MATH 248 WRITTEN ASSIGNMENT 1

DUE TUESDAY OCTOBER 8

1. Does there exist a function $u : \mathbb{R}^2 \to \mathbb{R}$ that is not continuous at $0 \in \mathbb{R}^2$, but whose restriction to every polynomial curve going through $0 \in \mathbb{R}^2$ is continuous? By a polynomial curve we mean the parameterized curve $(p(t), q(t))$ where p and q are polynomials, and by the restriction of u to the curve $(p(t), q(t))$, we mean the function $u(p(t), q(t))$ of the single variable t.
2. (a) Determine the values of the parameter $\alpha \in \mathbb{R}$ for which the function

$$f(x, y) = \begin{cases} |xy|^\alpha & \text{for } xy \neq 0 \\ 0 & \text{for } xy = 0 \end{cases}$$

is differentiable at every point $(x, y) \in \mathbb{R}^2$.

(b) Let $A \in \mathbb{R}^{n\times n}$, let $b \in \mathbb{R}^n$, and let

$$f(x) = x^\top Ax + b^\top x = \sum_{i,k=1}^{n} A_{ik}x_i x_k + \sum_{i=1}^{n} b_i x_i.$$

Show that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable in $\mathbb{R}^n$, and compute the derivative.

3. Let $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$\Psi(r, \phi, \theta) = \begin{pmatrix} x(r, \phi, \theta) \\ y(r, \phi, \theta) \\ z(r, \phi, \theta) \end{pmatrix} = \begin{pmatrix} r\cos\theta\cos\phi \\ r\cos\theta\sin\phi \\ r\sin\theta \end{pmatrix}, \tag{1}$$

where $(r, \phi, \theta) \in \mathbb{R}^3$ are the coordinates of the domain, and (x, y, z) are the components of the function Ψ.

(a) Compute the Jacobian of Ψ, and show that Ψ is differentiable everywhere in $\mathbb{R}^3$.

(b) Determine the points at which the Jacobian of Ψ is invertible. What would the inverse function theorem give if you apply this theorem near every such point?

(c) Let $u : \mathbb{R}^3 \to \mathbb{R}$ be differentiable in $\mathbb{R}^3$, and let $v = u \circ \Psi$. Find $Dv = (\partial_r v, \partial_\phi v, \partial_\theta v)$ in terms of $Du = (\partial_x u, \partial_y u, \partial_z u)$. Then compute Du in terms of Dv.

(d) Let $u : \mathbb{R}^3 \to \mathbb{R}$ be twice differentiable in $\mathbb{R}^3$, and let $v = u \circ \Psi$. Compute

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \tag{2}$$

in terms of the (first and second) derivatives of v.

Date: Fall 2019.

4. Sketch the level curves $f(x, y) = C$ of the function
$$f(x, y) = x^4 - x^2 + y^2,$$
for several different values of $C \in \mathbb{R}$. In particular, the case $C = 0$ yields the *lemniscate of Gerono*. Find the set of points at which the implicit function theorem does *not* apply to the equation $f(x, y) = C$ to yield $y = y(x)$.
5. Suppose that the real variables p, v, t, and u satisfy the equations
$$f(p, v, t, u) = 0, \qquad g(p, v, t, u) = 0,$$
and that these two equations can be solved for any two of the four variables as functions of the other two. Then the symbol $\partial u/\partial p$, for example, is ambiguous. We denote by $(\partial u/\partial p)_t$ the partial derivative of u with respect to p, with u and v considered as functions of p and t, and by $(\partial u/\partial p)_v$ the partial derivative of u with respect to p, with u and t considered as functions of p and v. With this notation, show that
$$\left(\frac{\partial u}{\partial p}\right)_v = \left(\frac{\partial u}{\partial t}\right)_v \left(\frac{\partial t}{\partial p}\right)_v = \left(\frac{\partial u}{\partial t}\right)_p \left(\frac{\partial t}{\partial p}\right)_v + \left(\frac{\partial u}{\partial p}\right)_t.$$
All functions that appear in this problem may be assumed to be defined and differentiable in a suitable set to make the arguments work.
6. Let $M \subset \mathbb{R}^3$ be the set obtained by rotating the circle $\{(x, 0, z) \in \mathbb{R}^3 : (x-2)^2 + z^2 = 1\}$ about the z-axis. By explicitly constructing coordinate charts, show that M is a smooth surface in $\mathbb{R}^3$. This surface is an example of a *torus*.

Homework policy

You are welcome to consult each other provided (1) you list all people and sources who aided you, or whom you aided and (2) you write-up the solutions independently, in your own language. If you seek help from other people, you should be seeking general advice, not specific solutions, and must disclose this help. This applies especially to internet fora such as `MathStackExchange`.

Similarly, if you consult books and papers outside your notes, you should be looking for better understanding of or different points of view on the material, not solutions to the problems.