y+h)} F \cdot d\gamma \approx -F_x(x,y+h)h, \qquad \int_{(x,y+h)}^{(x,y)} F \cdot d\gamma \approx -F_y(x,y)h, \end{aligned} \tag{40}$$

which yields

$$\begin{aligned} \frac{1}{h^2}\oint_\gamma F \cdot d\gamma &\approx \frac{F_y(x+h,y) - F_y(x,y)}{h} - \frac{F_x(x,y+h) - F_x(x,y)}{h} \\ &\approx \partial_x F_y(x,y) - \partial_y F_x(x,y) = \mathrm{curl}F(x,y). \end{aligned} \tag{41}$$

Since the approximation is exact as $h \to 0$, we get

$$\mathrm{curl}F(x,y) = \lim_{h\to 0} \frac{1}{h^2} \oint_\gamma F \cdot d\gamma. \tag{42}$$

Thus we could say that the curl is "circulation per unit area."

As the curl is "circulation per unit area," can we expect that integration of curl would give the total circulation? Green's theorem answers it in the affirmative.

Theorem 4.2 (Green's theorem)**.** *Let γ be a closed oriented curve, and let K be a region bounded by γ, such that K is on the left side of γ. Then we have*

$$\oint_\gamma F \cdot d\gamma = \int_K \mathrm{curl}F, \tag{43}$$

for any smooth vector field F in K.

Proof for polygons. Any polygon can be subdivided into triangles, and any triangle can be thought of as the sum or the difference of two right-angled triangles. Thus it suffices to prove the theorem for right-angled triangles. Consider the triangle T with vertices at $(0, 0)$, $(a, 0)$, and $(0, b)$. We compute

$$\int_T \partial_y F_x = \int_0^a \int_0^{g(x)} \partial_y F_x(x,y)dydx = \int_0^a F_x(x, g(x))dx - \int_0^a F_x(x,0)dx, \tag{44}$$

where $g(x) = b(1 - x/a)$, and

$$\int_T \partial_x F_y = \int_0^b \int_0^{h(y)} \partial_x F_y(x,y)dxdy = \int_0^b F_y(h(y),y)dy - \int_0^b F_y(0,y)dy, \tag{45}$$

where $h(y) = a(1 - y/b)$. Since $h(g(x)) = x$, under the substitution $y = g(x)$, we have

$$\int_0^b F_y(h(y),y)dy = \int_0^a F_y(x, g(x)) \cdot \frac{b}{a} \cdot dx \tag{46}$$

and hence

$$\begin{aligned}\int_T \operatorname{curl} F &= \int_0^a F_x(x,0)dx - \int_0^b F_y(0,y)dy + \int_0^a [F_y(x,g(x)) \cdot \frac{b}{a} - F_x(x,g(x))]dx \\ &= \oint_\gamma F \cdot d\gamma,\end{aligned} \tag{47}$$

where γ is the boundary of T. This establishes the proof. □

Remark 4.3. In the components of the vector field, Green's theorem is written as

$$\oint_\gamma Adx + Bdy = \int_K (\partial_x B - \partial_y A)dxdy. \tag{48}$$

Example 4.4. Let $F(x,y) = (0,x)$, so that $\operatorname{curl} F(x,y) = 1$. An application of Green's theorem yields a useful formula for computing areas:

$$|K| = \int_K \operatorname{curl} F = \oint_\gamma F \cdot d\gamma = \oint_\gamma xdy. \tag{49}$$

As an example, consider Gerono's lemniscate

$$\begin{cases} x = \cos t \\ y = \sin t \cos t \end{cases} \tag{50}$$

and take γ to be the right half of it, given by the parameter range $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. Then the area of the region enclosed by γ can be computed as

$$|K| = \oint_\gamma xdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cdot (\cos^2 t - \sin^2 t)dt = \frac{2}{3}. \tag{51}$$

Example 4.5. Let us compute the integral

$$\oint_C (2y + \sqrt{x^2+9})dx + (5x + e^{2\arctan y})dy, \tag{52}$$

where C is the circle of radius 2 centred at the origin. The curl of the vector field under integration is $\operatorname{curl} F = 3$, and hence

$$\oint_C (2y + \sqrt{x^2+9})dx + (5x + e^{2\arctan y})dy = \int_D 3dxdy = 12\pi, \tag{53}$$

where D is the disk of radius 2 centred at the origin.

We shall now prove Poincaré's lemma (Theorem 3.4) for scalar potentials. For convenience, we state it as a corollary to Green's theorem.

Corollary 4.6 (Poincaré's lemma)**.** *If F is irrotational in Ω, where $\Omega \subset \mathbb{R}^2$ is open and convex, then F is conservative in Ω.*

Proof. In view of Proposition 2.4, it suffices to prove that the circulation of F over any closed curve γ in Ω vanishes. Since any closed curve can be decomposed into non-self-intersecting loops, without loss of generality, we can assume that γ is non-self-intersecting. Then γ encloses a bounded region (this is a nontrivial point, requiring at least a smooth version of the Jordan curve theorem), and this region is contained in Ω due to convexity. By Green's theorem, the circulation over γ is equal to the integral of $\operatorname{curl} F = 0$ over the region. □

Remark 4.7. In fact, Poincaré's lemma is true for *simply connected* regions, that is, when any closed curve in Ω can be continuously deformed into a point. Intuitively, a simply connected region is a region without holes. This includes many non-convex sets, such as a crescent.

5. Curl in 3D

In 2 dimensions, a conservative vector field F must satisfy $\partial_x F_y = \partial_y F_x$, because its potential satisfies $\partial_x \partial_y \phi = \partial_y \partial_x \phi$. In other words, the criterion is based on the fact that the Hessian of a scalar function (if exists) is symmetric.

In 3 dimensions, if $F = (F_x, F_y, F_z) = (\partial_x \phi, \partial_y \phi, \partial_z \phi)$, then

$$DF = \begin{pmatrix} \partial_x F_x & \partial_y F_x & \partial_z F_x \\ \partial_x F_y & \partial_y F_y & \partial_z F_y \\ \partial_x F_z & \partial_y F_z & \partial_z F_z \end{pmatrix} = \begin{pmatrix} \partial_x \partial_x \phi & \partial_y \partial_x \phi & \partial_z \partial_x \phi \\ \partial_x \partial_x \phi & \partial_y \partial_x \phi & \partial_z \partial_x \phi \\ \partial_x \partial_z \phi & \partial_y \partial_z \phi & \partial_z \partial_z \phi \end{pmatrix} = D^2 \phi, \tag{54}$$

and the symmetricity of the Hessian implies that

$$\partial_x F_y = \partial_y F_x, \qquad \partial_x F_z = \partial_z F_x, \qquad \partial_y F_z = \partial_z F_y. \tag{55}$$

Another way to think about this is that in 3D, we have 3 coordinate planes: yz, zx, and xy, and the 2D curls in each coordinate plane must vanish for conservative vector fields.

Definition 5.1. In 3 dimensions, the *curl* of F is defined to be

$$\text{curl} F = (\partial_y F_z - \partial_z F_y, \partial_z F_x - \partial_x F_z, \partial_x F_y - \partial_y F_x). \tag{56}$$

The following alternative notations are also used:

$$\nabla \times F = \nabla \wedge F = \text{rot} F = \text{curl} F. \tag{57}$$

Example 5.2. For $F(x, y, z) = (3z, 5x + z^2, 7y)$, we have

$$\text{curl} F(x, y, z) = (7 - 2z, 3, 5). \tag{58}$$

Definition 5.3. If $\text{curl} F = 0$ in Ω, then the vector field F is called *irrotational*, *longitudinal* or *curl-free in* Ω.

We may summarize the preceding discussion as follows.

Proposition 5.4. *Conservative fields are irrotational.*

In the converse direction, we have the following.

Theorem 5.5 (Poincaré's lemma). *If F is irrotational in Ω, where $\Omega \subset \mathbb{R}^3$ is open and convex, then F is conservative in Ω.*

The proof needs to wait until we have a generalization of Green's theorem to 3 dimensions.

Example 5.6. (a) Simple examples of conservative fields are given by fields of the form

$$F(x, y, z) = (a(x), b(y), c(z)), \tag{59}$$

where a, b and c are scalar functions of a single variable. If A, B and C are primitives of a, b and c, respectively, then it is obvious that F is the gradient of $A(x) + B(y) + C(z)$.

(b) Another class of conservative fields is provided by radial vector fields. Let

$$F(x, y, z) = g(r)(x, y, z), \tag{60}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and $g(r)$ is a scalar function of r. Noting that

$$\partial_x r = \frac{x}{r}, \qquad \partial_y r = \frac{y}{r}, \qquad \partial_z r = \frac{z}{r}, \tag{61}$$

we have

$$\partial_y F_z - \partial_z F_y = z g'(r) \cdot \frac{y}{r} - y g'(r) \cdot \frac{z}{r} = 0, \quad \partial_z F_x - \partial_x F_z = 0, \quad \partial_x F_y - \partial_y F_x = 0, \tag{62}$$

that is, $\mathrm{curl} F = 0$. Let us look for a scalar potential in the form $\phi(x, y, z) = G(r)$. Its gradient is easily computed to be

$$\nabla\phi = \frac{G'(r)}{r}(x, y, z), \tag{63}$$

and thus $\nabla\phi = F$ is equivalent to

$$\frac{G'(r)}{r} = g(r). \tag{64}$$

For instance, when $g(r) = r$, we may solve it as

$$G(r) = \int_0^r g(r) r dr = \frac{r^3}{3}, \tag{65}$$

and when $g(r) = r^{-3}$, we may solve it as

$$G(r) = -\int_r^\infty g(r) r dr = -\frac{1}{r}. \tag{66}$$

6. Surface integrals and fluxes

A proper extension of Green's theorem to 3D should turn the circulation of F over a closed curve γ into some sort of integral of $\mathrm{curl} F$ over a surface bounded by γ. The correct notion of surface integrals in this context turns out to be the so-called flux.

As a local model of a parametrized surface, consider the map $\Phi(u, v) = va + ub$, where $u, v \in \mathbb{R}$ are the parameters, and $a, b \in \mathbb{R}^3$ are fixed vectors. We have $a = \partial_u \Phi$ and $b = \partial_v \Phi$. Let $Q = [0, 1]^2$ be the unit square in the parameter space. Then the area of its image $\Phi(Q)$ is equal to

$$|\Phi(Q)| = |\det(a, b, n)|, \tag{67}$$

where $n \in \mathbb{R}^3$ is a unit vector, orthogonal to both a and b. Note that the right hand side is the volume of the parallelepiped generated by the three vectors a, b, and n. Since this volume is maximized when n is orthogonal to the plane of a and b, we can also write it as

$$|\Phi(Q)| = |\det(a, b, n)| = \max_{|x|=1} \det(a, b, x) = \max_{|x|=1} \begin{vmatrix} a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \\ a_3 & b_3 & x_3 \end{vmatrix}, \tag{68}$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the norm of $x \in \mathbb{R}^3$. Expand the determinant by its last column:

$$\begin{vmatrix} a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \\ a_3 & b_3 & x_3 \end{vmatrix} = x_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + x_2 \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} + x_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \tag{69}$$

and introduce the vector (called the *cross product* between a and b)

$$c = a \times b := \left(\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right), \tag{70}$$

to arrive at

$$|\Phi(Q)| = |\det(a, b, n)| = \max_{|x|=1} \det(a, b, x) = \max_{|x|=1} c \cdot x. \tag{71}$$

From here it is obvious that the maximum is attained at

$$x^* = \frac{c}{|c|} = \frac{a \times b}{|a \times b|}, \tag{72}$$

with the value

$$|\Phi(Q)| = |\det(a, b, n)| = \det(a, b, x^*) = c \cdot x^* = |c| = |a \times b|. \tag{73}$$

Moreover, the two (unit) normal vectors are given by

$$n = \pm x^* = \pm \frac{a \times b}{|a \times b|}. \tag{74}$$

Remark 6.1. Consider a general parametrized surface $\Phi : K \to \mathbb{R}^3$, where $K \subset \mathbb{R}^2$ is the parameter domain. Then the *unit normal vectors* are given by

$$n = \pm \frac{\partial_u \Phi \times \partial_v \Phi}{|\partial_u \Phi \times \partial_v \Phi|}, \tag{75}$$

and the area of a small square $q \subset K$ transforms as

$$|\Phi(q)| \approx |\partial_u \Phi \times \partial_v \Phi| \cdot |q|, \tag{76}$$

with the approximation being exact in the limit $|q| \to 0$.

