THE STOKES THEOREM

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Abstract. We will extend the notion of integral to curves, surfaces, and more generally manifolds, and prove the Stokes theorem, which is one of the most beautiful and important theorems in all of mathematics.

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1. Line integrals and 1-forms

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a continuously differentiable function, representing a curve L in $\mathbb{R}^n$. Suppose that $u : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function. Then $f(t) = u(\gamma(t))$ is a function in $[a, b]$, and by the fundamental theorem of calculus, we have

$$u(\gamma(b)) - u(\gamma(a)) = f(b) - f(a) = \int_a^b f'(t)\mathrm{d}t = \int_a^b Du(\gamma(t))\gamma'(t)\mathrm{d}t. \tag{1}$$

This quantity does not depend on the curve L, let alone the parameterization γ, as long as the endpoints $\gamma(a)$ and $\gamma(b)$ stay fixed. We are going to interpret the integral in the right hand side as the integral of Du over the curve L, and attempt to generalize it to a class of objects broader than the derivatives of scalar functions. We make the following observations.

- The integral must depend not only on the curve L as a subset of $\mathbb{R}^n$, but also on a directionality property of the curve, since switching the endpoints $\gamma(a)$ and $\gamma(b)$ would flip the sign of (1).
- The derivative Du is a row vector at each point $x \in \mathbb{R}^n$, that is, $Du(x) \in \mathbb{R}^{n*}$ for $x \in \mathbb{R}^n$, or $Du : \mathbb{R}^n \to \mathbb{R}^{n*}$. Thus it might be possible to generalize (1) from integration of Du to that of $\alpha : \mathbb{R}^n \to \mathbb{R}^{n*}$.

Let $\alpha : \mathbb{R}^n \to \mathbb{R}^{n*}$ be smooth, and define

$$\int_\gamma \alpha = \int_a^b \alpha(\gamma(t))\gamma'(t)\mathrm{d}t. \tag{2}$$

The right hand side makes sense, because for each $t \in [a, b]$, we have $\alpha(\gamma(t)) \in \mathbb{R}^{n*}$ and $\gamma'(t) \in \mathbb{R}^n$, and so $\alpha(\gamma(t))\gamma'(t) \in \mathbb{R}$. Objects such as α are called differential 1-forms.

Now let us check if the integral (2) depends on the parameterization γ. Suppose that $\phi : [c, d] \to [a, b]$ is a continuously differentiable function, with $\phi([c, d]) = [a, b]$, $\phi(c) = a$ and

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$\phi(d) = b$. Then by applying the change of variables formula

$$\int_{\phi(c)}^{\phi(d)} f = \int_c^d (f \circ \phi)\phi', \tag{3}$$

to $f(t) = \alpha(\gamma(t))\gamma'(t)$, we have

$$\int_a^b \alpha(\gamma(t))\gamma'(t)\mathrm{d}t = \int_c^d \alpha(\gamma(\phi(s)))\gamma'(\phi(s))\phi'(s)\mathrm{d}s = \int_c^d \alpha(\eta(s))\eta'(s)\mathrm{d}s, \tag{4}$$

where $\eta(s) = \gamma(\phi(s))$ is the new parameterization. Hence the integral does not depend on the parameterization, as long as the endpoints of the curve are kept fixed. On the other hand, if $\phi(c) = b$ and $\phi(d) = a$, then we have

$$\int_a^b \alpha(\gamma(t))\gamma'(t)\mathrm{d}t = -\int_b^a \alpha(\gamma(t))\gamma'(t)\mathrm{d}t = -\int_c^d \alpha(\eta(s))\eta'(s)\mathrm{d}s, \tag{5}$$

which tells us that if the endpoints of the curve get switched under reparameterization, then the sign of the integral flips. Therefore, the integral (2) depends only on those aspects of the parameterization γ that specify a certain "directionality" property of the underlying curve. This "directionality" property is called orientation.

