# MATH 248 WRITTEN ASSIGNMENT 2

DUE TUESDAY OCTOBER 29

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = (y - x^2)(y - 2x^2)$. Show that the origin $(0, 0)$ is a local minimum of $f$ restricted to any line that passes through the origin, but the origin is *not* a local minimum of $f$.
2. Let $a, b, c$ and $d$ be real numbers. Find, with full justifications, the maximum and minimum values of the function
$$f(x, y, z) = \frac{ax + by + cz + d}{1 + x^2 + y^2 + z^2},$$
over $\mathbb{R}^3$.
3. In this exercise, we will study the critical points of the function
$$u(x, y) = -\frac{1}{R} - \frac{\mu}{\rho} - \frac{r^2}{2(1 + \mu^2)a^3},$$
where $r = \sqrt{x^2 + y^2}$, $\rho = \sqrt{(x - a)^2 + y^2}$, $R = \sqrt{(x + \mu a)^2 + y^2}$, and $a > 0$ and $\mu \geq 0$ are real parameters. This function is defined in $\Omega = \mathbb{R}^2 \setminus \{(-\mu a, 0), (a, 0)\}$, and appears as the effective potential field for a test mass (satellite) moving in the gravitational field generated by a small body (Earth) at $(a, 0)$ and a large body (Sun) at $(-\mu a, 0)$, with respect to the coordinate system that is following the rotation of both bodies. In particular, in this coordinate system, the two bodies are fixed, but the rotation of the coordinate system manifests itself through the centrifugal term $\frac{r^2}{2(1+\mu^2)a^3}$. With respect to a non-rotating coordinate system, the two bodies would be circling around their common centre of mass, which is the origin in our rotating coordinate system $xy$. The parameter $\mu$ is assumed to be *small*, and it represents the mass ratio between the bodies.
   (a) Show that $u$ has exactly one critical point of the form $p_1 = (x_1, 0)$ with $x_1 < -\mu a$. Show that $x_1 = x_1(\mu)$ as a function $\mu$ is differentiable at $\mu = 0$, with $x_1(0) = -a$ and $x_1'(0) = -\frac{17}{12}a$, meaning that
   $$x_1 \approx -a - \frac{17}{12}a\mu,$$
   for $\mu$ small. Classify this critical point.
   (b) For $\mu > 0$, show that $u$ has exactly one critical point of the form $p_2 = (x_2, 0)$ with $-\mu a < x_2 < a$, and exactly one critical point of the form $p_3 = (x_3, 0)$ with $x_3 > a$. Show that
   $$(x_{2,3} - a)^3 \approx \mp a^3 \mu,$$

*Date*: Fall 2019.

for $\mu$ small. Classify these critical points.

(c) Show that there are exactly two critical points (say, $p_4$ and $p_5$) off the $x$-axis. Compute the coordinates of these points exactly, and classify them. The classification might depend on the value of $\mu$.

(d) Sketch the graph of $u(x,0)$ as $x$ varies. Sketch some level curves of $u$ in two dimensions. You can use a graphing software. Give explanation relating the sketches with what you found in (a)–(c).

4. The maximum value of $f(x,y)$ subject to the constraint $g(x,y) = 270$ is 5100. The method of Lagrange multipliers gives $\lambda = 35$. Find an approximate value for the maximum of $f(x,y)$ subject to the constraint $g(x,y) = 267$. Give a detailed and convincing explanation why the general method you used to solve this problem works.
5. Compute the volume of the 4-dimensional ball
$$B_r = \{x \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 < r^2\},$$
of radius $r > 0$.
6. Compute the mass of the solid ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ if its mass density is given by $\mu(x,y,z) = x^2 + y^2 + z^2$.