Thus the following definition is natural.

Definition 6.2. The *integral of a scalar function* f over the parametrized surface $\Phi(K)$ is

$$\int_{\Phi(K)} f = \int_K (f \circ \Phi)|\partial_u \Phi \times \partial_v \Phi|, \tag{77}$$

and the *surface area* of $\Phi(K)$ is

$$|\Phi(K)| = \int_K |\partial_u \Phi \times \partial_v \Phi|. \tag{78}$$

Example 6.3. Let us compute the surface area of the torus

$$\begin{cases} x = (a + b\cos\theta)\cos\phi \\ y = (a + b\cos\theta)\sin\phi \\ z = b\sin\theta \end{cases} \tag{79}$$

where $0 \le \phi < 2\pi$, $0 \le \theta < 2\pi$, and $a > b > 0$ are constants. Denote by Φ the map given by (79), and compute

$$D\Phi(\phi, \theta) = \begin{bmatrix} -(a + b\cos\theta)\sin\phi & -b\sin\theta\cos\phi \\ (a + b\cos\theta)\cos\phi & -b\sin\theta\sin\phi \\ 0 & b\cos\theta \end{bmatrix}, \tag{80}$$

and hence

$$\partial_\phi \Phi \times \partial_\theta \Phi = \begin{bmatrix} b(a + b\cos\theta)\cos\phi\cos\theta \\ b(a + b\cos\theta)\sin\phi\cos\theta \\ b(a + b\cos\theta)\sin\theta \end{bmatrix}. \tag{81}$$

Thus the area of the torus is

$$A = \int_0^{2\pi} \int_0^{2\pi} |\partial_\phi \Phi \times \partial_\theta \Phi| d\phi d\theta = \int_0^{2\pi} \int_0^{2\pi} b(a + b\cos\theta) d\phi d\theta = 4\pi^2 ab. \tag{82}$$

Finally, we can discuss integration of vector fields over surfaces.

Definition 6.4. Let S be a smooth surface, and let n be a unit normal to S. Then the *flux of a vector field* F across S (in the direction of n) is

$$\int_S F \cdot dS = \int_S F \cdot n, \tag{83}$$

where the right hand side is to be understood as the integral of the scalar function $F \cdot n$.

Remark 6.5. Suppose that S is parametrized as $S = \Phi(K)$. Then by swapping $u \leftrightarrow v$ if necessary, we can assume

$$n = \frac{\partial_u \Phi \times \partial_v \Phi}{|\partial_u \Phi \times \partial_v \Phi|}, \tag{84}$$

in which case the flux can be written as

$$\int_S F \cdot dS = \int_K (F \circ \Phi) \cdot (\partial_u \Phi \times \partial_v \Phi) du dv = \int_K \det(\partial_u \Phi, \partial_v \Phi, F \circ \Phi) du dv. \tag{85}$$

Example 6.6. Let $S = \{(x, y, x^2 + y^2) : x^2 + y^2 \leq 4\}$, with n chosen as the unit normal pointing downward (i.e., with its z-component negative). We are to compute the flux of the vector field $F(x, y, z) = (x, y, 2)$ across S in the direction n. First off, introduce the parametrization

$$\Phi(u, v) = (u, v, u^2 + v^2), \tag{86}$$

with the domain $K = \{(u, v) : u^2 + v^2 \leq 4\}$. Then we have

$$D\Phi(u, v) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & 2v \end{bmatrix}, \qquad \partial_u \Phi \times \partial_v \Phi = \begin{bmatrix} -2u \\ -2v \\ 1 \end{bmatrix}. \tag{87}$$

Since the z-component of $\partial_u \Phi \times \partial_v \Phi$ is positive, we have n pointing in the *opposite* direction of $\partial_u \Phi \times \partial_v \Phi$. In other words, we have

$$\int_S F \cdot dS = -\int_K (F \circ \Phi) \cdot (\partial_u \Phi \times \partial_v \Phi) du dv = -\int_K \begin{bmatrix} u \\ v \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2u \\ -2v \\ 1 \end{bmatrix} du dv = 8\pi. \tag{88}$$

Exercise 6.7. Show that

$$\nabla \times (\phi X) = \phi \nabla \times X + (\nabla \phi) \times X, \tag{89}$$

for any scalar field ϕ and any 3-dimensional vector field X.

7. The Kelvin-Stokes theorem

Recall Green's theorem

$$\oint_\gamma F \cdot d\gamma = \int_S (\partial_x F_y - \partial_y F_x), \tag{90}$$

where F is a 2-dimensional vector field, and γ is the boundary of a 2-dimensional region S. Let us promote F to 3-dimensions by

$$F = (F_x, F_y, 0), \tag{91}$$

that is, we extend F from the xy-plane to all of $\mathbb{R}^3$ such that $\partial_z F_x = \partial_z F_y = 0$ and $F_z = 0$. Then its (3-dimensional) curl is

$$\text{curl} F = (0, 0, \partial_x F_y - \partial_y F_x). \tag{92}$$

Now we think of S as a surface lying in $\mathbb{R}^3$, and introduce an orientation on S by $n = (0, 0, 1)$. Keep the orientation of γ as it was (that is, counter-clockwise in the xy-plane). Then (90) can be rewritten in the new (3-dimensional) setting as

$$\oint_\gamma F \cdot d\gamma = \int_S \text{curl} F \cdot n. \tag{93}$$

This equality is in fact true in general.

Theorem 7.1 (Kelvin-Stokes theorem)**.** *Let S be a surface in $\mathbb{R}^3$, equipped with the normal n, and let γ be the boundary of S, oriented such that $n \times \gamma'$ points into S (or equivalently: seen from the tip of n, γ rotates counter-closkwise). Then we have*

$$\oint_\gamma F \cdot d\gamma = \int_S \mathrm{curl} F \cdot dS, \tag{94}$$

for any smooth vector field F.

Remark 7.2. Let D be a small disk centred at $p \in \mathbb{R}^3$, equipped with the unit normal n, and let $\gamma = \partial D$ be the boundary of D, oriented so that $n \times \gamma'$ points into D. Then by the Kelvin-Stokes theorem, we have

$$\oint_\gamma F \cdot d\gamma = \int_D \mathrm{curl} F \cdot n \approx |D| \mathrm{curl} F(p) \cdot n, \tag{95}$$

which yields

$$\mathrm{curl} F(p) \cdot n = \lim_{|D| \to 0} \frac{1}{|D|} \int_\gamma F \cdot d\gamma. \tag{96}$$

In particular, $n = \frac{\mathrm{curl} F(p)}{|\mathrm{curl} F(p)|}$ gives the spatial arrangement of γ corresponding to the maximum possible circulation.

Example 7.3. (a) Let S be the union of $S_1 : z = x^2 + y^2$, $0 \le z \le 1$, and $S_2 : x^2 + y^2 = 1$, $1 \le z \le 2$. Equip S with the normal n so that its z-component is negative on S_1. We want to compute the flux

$$\int_S \mathrm{curl} F \cdot n, \tag{97}$$

where $F(x, y, z) = (3z, 5x, -2y)$. The boundary $\gamma = \partial S$ of S is the circle $x^2 + y^2 = 1$, $z = 2$, oriented clockwise seen from above. We parametrize γ as $\gamma(t) = (\sin t, \cos t, 2)$, $0 \le t < 2\pi$. Now, with the help of the Kelvin-Stokes theorem, we compute

$$\int_S \mathrm{curl} F \cdot n = \int_\gamma F \cdot d\gamma = \int_0^{2\pi} \begin{bmatrix} 6 \\ 5 \sin t \\ -2 \cos t \end{bmatrix} \cdot \begin{bmatrix} \cos t \\ -\sin t \\ 0 \end{bmatrix} dt = -5\pi. \tag{98}$$

(b) Let γ be the circle $y^2 + z^2 = 1$, $x = 0$, oriented counter-clockwise seen from points on the positive x-axis. We want to compute the circulation of

$$F(x, y, z) = (x^2 z + \sqrt{x^3 + x^2 + 2}, xy, 2y + xz + \sqrt{z^3 + 2}), \tag{99}$$

over γ. Since

$$\mathrm{curl} F(x, y, z) = (2, x^2 - z, y), \tag{100}$$

it is advisable to use the Kelvin-Stokes theorem to turn the circulation into a flux integral. The simplest surface bounded by γ is the disk D given by $y^2 + z^2 \le 1$, $x = 0$, and the orientation consistent with that of γ is provided by the normal $n = (1, 0, 0)$. Thus

$$\int_\gamma F \cdot d\gamma = \int_D \mathrm{curl} F \cdot n = \int_D 2 dy dz = 2\pi. \tag{101}$$

Remark 7.4. (a) Let γ_1 and γ_2 be two curves connecting P to Q, and let the union $\gamma_1 \cup \gamma_2$ be the boundary of some surface S. If $\mathrm{curl} F = 0$ then the Kelvin-Stokes theorem yields

$$0 = \int_S \mathrm{curl} F \cdot dS = \int_{\gamma_1} F \cdot d\gamma_1 - \int_{\gamma_2} F \cdot d\gamma_2, \tag{102}$$

that is,

$$\int_{\gamma_1} F \cdot d\gamma_1 = \int_{\gamma_2} F \cdot d\gamma_2. \tag{103}$$

As in the 2-dimensional case, this implies the Poincaré lemma for scalar potentials in 3-dimensions.

(b) Let S_1 and S_2 be two surfaces, bounded by the same curve γ. If $G = \text{curl}E$, then we have

$$\int_{S_1} G \cdot dS_1 = \int_\gamma E \cdot d\gamma = \int_{S_2} G \cdot dS_2, \tag{104}$$

that is, the flux of G is independent of the surface, as long as the boundary is held fixed. This is strongly reminiscent of the fact that

$$\int_{\gamma_1} F \cdot d\gamma_1 = \int_{\gamma_2} F \cdot d\gamma_2, \tag{105}$$

for any vector field F with $F = \nabla\phi$, as long as the endpoints of γ_1 and γ_2 are the same.

Example 7.5. Let γ be the circle $x^2 + y^2 = 4$, $z = 0$, oriented counter-clockwise seen from above, and let S be a surface bounded by γ, with its orientation consistent with that of γ. We want to compute the flux of curl$F$ over S, where

$$F(x, y, z) = (\cos x \sin z + xy, x^3, e^{x^2+z^2} - e^{y^2+z^2} + \tan(xy)). \tag{106}$$

We can turn this flux into the circulation of F over γ, but it seems unwieldy. So we use Remark 7.4(b) to replace the surface S by a simpler surface bounded by γ. The simplest such surface is arguably the disk D given by $x^2 + y^2 \leq 4$, $z = 0$, with the normal $n = (0, 0, 1)$. Thus we have

$$I = \int_S \text{curl}F \cdot dS = \int_\gamma F \cdot d\gamma = \int_D \text{curl}F \cdot n = \int_D (\text{curl}F)_z dxdy, \tag{107}$$

where

$$(\text{curl}F)_z = 3x^2 - x, \tag{108}$$

is the z-component of curl$F$. We finish the computation as

$$I = \int_D (3x^2 - x)dxdy = 12\pi. \tag{109}$$

8. Fluxes, stream functions, and the divergence in 2D

Fluxes can also be defined in 2-dimensions.

