MATH 248 WRITTEN ASSIGNMENT 3

DUE THURSDAY NOVEMBER 14

1. Recall that a set $K \in \mathbb{R}^2$ is called *negligible* if for any given $\varepsilon > 0$, there exist finitely many rectangles $q_1, \ldots, q_m$, such that

$$K \subset q_1 \cup \ldots \cup q_m, \qquad \text{and} \qquad |q_1| + \ldots + |q_m| < \varepsilon.$$

   (a) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that the graph

$$\Gamma = \{(x, f(x)) : x \in [a, b]\},$$

   of f is a negligible set in $\mathbb{R}^2$.
   Note added on Nov 6: If you want, you can assume that f is continuously differentiable, with its derivative satisfying the estimate $|f'(x)| \leq M$ for all $x \in [a, b]$ and for some constant M.

   (b) Give an example of a function $f : [0, 1] \to \mathbb{R}$ whose graph is not negligible in $\mathbb{R}^2$.

2. Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} 2^{2n} & \text{for } (x, y) \in [2^{-n}, 2^{-n+1}) \times [2^{-n}, 2^{-n+1}),\ n \in \mathbb{N}, \\ -2^{2n+1} & \text{for } (x, y) \in [2^{-n-1}, 2^{-n}) \times [2^{-n}, 2^{-n+1}),\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

   Show that

$$\int_0^1 \int_0^1 f(x, y) dx dy \neq \int_0^1 \int_0^1 f(x, y) dy dx.$$

   How do we reconcile this with Fubini's theorem?

3. Let $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \in \mathbb{R}^2$, $t \in [a, b]$, be a smooth curve in $\mathbb{R}^2$, such that $\gamma_1(t) > 0$ for all t, and let $S \subset \mathbb{R}^3$ be the surface of revolution, given in cylindrical coordinates by

$$\begin{cases} r(t, s) = \gamma_1(t) \\ z(t, s) = \gamma_2(t) \\ \phi(t, s) = s \end{cases}$$

   with $t \in [a, b]$ and $s \in [0, 2\pi)$.

   (a) Show that the surface area of S is

$$|S| = 2\pi x_c |\gamma|,$$

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where $|\gamma|$ is the length of γ, and

$$x_c = \frac{1}{|\gamma|} \int_a^b \gamma_1(t)|\gamma'(t)|dt,$$

is the first coordinate of the center of mass of γ.

(b) Apply the formula to find the surface area of the spheroid

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1.$$

4. Introduce a notion of integral on 3 dimensional hypersurfaces in $\mathbb{R}^4$. Using it, compute the hyper-area of the 3-dimensional sphere

$$S_r = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2\}$$

of radius r.

5. Prove the following identities, where f is a scalar field, and X and Y are vector fields.
   (a) $\nabla \cdot (fX) = f\nabla \cdot X + (\nabla f) \cdot X$
   (b) $\nabla \times (fX) = f\nabla \times X + (\nabla f) \times X$
   (c) $\nabla \cdot (X \times Y) = (\nabla \times X) \cdot Y - X \cdot (\nabla \times Y)$
6. Find the area enclosed by the loop in the folium of Descartes, which is given by

$$\gamma(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right), \qquad t \in [0, \infty).$$