Intuitively, and in practice, an *oriented curve* is a curve given by some concrete parameterization γ, with the understanding that one can freely replace it by any other parameterization $\eta = \gamma \circ \phi$, as long as $\phi' > 0$. To define it precisely, we need some preparation. Let $L \subset \mathbb{R}^n$ be a curve, admitting a parameterization $\gamma : [a, b] \to \mathbb{R}^n$ which is a continuously differentiable function with $\gamma' \neq 0$ in (a, b). Suppose that P is the set of all such parameterizations of L. Then for any two parameterizations $\gamma_1 : [a_1, b_1] \to \mathbb{R}^n$ and $\gamma_2 : [a_2, b_2] \to \mathbb{R}^n$ from P, there exists a continuously differentiable function $\phi : [a_2, b_2] \to [a_1, b_1]$ with $\phi' \neq 0$ in (a_2, b_2), such that $\gamma_2 = \gamma_1 \circ \phi$. This gives a way to decompose P into two mutually disjoint classes P_1 and P_2: If $\phi' > 0$, then γ_1 and γ_2 are in the same class, and if $\phi' < 0$, then γ_1 and γ_2 are in different classes. The curve L, together with a choice of P_1 or P_2, is called an *oriented curve*. So the classes P_1 and P_2 are the possible *orientations* of the curve L. As mentioned before, in practice, we specify an orientated curve simply by giving a concrete parameterization.

Example 1.1. Let $\gamma : [0, \pi] \to \mathbb{R}^2$ be given by $\gamma(t) = (\cos t, \sin t)$, and let $\eta : [0, \sqrt{\pi}] \to \mathbb{R}^2$ be given by $\eta(s) = (\cos s^2, \sin s^2)$. Then we have $\eta = \gamma \circ \phi$ with $\phi(s) = s^2$ for $s \in [0, \sqrt{\pi}]$, and since $\phi' > 0$ in $(0, \sqrt{\pi})$, these two parameterizations define the same oriented curve. On the other hand, $\xi(\tau) = (-\cos \tau, \sin \tau)$, $\tau \in [0, \pi]$, gives the same curve as γ and η, but ξ is in the orientation opposite to that of γ and η, and thus as an *oriented* curve, ξ is different than γ.

Definition 1.2. Let $\Omega \subset \mathbb{R}^n$ be an open set. Then a *(differential) 1-form* on Ω is a continuously differentiable function $\alpha : \Omega \to \mathbb{R}^{n*}$.

For convenience, we restate the definition of integration of 1-forms over oriented curves.

Definition 1.3. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let α be a 1-form on Ω. Let $\gamma : [a, b] \to \Omega$ be an oriented curve. Then we define

$$\int_\gamma \alpha = \int_a^b \alpha(\gamma(t))\gamma'(t)\mathrm{d}t. \tag{6}$$

We have already established that the integral in the right hand side does not depend on the parameterization γ, as long as the orientation is kept fixed.

Example 1.4. Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^{2*}$ be given by $\alpha(x, y) = (-y, x)$, and let $\gamma(t) = (\cos t, \sin t)$, $t \in [0, \pi]$. Then we have

$$\int_\gamma \alpha = \int_0^\pi \begin{pmatrix} -\sin t & \cos t \end{pmatrix} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \mathrm{d}t = \int_0^\pi \mathrm{d}t = \pi. \tag{7}$$

A convenient way to generate 1-forms is to differentiate scalar functions. That is, if $\Omega \subset \mathbb{R}^n$ is an open set, and $u : \Omega \to \mathbb{R}$ is a smooth function, then Du is a 1-form on Ω. In the following, we will use the notation

$$\mathrm{d}u \equiv Du. \tag{8}$$

For example, if $u(x, y) = x^2 + y$, then $\mathrm{d}u(x, y) = (2x, 1) \in \mathbb{R}^{2*}$ for $(x, y) \in \mathbb{R}^2$.