Definition 8.1. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a smooth curve, and let n be a unit normal to γ. Then the *flux of a vector field* F across γ (in the direction of n) is

$$\int_\gamma F \cdot n = \int_a^b F \cdot n|\gamma'|dt. \tag{110}$$

Remark 8.2. Conventionally, we take

$$n = \frac{1}{|\gamma'|}(\gamma')^\perp, \tag{111}$$

where

$$(a, b)^\perp = (b, -a), \tag{112}$$

is the 90° clockwise rotation, so that

$$\int_\gamma F \cdot n = \int_a^b F \cdot (\gamma')^\perp dt = \int_\gamma F_x dy - F_y dx. \tag{113}$$

Example 8.3. We want to compute the outward flux of $F(x,y) = (x^3, y^3)$ across the circle $x^2 + y^2 = 9$. Let us use the parametrization $\gamma(t) = (3\cos t, 3\sin t)$, $0 \leq t \leq 2\pi$. We have

$$\gamma'(t) = (-3\sin t, 3\cos t), \qquad \text{and so} \qquad \gamma'(t)^\perp = (3\cos t, 3\sin t). \tag{114}$$

The flux is then computed as

$$\int_\gamma F \cdot n = \int_0^{2\pi} \begin{bmatrix} 27\cos^3 t \\ 27\sin^3 t \end{bmatrix} \cdot \begin{bmatrix} 3\cos t \\ 3\sin t \end{bmatrix} dt = 82 \int_0^{2\pi} (\cos^4 t + \sin^4 t) dt = \frac{243\pi}{2}. \tag{115}$$

Remark 8.4. Note that

$$(V^\perp)^\perp = -V, \qquad \text{or} \qquad -(V^\perp)^\perp = V. \tag{116}$$

This means that $F \cdot (\gamma')^\perp = -F^\perp \cdot \gamma'$, and hence the flux of F across γ can be written in terms if the line integral of $F^\perp$ over γ as

$$\int_\gamma F \cdot n = -\int_\gamma F^\perp \cdot d\gamma. \tag{117}$$

In view of this, if $-F^\perp = \nabla\phi$ for some scalar function ϕ, then the flux $\int_\gamma F \cdot n$ depends only on the endpoints of γ. That $-F^\perp = \nabla\phi$ means $F = (\nabla\phi)^\perp$, that is,

$$F = \begin{bmatrix} \partial_x \phi \\ \partial_y \phi \end{bmatrix}^\perp = \begin{bmatrix} \partial_y \phi \\ -\partial_x \phi \end{bmatrix}. \tag{118}$$

Definition 8.5. The operator $\nabla^\perp$ defined by

$$\nabla^\perp \phi = (\nabla\phi)^\perp = (\partial_y \phi, -\partial_x \phi), \tag{119}$$

is called the *grad-perp* or the *rotated gradient*. The notation

$$\mathrm{grad}^\perp \phi = \nabla^\perp \phi, \tag{120}$$

can also be used.

Definition 8.6. If $F = \nabla^\perp \phi$ in $\Omega \subset \mathbb{R}^2$ for some scalar function ϕ, then we say that ϕ is a *stream function* of F in Ω.

Example 8.7. Let $\phi(x,y) = x^2 + y^2$. Then we have

$$\nabla\phi = (2x, 2y), \qquad \text{and} \qquad \nabla^\perp\phi = (2y, -2x). \tag{121}$$

Remark 8.8. Recall that $G = \nabla\phi$ implies $\mathrm{curl}G = 0$, and $\mathrm{curl}G = 0$ implies $G = \nabla\phi$ in convex regions. If $G = -F^\perp = (-F_y, F_x)$, then

$$\mathrm{curl}G = \partial_x F_x + \partial_y F_y. \tag{122}$$

Thus if F admits a stream function, then $\partial_x F_x + \partial_y F_y = 0$. In the converse direction, if $\partial_x F_x + \partial_y F_y = 0$, then F admits a stream function in convex regions.

Definition 8.9. In 2 dimensions, the *divergence* of F is defined to be

$$\mathrm{div}F = \partial_x F_x + \partial_y F_y. \tag{123}$$

The notation

$$\nabla \cdot F = \mathrm{div}F, \tag{124}$$

can also be used. If $\nabla \cdot F = 0$ in Ω, then the vector field F is called *incompressible*, *transverse* or *divergence-free in* Ω.

Example 8.10. Let $F(x,y) = (2x, 2y)$, and $G(x,y) = (-y, x)$. Then we have

$$\mathrm{curl}F = 0, \qquad \mathrm{div}F = 4, \qquad \mathrm{curl}G = 2, \quad \text{and} \quad \mathrm{div}G = 0. \tag{125}$$

We may summarize Remark 8.8 as follows.

Proposition 8.11. *Fields admitting stream functions are incompressible.*

In the converse direction, Remark 8.8 says the following.

Theorem 8.12 (Poincaré's lemma)**.** *If F is incompressible in Ω, where $\Omega \subset \mathbb{R}^2$ is open and convex, then F admits a stream function in Ω.*

Proceeding further, recall Green's theorem

$$\oint_\gamma G \cdot d\gamma = \int_K \text{curl} G, \tag{126}$$

which, upon substituting $G = -F^\perp$, and hence $\text{curl} G = \text{div} F$, implies the following.

Theorem 8.13 (Divergence theorem in 2D)**.** *Let γ be a closed curve, with n the outward unit normal, and let K be a region bounded by γ. Then we have*

$$\oint_\gamma F \cdot n = \int_K \text{div} F, \tag{127}$$

for any smooth vector field F in K.

In the left hand side, the circle over the integral sign simply signifies that γ is a closed curve. The circle could have been omitted, as γ is already specified to be closed in the context.

Remark 8.14. Let D be a small disk centred at $(x, y) \in \mathbb{R}^2$, and let $\gamma = \partial D$ be the boundary of D, with n being the outward unit normal to γ. Then by the divergence theorem, we have

$$\oint_\gamma F \cdot n = \int_D \text{div} F \approx |D| \text{div} F(x, y), \tag{128}$$

which yields

$$\text{div} F(x, y) = \lim_{|D| \to 0} \frac{1}{|D|} \oint_\gamma F \cdot n. \tag{129}$$

This shows that the divergence is "outward flux across closed curves per unit area."

Exercise 8.15. Show that

$$\nabla \times (\phi X) = \phi \nabla \times X - X \cdot (\nabla^\perp \phi), \tag{130}$$

for any scalar field ϕ and any 2-dimensional vector field X.

9. Vector potentials and the divergence in 3D

Let S_1 and S_2 be two surfaces, bounded by the same curve γ, and let $F = \text{curl} A$. Then recall from Remark 7.4(b) that

$$\int_{S_1} F \cdot dS_1 = \int_\gamma A \cdot d\gamma = \int_{S_2} F \cdot dS_2, \tag{131}$$

that is, the flux of F is independent of the surface, as long as the boundary is held fixed.

Definition 9.1. If $F = \text{curl} A$ in $\Omega \subset \mathbb{R}^3$ for some vector field A, then we say that A is a *vector potential* of F in Ω.

Remark 9.2. Vector potentials generalize stream functions. Indeed, let $F = \nabla^\perp \phi$ be a 2-dimensional vector field, and extend it to 3-dimensions such that $F_z = 0$ and $\partial_z F_x = \partial_z F_y = 0$. Then we have $F = (F_x, F_y, 0) = (\partial_y \phi, -\partial_x \phi, 0) = \text{curl} A$ with $A = (0, 0, \phi)$.

Remark 9.3. Suppose that $F = \text{curl}A$. If F was a 2-dimensional vector field admitting a stream function, then we would have $\partial_x F_x + \partial_y F_y = 0$. Let us check what becomes of this quantity now:

$$\begin{aligned}\partial_x F_x + \partial_y F_y &= \partial_x(\partial_y A_z - \partial_z A_y) + \partial_y(\partial_z A_x - \partial_x A_z) = -\partial_x\partial_z A_y + \partial_y\partial_z A_x \\ &= -\partial_z(\partial_x A_y - \partial_y A_x) = -\partial_z F_z.\end{aligned} \tag{132}$$

This means that if $F = \text{curl}A$ then

$$\partial_x F_x + \partial_y F_y + \partial_z F_z = 0. \tag{133}$$

Definition 9.4. In 3 dimensions, the *divergence* of F is defined to be

$$\text{div}F = \nabla \cdot F = \partial_x F_x + \partial_y F_y + \partial_z F_z. \tag{134}$$

If $\text{div}F = 0$ in Ω, then the vector field F is called *incompressible*, *solenoidal*, *transverse* or *divergence-free in* Ω.

We may summarize Remark 9.3 as follows.

Proposition 9.5. *Fields admitting vector potentials are incompressible.*

In the converse direction, we have the following.

Theorem 9.6 (Poincaré's lemma)**.** *If F is incompressible in Ω, where $\Omega \subset \mathbb{R}^3$ is open and convex, then F admits a vector potential in Ω.*

The proof is more complicated than the proof of Poincaré's lemma for scalar potentials, so we postpone it to a later section. The overall logic still follows the same pattern: Poincaré's lemma for vector potentials hinges on the following Green-Kelvin-Stokes type theorem. This is the divergence theorem in 3-dimensions, also called the Gauss-Ostrogradsky theorem.

Theorem 9.7 (Gauss-Ostrogradsky theorem)**.** *Let K be a bounded region, and let $S = \partial K$ be its boundary, oriented by the outward normal. Then we have*

$$\int_S F \cdot dS = \int_K \text{div}F, \tag{135}$$

for any smooth vector field F in K.

Proof for box domains. Suppose that $K = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$. Let $\Sigma_1 = [a_1, a_2] \times [b_1, b_2] \times \{c_1\}$ be the lower face of K, and let $\Sigma_2 = [a_1, a_2] \times [b_1, b_2] \times \{c_2\}$ be the upper face of K. Then one of the terms of the divergence integrates as

$$\begin{aligned}\int_K \frac{\partial F_z}{\partial z} &= \int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_{c_1}^{c_2} \frac{\partial F_z}{\partial z} dz dy dx = \int_{a_1}^{a_2}\int_{b_1}^{b_2} \big(F_z(x,y,c_2) - F_z(x,y,c_1)\big) dy dx \\ &= \int_{\Sigma_2} F_z - \int_{\Sigma_1} F_z = \int_{\Sigma_2} F \cdot n + \int_{\Sigma_1} F \cdot n,\end{aligned} \tag{136}$$

where n is the outward unit normal to ∂K. Thus, the integral of the divergence is

$$\int_K \text{div}F = \int_K \frac{\partial F_x}{\partial x} + \int_K \frac{\partial F_y}{\partial y} + \int_K \frac{\partial F_z}{\partial z} = \int_{\partial K} F \cdot n, \tag{137}$$

which establishes the proof. □

Example 9.8. Let K be the half cylinder given by $x^2 + z^2 \le 4$, $z \ge 0$, $0 \le y \le 2$. What is the outward flux of $F(x,y,z) = (x + \cos y, y + xe^z, z + \sin x)$ across the boundary of K? Since $\text{div}F = 3$, with n denoting the outward unit normal to ∂K, we have

$$\int_{\partial K} F \cdot n = \int_K \text{div}F = 3|K| = 3 \cdot 4\pi = 12\pi. \tag{138}$$

Example 9.9 (Archimedes' principle)**.** We want to compute the force exerted by a fluid on a body K that is floating (or immersed) in the fluid. The pressure at the point (x, y, z) is

$$p = -\rho g z, \tag{139}$$

where ρ is the density of the fluid, g is the gravitational constant, and z is counted upward, so that $z = 0$ at the surface level of the fluid. The force exerted by the fluid on a small surface area ΔS of the boundary S of K is

$$\Delta F = -pn\Delta S = \rho g z n \Delta S, \tag{140}$$

where n is the outward unit normal to $S = \partial K$. Let S^- be the part of S that is immersed in the fluid, and let e_3 be the unit vector along the z-axis. Then the z-component of the total force exerted on the body is

$$F_z = \int_{S^-} \rho g z e_3 \cdot n dS. \tag{141}$$

This is the flux of the vector field $G = \rho g z e_3$ across S^-. If K^- is the part of K that is below the surface level of the fluid, then the boundary of K^- consists of S^- and $K \cap \{z = 0\}$. Since $G = 0$ at $K \cap \{z = 0\}$, we can write

$$F_z = \int_{S^-} G \cdot n = \int_{\partial K^-} G \cdot n = \int_{K^-} \mathrm{div} G. \tag{142}$$

Now we compute

$$\mathrm{div}(z e_3) = e_3 \cdot \nabla z = e_3 \cdot e_3 = 1, \tag{143}$$

yielding

$$F_z = \int_{S^-} \rho g z e_3 \cdot n dS = \int_{K^-} \mathrm{div}(\rho g z e_3) = \rho g |K^-|. \tag{144}$$

For the other components, since

$$\mathrm{div}(z e_1) = e_1 \cdot \nabla z = e_1 \cdot e_3 = 0, \qquad \mathrm{div}(z e_2) = e_2 \cdot \nabla z = e_2 \cdot e_3 = 0, \tag{145}$$

we have

$$F_x = \int_{S^-} \rho g z e_1 \cdot n dS = \int_{K^-} \mathrm{div}(\rho g z e_1) = 0, \tag{146}$$

and

$$F_y = \int_{S^-} \rho g z e_2 \cdot n dS = \int_{K^-} \mathrm{div}(\rho g z e_2) = 0. \tag{147}$$

Exercise 9.10. Prove the following identities, where f is a scalar field, and X and Y are 3-dimensional vector fields.

(a) $\nabla \cdot (fX) = f \nabla \cdot X + (\nabla f) \cdot X$
(b) $\nabla \cdot (X \times Y) = (\nabla \times X) \cdot Y - X \cdot (\nabla \times Y)$

10. De Rham diagrams

In this section, we will review the differential operators such as grad, curl, etc., by using the organizing principle known as the *de Rham diagram*, and extend them to 4-dimensions. Before that, however, let us give an overview of the *nabla notations*.

Remark 10.1 (Nabla notation)**.** In 2-dimensions, the *nabla* is the "vector"

$$\nabla = (\partial_x, \partial_y), \tag{148}$$

consisting of differential operators (or symbols) ∂_x and ∂_y as components. The nabla "multiplied" by a scalar function and by a vector field respectively yield gradient and divergence:

$$\nabla f = (\partial_x f, \partial_y f), \qquad \nabla \cdot F = \partial_x F_x + \partial_y F_y. \tag{149}$$

Rotating the nabla and multiplying subsequently by a scalar function gives

$$\nabla^\perp = (\partial_y, -\partial_x), \qquad \nabla^\perp f = (\partial_y f, -\partial_x f). \tag{150}$$

The 2-dimensional curl can be written as

$$\mathrm{curl} F = \partial_x F_y - \partial_y F_x = \nabla \cdot F^\perp = -\nabla^\perp \cdot F, \tag{151}$$

inspiring us to define the *2-dimensional cross product* as

$$A \times B = A \cdot B^\perp = -A^\perp \cdot B = A_x B_y - A_y B_x. \tag{152}$$

With this, we have

$$\mathrm{curl} F = \nabla \times F = \begin{bmatrix} -\partial_y & \partial_x \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}. \tag{153}$$

In 3-dimensions, the *nabla* is the "vector"

$$\nabla = (\partial_x, \partial_y, \partial_z). \tag{154}$$

Similarly to the 2-dimensional case, we have

$$\nabla f = (\partial_x f, \partial_y f, \partial_z f), \qquad \nabla \cdot F = \partial_x F_x + \partial_y F_y + \partial_z F_z, \tag{155}$$

and

$$\nabla \times F = \left(\begin{vmatrix} \partial_y & F_y \\ \partial_z & F_z \end{vmatrix}, \begin{vmatrix} \partial_z & F_z \\ \partial_x & F_x \end{vmatrix}, \begin{vmatrix} \partial_x & F_x \\ \partial_y & F_y \end{vmatrix} \right) = \begin{bmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}, \tag{156}$$

which are of course gradient, divergence, and curl, respectively.

After this digression, we return to the focus of the section. Let us fix an open region $\Omega \subset \mathbb{R}^2$. Denote by SF the set of all *scalar fields* in Ω, and by VF the set of all *vector fields* in Ω. Then we have the diagram

$$SF \xrightarrow{\text{grad}} VF \xrightarrow{\text{curl}} SF, \tag{157}$$

with the property that

$$\underbrace{\text{image of grad}}_{\text{conservative fields}} \subset \underbrace{\text{kernel of curl}}_{\text{irrotational fields}}. \tag{158}$$

The latter property can also be expressed succinctly as

$$\mathrm{curl}\,\mathrm{grad} = 0. \tag{159}$$

Moreover, Poincaré's lemma states that the inclusion in (158) is in fact an *equality* if Ω is convex. In this case, the diagram (157) is called an *exact sequence.*

We also have the "rotated diagram"

$$SF \xrightarrow{\text{grad}^\perp} VF \xrightarrow{\text{div}} SF, \tag{160}$$

with

$$\underbrace{\text{image of grad}^\perp}_{\text{admitting stream function}} \subset \underbrace{\text{kernel of div}}_{\text{incompressible fields}}, \tag{161}$$

or equivalently,

$$\mathrm{div}\,\mathrm{grad}^\perp = 0. \tag{162}$$

Poincaré's lemma states that the inclusion in (161) becomes an *equality* (and hence (160) is an exact sequence) if Ω is convex.

In 3-dimensions, obviously with $\Omega \subset \mathbb{R}^3$, we have the diagram

$$SF \xrightarrow{\text{grad}} VF \xrightarrow{\text{curl}} VF \xrightarrow{\text{div}} SF, \tag{163}$$

with

$$\underbrace{\text{image of grad}}_{\text{conservative fields}} \subset \underbrace{\text{kernel of curl}}_{\text{irrotational fields}}, \qquad \underbrace{\text{image of curl}}_{\text{admitting vector potential}} \subset \underbrace{\text{kernel of div}}_{\text{incompressible fields}}, \tag{164}$$

or equivalently,

$$\operatorname{curl}\operatorname{grad} = 0 \qquad \text{and} \qquad \operatorname{div}\operatorname{curl} = 0. \tag{165}$$

Again, Poincaré's lemma states that both inclusions in (164) become *equalities* (and hence (163) is an exact sequence) if Ω is convex.

In order to extend the framework to 4-dimensions, we need to look at the 3-dimensional case in a slightly different light. The core idea is to consider the curl as an *antisymmetric matrix*, rather than a vector (at each point). The antisymmetric part of the Jacobian of a vector field F is

$$\begin{aligned} DF - (DF)^{\mathsf{T}} &= \begin{bmatrix} 0 & \partial_y F_x - \partial_x F_y & \partial_z F_x - \partial_x F_z \\ \partial_x F_y - \partial_y F_x & 0 & \partial_z F_y - \partial_y F_z \\ \partial_x F_z - \partial_z F_x & \partial_y F_z - \partial_z F_y & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -(\operatorname{curl}F)_z & (\operatorname{curl}F)_y \\ (\operatorname{curl}F)_z & 0 & -(\operatorname{curl}F)_x \\ -(\operatorname{curl}F)_y & (\operatorname{curl}F)_x & 0 \end{bmatrix}, \end{aligned} \tag{166}$$

that is,

$$DF - (DF)^{\mathsf{T}} = \star\operatorname{curl}F \tag{167}$$

where we have introduced the map $\star$ that turns vectors into antisymmetric matrices by

$$\star V = \begin{bmatrix} 0 & -V_z & V_y \\ V_z & 0 & -V_x \\ -V_y & V_x & 0 \end{bmatrix}. \tag{168}$$

Now we *redefine* the 3-dimensional curl as

$$\operatorname{curl}F = DF - (DF)^{\mathsf{T}}, \qquad \text{i.e.,} \qquad (\operatorname{curl}F)_{ik} = \partial_k F_i - \partial_i F_k. \tag{169}$$

Thus, the curl of a vector field is an *antisymmetric matrix field* (i.e., an antisymmetric matrix at each point of Ω), and the former vector-curl can now be written as $\star^{-1}\operatorname{curl}F$. As the former property $\operatorname{div}\operatorname{curl}F = 0$ becomes $\operatorname{div}\star^{-1}\operatorname{curl}F = 0$, the role of the divergence should now be played by the operator

$$\widetilde{\operatorname{div}} M = \operatorname{div}\star^{-1} M, \tag{170}$$

acting on antisymmetric matrix fields. Since

$$\star^{-1}\begin{bmatrix} 0 & -V_z & V_y \\ V_z & 0 & -V_x \\ -V_y & V_x & 0 \end{bmatrix} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}, \quad \text{or} \quad \star^{-1}\begin{bmatrix} 0 & M_{12} & M_{13} \\ M_{21} & 0 & M_{23} \\ M_{31} & M_{32} & 0 \end{bmatrix} = \begin{bmatrix} -M_{23} \\ M_{13} \\ -M_{12} \end{bmatrix}, \tag{171}$$

we can write explicitly

$$\widetilde{\operatorname{div}} M = -\partial_1 M_{23} + \partial_2 M_{13} - \partial_3 M_{12}. \tag{172}$$

Finally, the updated 3D diagram is

$$SF \xrightarrow{\operatorname{grad}} VF \xrightarrow{\operatorname{curl}} AMF \xrightarrow{\widetilde{\operatorname{div}}} SF, \tag{173}$$

where AMF denotes the set of all antisymmetric matrix fields in Ω. The usual properties

$$\operatorname{curl}\operatorname{grad} = 0 \qquad \text{and} \qquad \widetilde{\operatorname{div}}\operatorname{curl} = 0, \tag{174}$$

as well as the Poincaré lemma hold for the new diagram.

In 4-dimensions, we define

$$\operatorname{curl}F = DF - (DF)^{\mathsf{T}}, \qquad \text{i.e.,} \qquad (\operatorname{curl}F)_{ik} = \partial_k F_i - \partial_i F_k. \tag{175}$$

Corresponding to each coordinate axis, we have a coordinate 3-plane, which has a 3-dimensional curl. The divergence of each of those 3-dimensional curls must be zero. Assembling those divergences, we get the operator

$$\widetilde{\operatorname{div}} M = \begin{bmatrix} \partial_2 M_{34} - \partial_3 M_{24} + \partial_4 M_{23} \\ -\partial_1 M_{34} + \partial_3 M_{14} - \partial_4 M_{13} \\ \partial_1 M_{24} - \partial_2 M_{14} + \partial_4 M_{12} \\ -\partial_1 M_{23} + \partial_2 M_{13} - \partial_3 M_{12} \end{bmatrix}, \tag{176}$$

satisfying $\widetilde{\operatorname{div}}\operatorname{curl} F = 0$. Moreover, it is clear that $\operatorname{div}\widetilde{\operatorname{div}} M = 0$ for any matrix field M, where the 4-dimensional divergence of a vector field is defined by

$$\operatorname{div} F = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3 + \partial_4 F_4. \tag{177}$$

Thus we have constructed the diagram

$$SF \xrightarrow{\text{grad}} VF \xrightarrow{\text{curl}} AMF \xrightarrow{\widetilde{\text{div}}} VF \xrightarrow{\text{div}} SF \tag{178}$$

with the expected properties

$$\operatorname{curl}\operatorname{grad} = 0, \qquad \widetilde{\operatorname{div}}\operatorname{curl} = 0, \qquad \text{and} \qquad \operatorname{div}\widetilde{\operatorname{div}} = 0. \tag{179}$$

We have not discussed them, but the corresponding Poincaré lemmas would also hold.