Remark 1.5. If $u(x, y) = x$, then $\mathrm{d}u = (1, 0)$. In other words, we have $\mathrm{d}x = (1, 0)$. If $u(x, y) = y$, then $\mathrm{d}u = (0, 1)$, or $\mathrm{d}y = (0, 1)$. Hence any 1-form $\alpha(x, y) = (\alpha_1(x, y), \alpha_2(x, y))$ can be written as

$$\alpha(x, y) = \alpha_1(x, y)\mathrm{d}x + \alpha_2(x, y)\mathrm{d}y. \tag{9}$$

For example, if $u(x, y) = x^2+y$, then $\mathrm{d}u(x, y) = 2x\mathrm{d}x+\mathrm{d}y$. In general, any 1-form $\alpha : \Omega \to \mathbb{R}^{n*}$ with components $\alpha(x) = (\alpha_1(x), \ldots, \alpha_n(x)) \in \mathbb{R}^{n*}$ can be written as

$$\alpha(x) = \alpha_1(x)\mathrm{d}x_1 + \ldots + \alpha_n(x)\mathrm{d}x_n. \tag{10}$$

Example 1.6. Let $\alpha = x\mathrm{d}x + (x + y)\mathrm{d}y$, and let $\gamma(t) = (t, 2t)$, $t \in [0, 1]$. Then we have

$$\int_\gamma x\mathrm{d}x + (x + y)\mathrm{d}y = \int_0^1 (t + (t + 2t) \cdot 2)\mathrm{d}t = \int_0^1 7t\,\mathrm{d}t = \frac{7}{2}. \tag{11}$$

Definition 1.7. Let $\Omega \subset \mathbb{R}^n$ be an open set. Then a *vector field* on Ω is a continuously differentiable function $V : \Omega \to \mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ be open sets, and let $\Psi : \Omega \to U$ be a differentiable map. Then any oriented curve $\gamma : [a, b] \to \Omega$ gets sent to an oriented curve $\tilde{\gamma} = \Psi \circ \gamma$ in U. By the chain rule, the velocity vectors transform as

$$\tilde{\gamma}'(t) = D\Psi(\gamma(t))\gamma'(t). \tag{12}$$

Since any vector is a velocity vector of some curve, we are led to the transformation rule

$$\tilde{V}(\tilde{x}) = D\Psi(x)V(x), \qquad \text{or in components,} \qquad \tilde{V}_i = \sum_{k=1}^{n} \frac{\partial \tilde{x}_i}{\partial x_k} V_k, \quad (i = 1, \ldots, m), \tag{13}$$

for vector fields, where $V : \Omega \to \mathbb{R}^n$ and $\tilde{V} : U \to \mathbb{R}^m$ are vector fields, $x \in \Omega$ is arbitrary, and $\tilde{x} = \Psi(x) \in U$. The vector field $\tilde{V}$ is called the *push-forward* of V under the mapping Ψ, and denoted by $\tilde{V} = \Psi_* V$. In particular, taking $V = e_j \in \mathbb{R}^n$, the j-th standard basis vector in $\mathbb{R}^n$, we get $V_k = \delta_{jk}$, and hence[1]

$$\Psi_* e_j = \frac{\partial \tilde{x}_1}{\partial x_j}\,\tilde{e}_1 + \ldots + \frac{\partial \tilde{x}_m}{\partial x_j}\,\tilde{e}_m, \tag{14}$$

where $\tilde{e}_k \in \mathbb{R}^m$ denotes the m-th standard basis vector in $\mathbb{R}^m$.

Now suppose that $\tilde{\alpha} : U \to \mathbb{R}^{m*}$ is a 1-form on U. Then it is natural to define a 1-form α on Ω by requiring

$$\alpha(x)V(x) = \tilde{\alpha}(\tilde{x})\tilde{V}(\tilde{x}) = \tilde{\alpha}((\tilde{x}))D\Psi(x)V(x), \qquad (x \in \Omega), \tag{15}$$