Remark 10.2. Any 4×4 antisymmetric matrix can be written as

$$M = \begin{bmatrix} 0 & -X_3 & X_2 & Y_1 \\ X_3 & 0 & -X_1 & Y_2 \\ -X_2 & X_1 & 0 & Y_3 \\ -Y_1 & -Y_2 & -Y_3 & 0 \end{bmatrix}, \tag{180}$$

with $X, Y \in \mathbb{R}^3$. If M is an antisymmetric matrix field, then X and Y would be 3-component vectors depending on x_1, x_2, x_3 and x_4. Thus we can rewrite (176) as

$$\widetilde{\operatorname{div}} M = \begin{bmatrix} \partial_2 Y_3 - \partial_3 Y_2 - \partial_4 X_1 \\ -\partial_1 Y_3 + \partial_3 Y_1 - \partial_4 X_2 \\ \partial_1 Y_2 - \partial_2 Y_1 - \partial_4 X_3 \\ \partial_1 X_1 + \partial_2 X_2 + \partial_3 X_3 \end{bmatrix}. \tag{181}$$

Observe that we have the 3D curl of Y minus the x_4-derivative of X, that is, $\nabla \times Y - \partial_4 X$, in the first 3 rows, and the 3D divergence of X in the last row. That is to say, if we define the operations

$$X \boxplus Y = \begin{bmatrix} 0 & -X_3 & X_2 & Y_1 \\ X_3 & 0 & -X_1 & Y_2 \\ -X_2 & X_1 & 0 & Y_3 \\ -Y_1 & -Y_2 & -Y_3 & 0 \end{bmatrix}, \qquad X \boxplus \phi = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \phi \end{bmatrix}, \tag{182}$$

then (181) becomes

$$\widetilde{\operatorname{div}}(X \boxplus Y) = \big(\nabla \times Y - \partial_4 X\big) \boxplus (\nabla \cdot X). \tag{183}$$

Furthermore, looking at the 4D curl

$$\operatorname{curl} F = \begin{bmatrix} 0 & \partial_2 F_1 - \partial_1 F_2 & \partial_3 F_1 - \partial_1 F_3 & \partial_4 F_1 - \partial_1 F_4 \\ \partial_1 F_2 - \partial_2 F_1 & 0 & \partial_3 F_2 - \partial_2 F_3 & \partial_4 F_2 - \partial_2 F_4 \\ \partial_1 F_3 - \partial_3 F_1 & \partial_2 F_3 - \partial_3 F_2 & 0 & \partial_4 F_3 - \partial_3 F_4 \\ \partial_1 F_4 - \partial_4 F_1 & \partial_2 F_4 - \partial_4 F_2 & \partial_3 F_4 - \partial_4 F_3 & 0 \end{bmatrix}, \tag{184}$$

we notice that

$$\operatorname{curl}(X \boxplus \phi) = (\nabla \times X) \boxplus \big(\partial_4 X - \nabla\phi\big). \tag{185}$$

Finally, it is obvious that

$$\mathrm{div}(X \boxplus \phi) = \nabla \cdot X + \partial_4 \phi \qquad \text{and} \qquad \mathrm{grad}\phi = \nabla\phi \boxplus \partial_4 \phi. \tag{186}$$

These formulas give us a way to think about 4D operations in terms of 3D operations.

Exercise 10.3. What would be the analogue of the preceding remark for reducing 3D operations into those in 2D?

Exercise 10.4. Let X, Y and Z be 3-dimensional vectors, and let ϕ be a scalar. Show that

$$(\star X)Y = X \times Y, \qquad (X \boxplus Y)(Z \boxplus \phi) = (X \times Z + \phi Y) \boxplus (-Y \cdot Z). \tag{187}$$

Exercise 10.5. Construct a 5-dimensional de Rham diagram.

11. Duality and the Hodge star

While we have 2 diagrams (the de Rham diagram and its rotated version) in 2D, we have seen only one diagram in 3D. Then a natural question is: Is there a "rotated version" of the 3D diagram? In order to answer this question, we look for relationships between the two versions of the 2D diagram that can be extended to the 3-dimensional situation.

One possible clue is the following. If F is a 3D vector field with $\partial_z F_x = \partial_z F_y = 0$ and $F_z = 0$, then its curl and divergence are essentially the 2D curl and divergence of (F_x, F_y):

$$\mathrm{curl}F = (0, 0, \partial_x F_y - \partial_y F_x), \qquad \mathrm{div}F = \partial_x F_x + \partial_z F_z. \tag{188}$$

On the other hand, if F is a 3D vector field with $F_x = F_y = 0$ and $\partial_z F_z = 0$, then its curl is essentially the rotated gradient of F_z:

$$\mathrm{curl}F = (\partial_y F_z, -\partial_x F_z, 0). \tag{189}$$

Hence we see that both 2D diagrams are included in the 3D diagram as special cases, as the following commutative diagram describes.

$$\begin{array}{ccccccc}
SF & \xrightarrow{\mathrm{grad}} & VF & \xrightarrow{\mathrm{curl}} & SF & & \\
\downarrow{\scriptstyle j_0} & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} & & \\
SF & \xrightarrow{\mathrm{grad}} & VF & \xrightarrow{\mathrm{curl}} & VF & \xrightarrow{\mathrm{div}} & SF \\
 & & \uparrow{\scriptstyle j_2} & & \uparrow{\scriptstyle j_1} & & \uparrow{\scriptstyle j_0} \\
 & & SF & \xrightarrow[\mathrm{grad}^\perp]{} & VF & \xrightarrow[\mathrm{div}]{} & SF
\end{array} \tag{190}$$

Here, j_0 sends 2D scalar fields to 3D scalar fields, j_1 sends 2D vector fields to 3D vector fields, j_2 sends 2D scalar fields to 3D vector fields, and their actions are defined by

$$(j_0\phi)(x,y,z) = \phi(x,y), \quad (j_1 F)(x,y,z) = \begin{bmatrix} F_x(x,y) \\ F_y(x,y) \\ 0 \end{bmatrix}, \quad (j_2\phi)(x,y,z) = \begin{bmatrix} 0 \\ 0 \\ \phi(x,y) \end{bmatrix}. \tag{191}$$

The diagram *commutes*, means that if there are several ways to traverse the diagram from one place to another, then all these ways should give the same result. For example, the two squares in the middle of the diagram give

$$j_2 \mathrm{curl} F = \mathrm{curl} j_1 F \qquad \text{and} \qquad j_1 \mathrm{grad}^\perp \phi = \mathrm{curl} j_0 \phi, \tag{192}$$

for all 2D vector fields F and all 2D scalar fields ϕ.

Another possible clue is that the operators appearing in the rotated diagram and those appearing in the original diagram are related to each other via the 90° rotation as

$$\mathrm{grad}^\perp \phi = (\mathrm{grad}\phi)^\perp, \qquad \mathrm{curl}F = \mathrm{div}(F^\perp), \tag{193}$$

cf. (151), which can be depicted by the following commutative diagram.

$$\begin{array}{ccccc} SF & \xrightarrow{\text{grad}} & VF & \xrightarrow{\text{curl}} & SF \\ {\scriptstyle\text{id}}\downarrow & & \downarrow{\scriptstyle\perp} & & \downarrow{\scriptstyle\text{id}} \\ SF & \xrightarrow{\text{grad}^\perp} & VF & \xrightarrow{\text{div}} & SF \end{array} \tag{194}$$

Here "id" is the so-called *identity map* that sends ϕ to ϕ, i.e., it simply signifies that SF in the upper row should be identified with SF in the lower row.

The preceding two clues are interesting, but they seem to be not specific enough to produce a way to generate the "rotated version" of the 3D diagram. In particular, it is not clear what the extension of the 90° rotation should be in 3D. Now it turns out that there is a third clue that is most useful. First, we fit both diagrams in one picture as

$$SF \underset{-\text{div}}{\overset{\text{grad}}{\rightleftarrows}} VF \underset{\text{grad}^\perp}{\overset{\text{curl}}{\rightleftarrows}} SF \tag{195}$$

Then we claim that each operator in the lower row is the *transpose* of the corresponding operator in the upper row. "Transpose" is exactly what it sounds like: It is an extension of the notion of matrix transpose to operators such as grad and curl.

Let us recall what a matrix transpose is. If A is an $m \times n$ matrix, whose entries are denoted by A_{ik}, then the entries of its transpose A^T are

$$(A^\mathsf{T})_{ik} = A_{ki}. \tag{196}$$

However, since it is not clear what the "entries" of operators such as grad and curl would be, we want to focus on the viewpoint that an $m \times n$ matrix A is a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$. How do we identify the transpose of A? For any $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, we have

$$x \cdot Ay = x^\mathsf{T}Ay = (x^\mathsf{T}Ay)^\mathsf{T} = y^\mathsf{T}A^\mathsf{T}x = y \cdot A^\mathsf{T}x, \tag{197}$$

where the dot product in "$x \cdot Ay$" is happening in $\mathbb{R}^m$, and the dot product in "$y \cdot A^\mathsf{T}x$" is in $\mathbb{R}^n$. Thus, a *transpose* of $A : \mathbb{R}^n \to \mathbb{R}^m$ is any matrix $B : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$x \cdot Ay = y \cdot Bx, \qquad \text{for all} \quad x \in \mathbb{R}^m \quad \text{and} \quad y \in \mathbb{R}^n. \tag{198}$$

Since taking $x = e_k$ and $y = e_i$ in (198) reduces it to (196), the transpose of a matrix is unique.

To extend the transpose to the operator grad : $SF \to VF$, we need to have inner products in SF and in VF. We use

$$\langle \phi, \psi \rangle = \int_\Omega \phi\psi, \tag{199}$$

for scalar fields ϕ and ψ in Ω, and

$$\langle X, Y \rangle = \int_\Omega X \cdot Y, \tag{200}$$

for vector fields X and Y in Ω. Then the *transpose* of grad is an operator $\text{grad}^* : VF \to SF$ such that

$$\langle X, \text{grad}\phi \rangle = \langle \phi, \text{grad}^* X \rangle, \tag{201}$$

for all vector fields X and scalar fields ϕ vanishing on the boundary of Ω. To identify this operator, intergate the identity

$$\nabla \cdot (\phi X) = \phi \nabla \cdot X + X \cdot \nabla\phi, \tag{202}$$

and apply the divergence theorem, to get

$$\int_{\partial\Omega} \phi X \cdot n = \int_\Omega \nabla \cdot (\phi X) = \int_\Omega \phi \nabla \cdot X + \int_\Omega X \cdot \nabla\phi. \tag{203}$$

Since $\phi = 0$ on $\partial\Omega$, we infer

$$\langle \phi, \nabla \cdot X \rangle + \langle X, \nabla \phi \rangle = 0, \tag{204}$$

implying that the transpose of the gradient is the negative divergence, and the transpose of the divergence is the negative gradient:

$$\mathrm{grad}^* = -\mathrm{div}, \qquad \mathrm{div}^* = -\mathrm{grad}. \tag{205}$$

This result is dimension-independent, because the divergence theorem holds in all dimensions. To compute the transpose of the 2-dimensional curl, we proceed as

$$\int_\Omega \phi\, \mathrm{curl} X = \int_\Omega \phi\, \nabla \cdot X^\perp = -\int_\Omega X^\perp \cdot \nabla \phi = \int_\Omega X \cdot (\nabla \phi)^\perp = \int_\Omega X \cdot \nabla^\perp \phi, \tag{206}$$

yielding

$$\mathrm{curl}^* = \mathrm{grad}^\perp, \qquad (\mathrm{grad}^\perp)^* = \mathrm{curl}. \tag{207}$$

This establishes the diagram (195).

Moving onto 3-dimensions, (205) already populates most of the diagram

$$SF \underset{-\mathrm{div}}{\overset{\mathrm{grad}}{\rightleftarrows}} VF \underset{\mathrm{curl}}{\overset{\mathrm{curl}}{\rightleftarrows}} VF \underset{-\mathrm{grad}}{\overset{\mathrm{div}}{\rightleftarrows}} SF \tag{208}$$

leaving only the transpose of curl. To this end, recall the identity

$$\nabla \cdot (X \times Y) = (\nabla \times X) \cdot Y - X \cdot (\nabla \times Y), \tag{209}$$

cf. Exercise 9.10, and employ the divergence theorem to get

$$\int_{\partial\Omega} (X \times Y) \cdot n = \int_\Omega \nabla \cdot (X \times Y) = \int_\Omega X \cdot \nabla \times Y - \int_\Omega Y \cdot \nabla \times X, \tag{210}$$

which identifies the curl as the transpose of itself in 3-dimensions:

$$\mathrm{curl}^* = \mathrm{curl}. \tag{211}$$

Hence, transposing the 3D diagram has produced nothing new.

However, we know that there is a more fundamental diagram, namely (173). We claim that transposing (173) gives the following.

$$SF \underset{-\mathrm{div}}{\overset{\mathrm{grad}}{\rightleftarrows}} VF \underset{\mathrm{div}_c}{\overset{\mathrm{curl}}{\rightleftarrows}} AMF \underset{-\star\mathrm{grad}}{\overset{\widetilde{\mathrm{div}}}{\rightleftarrows}} SF \tag{212}$$

To verify this, we recall the so-called *Frobenius inner product* for matrices

$$A \cdot B = \sum_{i,k} A_{ik} B_{ik} = \mathrm{Tr}(A^\mathsf{T} B), \tag{213}$$

and introduce the following inner product for antisymmetric matrix fields:

$$\langle A, B \rangle = \frac{1}{2} \int_\Omega A \cdot B. \tag{214}$$

The factor $\frac{1}{2}$ ensures that

$$\langle \star F, \star G \rangle = \langle F, G \rangle, \tag{215}$$

for 3-dimensional vector fields F and G. For an antisymmetric matrix field A and a vector field F, we have

$$\begin{aligned}
\nabla\cdot(AF) &= \sum_{i,k}\partial_i(A_{ik}F_k) = \sum_{i,k}(\partial_i A_{ik})F_k + \sum_{i,k} A_{ik}\partial_i F_k \\
&= \sum_{i,k}(\partial_i A_{ik})F_k + \sum_{i<k} A_{ik}\partial_i F_k + \sum_{i>k} A_{ik}\partial_i F_k \\
&= \sum_{i,k}(\partial_i A_{ik})F_k + \sum_{i<k} A_{ik}\partial_i F_k + \sum_{k>i} A_{ki}\partial_k F_i \\
&= \sum_{i,k}(\partial_i A_{ik})F_k + \sum_{i<k} A_{ik}(\partial_i F_k - \partial_k F_i) \\
&= (\mathrm{div}_c A)\cdot F - \tfrac{1}{2} A\cdot \mathrm{curl} F,
\end{aligned} \tag{216}$$

where $\mathrm{div}_c A$ is the column-wise divergence of A. An integration followed by the divergence theorem gives

$$\int_{\partial\Omega}(AF)\cdot n = \int_\Omega (\mathrm{div}_c A)\cdot F - \frac{1}{2}\int_\Omega A\cdot\mathrm{curl}F = \langle \mathrm{div}_c A, F\rangle - \langle A, \mathrm{curl}F\rangle, \tag{217}$$

implying that

$$\mathrm{curl}^* = \mathrm{div}_c. \tag{218}$$

Note that this result is dimension-independent. Next, we recall

$$\widetilde{\mathrm{div}} M = \mathrm{div}\star^{-1} M, \tag{219}$$

from (170), and invoke (215) to get

$$\langle \widetilde{\mathrm{div}} M, \phi\rangle = \langle \mathrm{div}\star^{-1} M, \phi\rangle = -\langle \star^{-1} M, \mathrm{grad}\phi\rangle = -\langle M, \star\mathrm{grad}\phi\rangle, \tag{220}$$

yielding

$$(\widetilde{\mathrm{div}})^* = -\star\mathrm{grad}. \tag{221}$$

This establishes the diagram (212).

We can rewrite (219) as

$$\mathrm{div} F = \widetilde{\mathrm{div}}\star F. \tag{222}$$

Moreover, for any vector field G, we have

$$\begin{aligned}
\mathrm{div}_c \star G &= \mathrm{div}_c \begin{bmatrix} 0 & -G_z & G_y \\ G_z & 0 & -G_x \\ -G_y & G_x & 0 \end{bmatrix} \\
&= (\partial_y G_z - \partial_z G_y, \partial_z G_x - \partial_x G_z, \partial_x G_y - \partial_y G_x) \\
&= \nabla\times G,
\end{aligned} \tag{223}$$

meaning that

$$\mathrm{curl}\, G = \star\,\mathrm{div}_c \star G. \tag{224}$$

These observations are summarized in the following commutative diagram.

$$\begin{array}{ccccccc}
SF & \xrightarrow{\mathrm{grad}} & VF & \xrightarrow{\mathrm{curl}} & AMF & \xrightarrow{\widetilde{\mathrm{div}}} & SF \\
\uparrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \star} & & \uparrow{\scriptstyle \star} & & \downarrow{\scriptstyle \mathrm{id}} \\
SF & \xrightarrow[\star\mathrm{grad}]{} & AMF & \xrightarrow[\mathrm{div}_c]{} & VF & \xrightarrow[\mathrm{div}]{} & SF
\end{array} \tag{225}$$

Note that the Hodge star operator here can be compared to the 90° rotation in 2D, cf. (194).

At this point, transposing the 4D diagram (178) is not difficult. The dimension-independent results (205) and (218) leave only the transpose of $\widetilde{\mathrm{div}}$ to be identified:

$$SF \underset{-\mathrm{div}}{\overset{\mathrm{grad}}{\rightleftarrows}} VF \underset{\mathrm{div}_c}{\overset{\mathrm{curl}}{\rightleftarrows}} AMF \underset{\star\mathrm{curl}}{\overset{\widetilde{\mathrm{div}}}{\rightleftarrows}} VF \underset{-\mathrm{grad}}{\overset{\mathrm{div}}{\rightleftarrows}} SF \tag{226}$$

We proceed by writing (181) as

$$\widetilde{\mathrm{div}} \begin{bmatrix} 0 & -X_3 & X_2 & Y_1 \\ X_3 & 0 & -X_1 & Y_2 \\ -X_2 & X_1 & 0 & Y_3 \\ -Y_1 & -Y_2 & -Y_3 & 0 \end{bmatrix} = \mathrm{div}_c \begin{bmatrix} 0 & -Y_3 & Y_2 & X_1 \\ Y_3 & 0 & -Y_1 & X_2 \\ -Y_2 & Y_1 & 0 & X_3 \\ -X_1 & -X_2 & -X_3 & 0 \end{bmatrix}. \tag{227}$$

Thus, with the introduction of the (Euclidean) *Hodge star*

$$\star\,(X \boxplus Y) = Y \boxplus X, \tag{228}$$

where the operation $\boxplus$ has been defined in (182), we have

$$\widetilde{\mathrm{div}} M = \mathrm{div}_c \star M. \tag{229}$$

Then (217) yields

$$\langle \widetilde{\mathrm{div}} M, X\rangle = \langle \mathrm{div}_c \star M, X\rangle = \langle \star M, \mathrm{curl} X\rangle = \langle M, \star\mathrm{curl} X\rangle, \tag{230}$$

where in the last step we have used the properties

$$\star\star M = M, \qquad \text{and} \qquad (\star A)\cdot(\star B) = A \cdot B. \tag{231}$$

Finally, we note the following commutative diagram, whose only nontrivial content is (229).

$$\begin{array}{ccccccccc} SF & \xrightarrow{\mathrm{grad}} & VF & \xrightarrow{\mathrm{curl}} & AMF & \xrightarrow{\widetilde{\mathrm{div}}} & VF & \xrightarrow{\mathrm{div}} & SF \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \star} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ SF & \xrightarrow[\mathrm{grad}]{} & VF & \xrightarrow[\star\mathrm{curl}]{} & AMF & \xrightarrow[\mathrm{div}_c]{} & VF & \xrightarrow[\mathrm{div}]{} & SF \end{array} \tag{232}$$

Exercise 11.1. Similarly to (190), fit the two 3D diagrams (212) into the 4D diagram (178).

12. Electrostatics and Newtonian gravity

Newton's *law of universal gravitation*, first published in his *Principia* in 1687, asserts that the force exerted on a point mass Q at $x \in \mathbb{R}^3$ by the system of finitely many point masses q_i at $y_i \in \mathbb{R}^3$, $(i = 1, \ldots, m)$, is equal to

$$F = \sum_{i=1}^{m} \frac{Cq_iQ}{|x-y_i|^2}\frac{x-y_i}{|x-y_i|}, \tag{233}$$

with a constant $C < 0$ (like masses attract). Here Q and q_i are understood as real numbers that measure how much mass the corresponding points have, and $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ is the Euclidean length of the vector $a \in \mathbb{R}^3$. The same law of interaction between point charges was discovered experimentally by Charles Augustin de Coulomb and announced in 1785, now with $C > 0$ (like charges repel). Note that the numerical value of the constant C depends on the unit system one is using to measure force, mass (or charge), and distance.

It is convenient to view the force $F = F(x)$ as a vector function of x, that is, a vector field. This means that we fix the configuration of the point masses $\{q_i\}$, and think of Q as a test mass, that can be placed at any point in space to "probe the field." The vector field

$$E(x) = \sum_{i=1}^{m} \frac{Cq_i}{|x-y_i|^2}\frac{x-y_i}{|x-y_i|}, \tag{234}$$

does not depend on the test mass Q, and given any test mass Q at $x \in \mathbb{R}^3$, the force can be recovered as $F = QE(x)$. Therefore E can be thought of as a preexisting entity that characterizes the gravitational (or electric) field generated by the point masses $\{q_i\}$. In fact, we call E the *gravitational field* (or the *electric field*).

For a continuous distribution of mass, the sum in (234) must be replaced by an integral, as

$$E(x) = C\int_\Omega \frac{\rho(y)}{|x-y|^2}\frac{x-y}{|x-y|}\mathrm{d}y, \tag{235}$$

where ρ is the mass (or charge) density, and $\Omega \subset \mathbb{R}^3$ is the region occupied by the body. By defining $\rho = 0$ outside Ω, in (235) we may integrate over $\mathbb{R}^3$.

Early theoretical studies on the properties of (234) and (235) are connected to such names as Joseph-Louis Lagrange, Pierre-Simon Laplace, Adrien-Marie Legendre, Siméon Denis Poisson, Carl Friedrich Gauss, and George Green. We shall look at some of these developments now.