[1] If we identify a vector V with the directional derivative operator D_V, then the standard basis vectors are simply the partial derivative operators $\frac{\partial}{\partial x_j}$, etc. With this convention, (14) takes the convenient form

$$\Psi_* \frac{\partial}{\partial x_j} = \frac{\partial \tilde{x}_1}{\partial x_j}\,\frac{\partial}{\partial \tilde{x}_1} + \ldots + \frac{\partial \tilde{x}_m}{\partial x_j}\,\frac{\partial}{\partial \tilde{x}_m}.$$

for any vector field $V : \Omega \to \mathbb{R}^n$. By choosing V to be the standard basis vectors of $\mathbb{R}^n$, we get the transformation law

$$\alpha(x) = \tilde{\alpha}(\tilde{x}) D\Psi(x), \qquad \text{or in components,} \qquad \alpha_k = \sum_{i=1}^{m} \frac{\partial \tilde{x}_i}{\partial x_k} \tilde{\alpha}_i, \quad (k = 1, \ldots, n), \tag{16}$$

The 1-form α is called the *pull-back* of $\tilde{\alpha}$ under the mapping Ψ, and denoted by $\alpha = \Psi^* \tilde{\alpha}$. In particular, if $\tilde{\alpha} = \mathrm{d}\tilde{x}_j$, then $\tilde{\alpha}_i = \delta_{ij}$, and hence

$$\Psi^* \mathrm{d}\tilde{x}_j = \frac{\partial \tilde{x}_j}{\partial x_1} \mathrm{d}x_1 + \ldots + \frac{\partial \tilde{x}_j}{\partial x_n} \mathrm{d}x_n. \tag{17}$$

Remark 1.8. The aforementioned transformation laws include coordinate transformation formulas as special cases. Suppose that we have the relations $\tilde{x} = \tilde{x}(x)$ and $x = x(\tilde{x})$ between two coordinate systems, and we want to express vector fields and 1-forms in the $\tilde{x}$-coordinate system, assuming that they are available in the x-coordinate system. Then vector fields transform as

$$\tilde{V}_i = \frac{\partial \tilde{x}_i}{\partial x_1} V_1 + \ldots + \frac{\partial \tilde{x}_i}{\partial x_n} V_n, \qquad (i = 1, \ldots, n). \tag{18}$$

In this setting, (14) should be thought of as the expression of the vector field e_j in the $\tilde{x}$-coordinate system

$$e_j = \frac{\partial \tilde{x}_1}{\partial x_j} \tilde{e}_1 + \ldots + \frac{\partial \tilde{x}_n}{\partial x_j} \tilde{e}_n, \qquad (j = 1, \ldots, n). \tag{19}$$

On the other hand, for 1-forms, we need to switch the roles of x and $\tilde{x}$ in (16), to infer the transformation law

$$\tilde{\alpha}_k = \frac{\partial x_1}{\partial \tilde{x}_k} \alpha_1 + \ldots + \frac{\partial x_n}{\partial \tilde{x}_k} \alpha_n, \qquad (k = 1, \ldots, n). \tag{20}$$

Using (17), we can also derive the expression of $\mathrm{d}x_j$ in the $\tilde{x}$-coordinate system

$$\mathrm{d}x_j = \frac{\partial x_j}{\partial \tilde{x}_1} \mathrm{d}\tilde{x}_1 + \ldots + \frac{\partial x_j}{\partial \tilde{x}_n} \mathrm{d}\tilde{x}_n, \qquad (j = 1, \ldots, n). \tag{21}$$

We note that while the Jacobian matrix of the transformation $x = x(\tilde{x})$ enters in the transformation law for 1-forms (20), the inverse of the same matrix is needed in the transformation law for vectors (18).

Example 1.9. Consider $(x, y) = (r \cos\phi, r \sin\phi)$. Invoking (21), we have

$$\begin{aligned} \mathrm{d}x &= \cos\phi \, \mathrm{d}r - r \sin\phi \, \mathrm{d}\phi, \\ \mathrm{d}y &= \sin\phi \, \mathrm{d}r + r \cos\phi \, \mathrm{d}\phi. \end{aligned} \tag{22}$$