Proposition 12.1 (Lagrange 1773)**.** *Let E be given by either (234) or (235). Then E is conservative, i.e., there exists a scalar function ϕ such that $E = -\nabla\phi$.*

Proof. In order to confirm the existence of a potential ϕ, we simply define

$$\phi(x) = \sum_{i=1}^m \frac{Cq_i}{|x-y_i|}, \tag{236}$$

for a system of point charges, and

$$\phi(x) = C\int_{\mathbb{R}^3} \frac{\rho(y)\mathrm{d}y}{|x-y|}, \tag{237}$$

for a continuous distribution of charge, and check that $E = -\nabla\phi$ indeed gives (234) and (235), respectively. To this end, we compute

$$\frac{\partial}{\partial x_1}\frac{1}{|x|} = \frac{\partial}{\partial x_1}\left(|x|^2\right)^{-1/2} = -\frac{1}{2}\left(|x|^2\right)^{-3/2}\cdot 2x_1 = -\frac{1}{|x|^2}\cdot\frac{x_1}{|x|}, \tag{238}$$

and so

$$\nabla\frac{1}{|x|} = -\frac{1}{|x|^2}\cdot\frac{x}{|x|}, \qquad \text{or} \qquad \nabla\frac{1}{|x-y|} = -\frac{1}{|x-y|^2}\cdot\frac{x-y}{|x-y|}, \tag{239}$$

for a fixed $y \in \mathbb{R}^3$. This means that if ϕ is given by (236), then

$$\nabla\phi(x) = \sum_{i=1}^m \nabla\frac{Cq_i}{|x-y_i|} = -\sum_{i=1}^m \frac{Cq_i}{|x-y_i|^2}\cdot\frac{x-y_i}{|x-y_i|} = -E(x). \tag{240}$$

The case (237) can be treated similarly. □

Proposition 12.2 (Laplace 1782)**.** *Let E be given by either (234) or (235). Then E is solenoidal in free space, i.e., $\nabla\cdot E = 0$ in free space. Here "free space" means a place where there is no mass (or charge).*

Proof. When E is generated by a single particle at the origin, we have

$$\frac{\partial}{\partial x_1}\left(|x|^2\right)^{-3/2}x_1 = -\frac{3}{2}\left(|x|^2\right)^{-5/2}\cdot 2x_1\cdot x_1 + \left(|x|^2\right)^{-3/2} = \frac{-2x_1^2+x_2^2+x_3^2}{|x|^5}, \tag{241}$$

and so

$$\begin{aligned}\operatorname{div}\frac{x}{|x|^3} &= \frac{\partial}{\partial x_1}\left(|x|^2\right)^{-3/2}x_1 + \frac{\partial}{\partial x_2}\left(|x|^2\right)^{-3/2}x_2 + \frac{\partial}{\partial x_3}\left(|x|^2\right)^{-3/2}x_3 \\ &= \frac{-2x_1^2+x_2^2+x_3^2}{|x|^5} + \frac{-2x_2^2+x_1^2+x_3^2}{|x|^5} + \frac{-2x_3^2+x_1^2+x_2^2}{|x|^5} = 0,\end{aligned}$$

for $x \neq 0$. Therefore, for either (234) or (235), we have $\nabla\cdot E = 0$ in free space. □

If we combine this with $E = -\nabla\phi$, we get

$$\Delta\phi = \nabla \cdot \nabla\phi = 0 \qquad \text{in free space,} \tag{242}$$

where

$$\Delta = \nabla \cdot \nabla = \partial_1^2 + \partial_2^2 + \partial_3^2, \tag{243}$$

is called the *Laplace operator*. The equation (242) is the *Laplace equation*, and its solutions are called *harmonic functions*.

Laplace's result (242) was completed by his student Siméon Denis Poisson. The equation (245) appearing in the following proposition is known as the *Poisson equation*, and is valid everywhere, as opposed to (242), which is only valid in free space.

Proposition 12.3 (Poisson 1813)**.** *Let ϕ be given by*

$$\phi(x) = C \int_{\mathbb{R}^3} \frac{\rho(y)\mathrm{d}y}{|x-y|}, \tag{244}$$

with ρ smooth and vanishing outside some bounded set. Then we have[1]*.*

$$\Delta\phi = -4\pi C\rho \qquad \text{in } \mathbb{R}^3. \tag{245}$$

Sketch of proof. Let $U \subset \mathbb{R}^3$ be a region with smooth boundary, and let n be the unit outward normal to ∂U. Then for the vector field $F = \nabla u$, we have

$$\nabla \cdot F = \Delta u, \qquad \text{and} \qquad F \cdot n = n \cdot \nabla u = \partial_n u, \tag{246}$$

where $\partial_n u$ is the normal derivative of u at ∂U, and so the divergence theorem yields

$$\int_U \Delta u = \int_{\partial U} \partial_n u. \tag{247}$$

Now consider the potential generated by a point mass q at some point $y \in U$:

$$u(x) = \frac{Cq}{|x-y|}. \tag{248}$$

Let $B_\varepsilon = \{x \in \mathbb{R}^3 : |x-y| < \varepsilon\}$, and let us apply (247) to the domain $U_\varepsilon = U \setminus B_\varepsilon$. Since u is harmonic in U_ε, we infer

$$0 = \int_{U_\varepsilon} \Delta u = \int_{\partial U_\varepsilon} \partial_n u = \int_{\partial U} \partial_n u - \int_{\partial B_\varepsilon} \frac{\partial u}{\partial r}, \tag{249}$$

where $r = |x-y|$ is the radial variable centred at y. We can compute

$$\int_{\partial B_\varepsilon} \frac{\partial u}{\partial r} = \int_{\{r=\varepsilon\}} \frac{\partial}{\partial r}\frac{Cq}{r} = -\int_{\{r=\varepsilon\}} \frac{Cq}{r^2} = -4\pi\varepsilon^2 \frac{Cq}{\varepsilon^2} = -4\pi Cq, \tag{250}$$

and thus

$$\int_{\partial U} \partial_n u = -4\pi Cq. \tag{251}$$

This is for the potential generated by a charge at $y \in U$. On the other hand, if $y \notin U$, then $\Delta u = 0$ in U, and hence we have

$$\int_{\partial U} \partial_n u = \int_U \Delta u = 0. \tag{252}$$

Therefore, for the potential given (244), we get

$$\int_{\partial U} \partial_n \phi = -4\pi C \int_U \rho. \tag{253}$$

[1]In Gaussian type unit systems, one sets up the units so that $C = \pm 1$, and hence the Newtonian/Coulomb formulas (234) and (235) have simple expressions. In other systems such as SI, one sets up C such that the appearance of 4π in the Poisson equation (245) is avoided.

Finally, let $B_\varepsilon = \{y \in \mathbb{R}^3 : |y - x| < \varepsilon\}$ for $\varepsilon > 0$ small, and note that for any continuous function f, one has

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} f, \tag{254}$$

where $|B_\varepsilon|$ is the volume of the ball B_ε. By applying this result to $\Delta\phi$, we infer

$$\Delta\phi(x) \approx \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} \Delta\phi = \frac{1}{|B_\varepsilon|} \int_{\partial B_\varepsilon} \partial_n \phi = -\frac{4\pi C}{|B_\varepsilon|} \int_{B_\varepsilon} \rho \approx -4\pi C \rho(x), \tag{255}$$

where we used the identity (247) in the second step, and (253) in the third step. Upon taking the limit $\varepsilon \to 0$, we get

$$\Delta\phi(x) = -4\pi C \rho(x), \tag{256}$$

which is Poisson's equation. □

Remark 12.4. Note that in terms of the field E, the Poisson equation (245) is simply

$$\nabla \cdot E = 4\pi C \rho, \tag{257}$$

which is called the *Gauss law*. By integrating both sides over a region K and applying the divergence theorem to the left hand side, we discover the *integral form* of the Gauss law:

$$\int_{\partial K} E \cdot n = 4\pi C \int_K \rho. \tag{258}$$

This was discovered by Lagrange in 1773, and rediscovered by Gauss in 1813.

Remark 12.5. If f is a smooth function decaying sufficiently fast at infinity, then the *Newtonian potential* of f, defined by

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)\mathrm{d}y}{|x - y|}, \tag{259}$$

is a solution of the Poisson equation

$$\Delta u = f. \tag{260}$$

This observation will be used at crucial junctions many times later.

13. Magnetostatics

The qualitative effect of electric currents generating the magnetic field was discovered by Hans Christian Ørsted in 1820. In the same year, Jean Baptiste Biot and Félix Savart discovered the quantitative law

$$B(x) = \int_{\mathbb{R}^3} j(y) \times \frac{x - y}{|x - y|^3} d^3y, \tag{261}$$

which is the magnetic equivalent of the Coulomb law. Here, j is the *current density*, whose flux across a surface S gives the current flowing across S:

$$I = \int_S j \cdot dS. \tag{262}$$

The meaning of the vector field B is that when there is no electric field ($E = 0$), the force acting on a charge Q with velocity v is given by

$$F = Qv \times B. \tag{263}$$

By using Lagrange's observation

$$\nabla \frac{1}{|x - y|} = -\frac{x - y}{|x - y|^3}, \tag{264}$$

we can write the Biot-Savart law as

$$B(x) = -\int_{\mathbb{R}^3} j(y) \times \nabla_x \frac{1}{|x-y|} d^3y = \nabla \times \int_{\mathbb{R}^3} \frac{j(y)}{|x-y|} d^3y, \tag{265}$$

that is, with the introduction of the vector potential

$$A(x) = \int_{\mathbb{R}^3} \frac{j(y)}{|x-y|} d^3y, \tag{266}$$

we have

$$B = \nabla \times A. \tag{267}$$

From here, we immediately get the *Gauss law for magnetism*:

$$\nabla \cdot B = 0. \tag{268}$$

Note that by Remark 12.5, we have

$$\Delta A = -4\pi j, \tag{269}$$

where we understand ΔA in the component-wise fashion. Thus the curl of B is

$$\nabla \times B = \nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \Delta A = \nabla(\nabla \cdot A) + 4\pi j. \tag{270}$$

We can compute

$$\begin{aligned} \nabla \cdot A(x) = \nabla \cdot \int_{\mathbb{R}^3} \frac{j(y)}{|x-y|} d^3y &= \int_{\mathbb{R}^3} j(y) \cdot \nabla_x \frac{1}{|x-y|} d^3y \\ = -\int_{\mathbb{R}^3} j(y) \cdot \nabla_y \frac{1}{|x-y|} d^3y &= \int_{\mathbb{R}^3} \frac{\nabla \cdot j(y)}{|x-y|} d^3y, \end{aligned} \tag{271}$$

that is, $\nabla \cdot A$ is the Newtonian potential of $\nabla \cdot j$. To compute $\nabla \cdot j$, consider a region K, with n denoting its outer unit normal. Then the total charge contained in K at time $t + \Delta t$ is

$$q(t + \Delta t) \approx q(t) - \Delta t \int_{\partial K} j \cdot n = q(t) - \Delta t \int_K \nabla \cdot j, \tag{272}$$

where the integrals are performed at time t, and the approximation exact in the limit $\Delta t \to 0$. On the other hand, we have

$$q(t + \Delta t) - q(t) = \int_K \rho(t + \Delta t) - \int_K \rho(t), \tag{273}$$

and combining it with the previous formula, we get

$$\int_K \frac{\rho(t + \Delta t) - \rho(t)}{\Delta t} \approx -\int_K \nabla \cdot j, \tag{274}$$

yielding

$$\partial_t \rho + \nabla \cdot j = 0, \tag{275}$$

in the limit $\Delta t \to 0$. This equation expresses *charge conservation*, that is, the fact that electric charge cannot be produced out of nothing.

Charge conservation implies that if $\partial_t \rho = 0$, then $\nabla \cdot j = 0$, and therefore

$$\nabla \times B = 4\pi j, \tag{276}$$

by (270). This is called *Ampere's law*, named after André Marie Ampère, who did pioneering studies on magnetism, obtaining results equivalent to those of Biot and Savart. By integrating (276) over a surface S and applying the Kelvin-Stokes theorem to the left hand side, we discover the *integral form* of Ampere's law:

$$\int_\gamma B \cdot d\gamma = 4\pi \int_S j \cdot dS, \tag{277}$$

where $\gamma = \partial S$, suitably oriented. In closing, we want to emphasize that in the derivation of Ampere's law we have used the so-called *magnetostatic assumption* $\partial_t \rho = 0$.

14. Faraday's law and the Maxwell equations

Coulomb's law was discovered in the setting of static (or at least slowly moving) charges, and the Biot-Savart law was established for steady currents, which is embodied by the magnetostatic assumption. There is no *a priori* reason why they should hold also in highly dynamic situations. What we can say however is that their differential formulations, namely, Gauss' and Ampere's laws have more chance of being true in general, because of their local character.

So far, we have seen that electricity and magnetism both have Gauss laws, but Ampere's law is only for magnetism. The counterpart of Ampere's law for electricity, *Faraday's law*, was discovered in a qualitative form by Michael Faraday in 1831, and with some additions from Emil Lenz in 1834, written down in a final quantitative form by Franz Neumann in 1845. The integral form of Faraday's law is

$$\int_\gamma E \cdot d\gamma = -\frac{d}{dt} \int_S B \cdot dS, \tag{278}$$

where $\gamma = \partial S$, oriented as in the Kelvin-Stokes theorem. This can be manipulated as

$$\int_S \nabla \times E = \int_\gamma E \cdot d\gamma = \frac{d}{dt} \int_S B \cdot dS = -\int_S \partial_t B \cdot dS, \tag{279}$$

yielding Faraday's law in its differential form

$$\nabla \times E = -\partial_t B. \tag{280}$$

The full set of equations becomes thus

$$\begin{cases} \nabla \cdot E = 4\pi\rho, \\ \nabla \cdot B = 0, \\ \nabla \times B = 4\pi j, \\ \nabla \times E = -\partial_t B, \\ \partial_t \rho + \nabla \cdot j = 0. \end{cases} \tag{281}$$

However, this set has a serious limitation: Ampere's law implies

$$4\pi \nabla \cdot j = \nabla \cdot (\nabla \times B) = 0, \tag{282}$$

meaning that $\partial_t \rho = 0$ by charge conservation.

A way out was suggested by James Clerk Maxwell in 1861, when he replaced Ampere's law by the new equation

$$\nabla \times B = 4\pi j + \partial_t E. \tag{283}$$

This equation is now called the *Maxwell-Ampere equation*. If we take the divergence of the two sides of (283), we get

$$\nabla \cdot (\nabla \times B) = 4\pi \nabla \cdot j + \nabla \cdot \partial_t E = 4\pi \nabla \cdot j + \partial_t \nabla \cdot E = 4\pi \nabla \cdot j + 4\pi \partial_t \rho, \tag{284}$$

where in the last step we have used the Gauss law $\nabla \cdot E = 4\pi\rho$. Thus, the Maxwell-Ampere equation and the Gauss law together imply charge conservation as a consequence! The resulting set of equations

$$\begin{cases} \nabla \cdot B = 0 \\ \nabla \times E + \partial_t B = 0 \\ \nabla \cdot E = 4\pi\rho \\ \nabla \times B - \partial_t E = 4\pi j \end{cases} \tag{285}$$

is called *Maxwell's equations*.

Recall from (178) the 4-dimensional de Rham diagram

$$SF \xrightarrow{\text{grad}} VF \xrightarrow{\text{curl}} AMF \xrightarrow{\widetilde{\text{div}}} VF \xrightarrow{\text{div}} SF, \tag{286}$$

where the 4D operations can be expressed in terms of 3D operations as

$$\begin{aligned} \text{grad}\phi &= \nabla\phi \boxplus \partial_4\phi, \\ \text{curl}(X \boxplus \phi) &= (\nabla \times X) \boxplus \big(\partial_4 X - \nabla\phi\big), \\ \widetilde{\text{div}}(X \boxplus Y) &= \big(\nabla \times Y - \partial_4 X\big) \boxplus (\nabla \cdot X), \\ \text{div}(X \boxplus \phi) &= \nabla \cdot X + \partial_4\phi, \end{aligned} \tag{287}$$

cf. Remark 10.2. Thus, identifying the 4-th coordinate x_4 with time t, we see that the first pair of Maxwell's equations can be written as

$$\widetilde{\text{div}}(B \boxplus (-E)) = \big(-\nabla \times E - \partial_t B\big) \boxplus (\nabla \cdot B) = 0, \tag{288}$$

while the second pair is

$$\widetilde{\text{div}}(E \boxplus B) = \big(\nabla \times B - \partial_t E\big) \boxplus (\nabla \cdot E) = 4\pi(j \boxplus \rho). \tag{289}$$

Now, introduce the 4-dimensional vector field

$$J = j \boxplus \rho, \tag{290}$$

called the *4-current*, the 4×4 antisymmetric matrix field

$$F = B \boxplus (-E), \tag{291}$$

called the *Faraday tensor*, the new type of Hodge star operator

$$\star (X \boxplus Y) = (-Y) \boxplus X, \tag{292}$$

called the *Minkowskian Hodge star*, cf. (228). With these preparations at hand, the Maxwell equations (288)-(289) become

$$\widetilde{\text{div}}F = 0, \qquad \widetilde{\text{div}} \star F = 4\pi J. \tag{293}$$

By the property $\text{div}\,\widetilde{\text{div}} = 0$ of the de Rham diagram (286), we have

$$\text{div}J = \nabla \cdot j + \partial_t\rho = 0, \tag{294}$$

which is the 4-dimensional perspective on charge conservation. Moreover, since $\widetilde{\text{div}}F = 0$, by a 4D version of Poincaré's lemma (which we assume), there exists a 4-vector potential $Z = A \boxplus \psi$ such that

$$F = \text{curl}Z = (\nabla \times A) \boxplus \big(\partial_t A - \nabla\psi\big), \tag{295}$$

that is,

$$B = \nabla \times A, \qquad E = \nabla\psi - \partial_t A. \tag{296}$$

Thus we see that the vector potential for B survived Maxwell's modification (283), while the scalar potential formulation of E needed an upgrade.

The 4-dimensional formulation also gives an illuminating perspective on the *gauge freedom*. Since $\text{curl}\,\text{grad} = 0$, modifying the 4-vector potential by

$$Z' = Z + \text{grad}\chi, \tag{297}$$

for any scalar field χ does not affect the Faraday tensor:

$$\text{curl}Z' = \text{curl}Z + \text{curl}\,\text{grad}\chi = \text{curl}Z. \tag{298}$$

The transform (297) is called the *gauge transform*. In terms of A and ψ, this is

$$A' = A + \nabla\chi, \qquad \psi' = \psi + \partial_t\chi. \tag{299}$$

Note that traditionally, we take $\phi = -\psi$ as the scalar potential, which transforms as

$$\phi' = \phi - \partial_t \chi. \tag{300}$$

Finally, let us see how the Maxwell equations look in terms of the potentials A and ψ. The first pair of equations are automatically satisfied. The second pair is

$$\begin{aligned} \nabla \cdot E = \Delta\psi - \partial_t \nabla \cdot A = 4\pi\rho, \\ \nabla \times B - \partial_t E = \nabla \times \nabla \times A - \partial_t \nabla\psi + \partial_t^2 A = 4\pi j. \end{aligned} \tag{301}$$

Invoking the identity

$$\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \Delta A, \tag{302}$$

the second of the preceding equations can be rewritten as

$$\nabla(\nabla \cdot A) - \Delta A - \partial_t \nabla\psi + \partial_t^2 A = 4\pi j. \tag{303}$$

Thus, if we can ensure that

$$\nabla \cdot A = \partial_t \psi, \tag{304}$$

then (301) would become

$$\begin{aligned} \Delta\psi - \partial_t^2 \psi = 4\pi\rho, \\ \partial_t^2 A - \Delta A = 4\pi j, \end{aligned} \tag{305}$$

that is, the Maxwell equations would turn into wave equations for the potentials. Now, the condition (304) is called the *Lorenz gauge*, introduced by Ludvig Lorenz in 1867. This can always be satisfied, because under the gauge transform (299), we have

$$\nabla \cdot A' - \partial_t \psi' = \nabla \cdot A - \partial_t \psi + \Delta\chi - \partial_t^2 \chi, \tag{306}$$

and so by solving the wave equation

$$\partial_t^2 \chi - \Delta\chi = \nabla \cdot A - \partial_t \psi, \tag{307}$$

for χ, we can ensure (304).

Exercise 14.1. Another popular gauge is the *Coulomb gauge*

$$\nabla \cdot A = 0. \tag{308}$$

Show that there exists a gauge transform that ensures $\nabla \cdot A = 0$. What would the field equations (301) become under this gauge?

15. The Helmholtz decomposition

In the preceding section, we have derived the existence of potentials (296) for electromagnetism by applying a 4D Poincaré's lemma, which we have simply assumed. In this section, we give an entirely 3D approach to reach the same conclusion.

Let F be a smooth 3-dimensional vector field decaying sufficiently fast at infinity. Then Remark 12.5 guarantees that the vector field

$$U(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{F(y)\mathrm{d}y}{|x-y|}, \tag{309}$$

satisfies the vector Poisson equation

$$\Delta U = F. \tag{310}$$

Applying the curl-curl identity (302), we infer

$$F = \Delta U = \nabla(\nabla \cdot U) - \nabla \times \nabla \times U. \tag{311}$$

Thus, letting

$$\psi(x) = \nabla \cdot U(x) = -\frac{1}{4\pi} \nabla \cdot \int_{\mathbb{R}^3} \frac{F(y)\mathrm{d}y}{|x-y|}, \tag{312}$$

and

$$A(x) = -\nabla \times U(x) = \frac{1}{4\pi}\nabla \times \int_{\mathbb{R}^3} \frac{F(y)\mathrm{d}y}{|x-y|}, \tag{313}$$

we derive the fundamental decomposition

$$F = \nabla\psi + \nabla \times A, \tag{314}$$

known as the *Helmholtz decomposition*. This says that any vector field can be decomposed into the sum of a gradient and a curl. Note that since A itself is a curl, we have

$$\nabla \cdot A = 0. \tag{315}$$

However, given one particular decomposition (314), we can transform A into

$$A' = A + \nabla\chi, \tag{316}$$

for any scalar function χ, and obtain another decomposition

$$F = \nabla\psi + \nabla \times A', \tag{317}$$

which may have $\nabla \cdot A' \neq 0$.

Let us compute

$$\begin{aligned} 4\pi\psi(x) &= -\nabla \cdot \int_{\mathbb{R}^3} \frac{F(y)\mathrm{d}y}{|x-y|} = -\int_{\mathbb{R}^3} F(y) \cdot \nabla_x \frac{1}{|x-y|}\mathrm{d}y \\ &= \int_{\mathbb{R}^3} F(y) \cdot \nabla_y \frac{1}{|x-y|}\mathrm{d}y = -\int_{\mathbb{R}^3} \frac{\nabla \cdot F(y)}{|x-y|}\mathrm{d}y, \end{aligned} \tag{318}$$

which implies that if $\nabla \cdot F = 0$ then $\psi = 0$ in (312). In other words, if $\nabla \cdot F = 0$ in $\mathbb{R}^3$ then there exists a vector field A such that

$$F = \nabla \times A. \tag{319}$$

This is simply Poincaré's lemma for vector potentials in $\mathbb{R}^3$, cf Theorem 9.6.

Exercise 15.1. Show that if $\nabla \times F = 0$ then $A = 0$ in (313).

Now we apply these results to the Maxwell equations (285). Since $\nabla \cdot B = 0$, we have

$$B = \nabla \times A. \tag{320}$$

Substituting this into $\nabla \times E + \partial_t B = 0$, we get $\nabla \times E + \partial_t \nabla \times A = 0$, that is

$$\nabla \times (E + \partial_t A) = 0. \tag{321}$$

Hence there is a scalar function ϕ such that

$$E + \partial_t A = -\nabla\phi, \qquad \text{i.e.,} \qquad E = -\nabla\phi - \partial_t A. \tag{322}$$