So, for example, the 1-form $\alpha = x\mathrm{d}x + xy\mathrm{d}y$ given in the xy-coordinate system can immediately be written in the $r\phi$-coordinate system as

$$\begin{aligned} \alpha &= r\cos\phi(\cos\phi \, \mathrm{d}r - r \sin\phi \, \mathrm{d}\phi) + r^2 \sin\phi \cos\phi(\sin\phi \, \mathrm{d}r + r\cos\phi \, \mathrm{d}\phi) \\ &= (r\cos^2\phi + r^2 \sin^2\phi \cos\phi)\mathrm{d}r + r^2 \sin\phi \cos\phi(r\cos\phi - 1)\mathrm{d}\phi. \end{aligned} \tag{23}$$

Definition 1.10. For $V, W \in \mathbb{R}^n$, their *Euclidean inner product* (or *dot product*) is

$$V \cdot W = V_1 W_1 + \ldots + V_n W_n. \tag{24}$$

Furthermore, the *Euclidean norm* or the *2-norm* of a vector $V \in \mathbb{R}^n$ is

$$|V| = \sqrt{V \cdot V}. \tag{25}$$

If we fix $V \in \mathbb{R}^n$, and consider $f(W) = V \cdot W$ as a function of $W \in \mathbb{R}^n$, then it is a linear function: $f(\lambda W_1 + W_2) = \lambda f(W_1) + f(W_2)$ for $W_1, W_1 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, and can be represented by the row vector V^{T}, in the sense that $f(W) = V^{\mathsf{T}} W$. Hence the Euclidean inner product establishes the correspondence between $\mathbb{R}^n$ and $\mathbb{R}^{n*}$ given by $\kappa(V) = V^{\mathsf{T}}$. Using this correspondence, we can go between 1-forms and vector fields. Thus given a 1-form α, the vector field α^{T} is called the vector field *associated to* α. Similarly, given a vector field F, the 1-form F^{T} is called the 1-form *associated to* F. One of the applications of this is that we can make sense of the integral of a vector field F over an oriented curve γ as

$$\int_\gamma F \cdot \mathrm{d}\ell := \int_\gamma F^{\mathsf{T}} = \int_a^b F(\gamma(t))^{\mathsf{T}} \gamma'(t) \mathrm{d}t = \int_a^b F(\gamma(t)) \cdot \gamma'(t) \mathrm{d}t. \tag{26}$$

Remark 1.11. The expression (24) for the Euclidean inner product is valid only in a special class of coordinate systems, and *cannot* be valid in general coordinate systems. Even under the simple scaling $x = \lambda \tilde{x}$ for a constant $\lambda \neq 0$, we get

$$V \cdot W = V_1 W_1 + \ldots + V_n W_n = \lambda^2 (\tilde{V}_1 \tilde{W}_1 + \ldots + \tilde{V}_n \tilde{W}_n) = \lambda^2 V^{\mathsf{T}} W, \tag{27}$$

because $V = \lambda \tilde{V}$ and $W = \lambda \tilde{W}$ by (18). Note that we have to apply (18) with the roles of the quantities with tilde and the ones without reversed. This would induce the correspondence $\tilde{\kappa}(V) = \lambda^2 V^{\mathsf{T}}$ between $\mathbb{R}^n$ and $\mathbb{R}^{n*}$. Therefore, if we are using $\kappa(V) = V^{\mathsf{T}}$ to go between $\mathbb{R}^n$ and $\mathbb{R}^{n*}$, then we are effectively declaring that the current coordinate system is a special one.

2. Oriented manifolds

Our next goal is to define integration on surfaces and higher dimensional manifolds. In the case of line integrals, what guided us were properties of the derivative of a scalar function and the change of variables formula. It is not clear what the former would be in higher dimensions, so we start our investigation by looking once again at the change of variables formula. Let $U \subset \mathbb{R}^n$ be a bounded open set, and let $\Phi : Q \to \mathbb{R}^n$ be a map for which the change of variables formula holds, where $Q \subset \mathbb{R}^n$ is an oriented rectangle. We assume that $U \subset \Phi(Q)$ and that U has the same orientation as that of $\Phi(Q)$. Then for any integrable function $u : \mathbb{R}^n \to \mathbb{R}$ with $u = 0$ outside U, we have

$$\int_U u = \int_{\Phi(Q)} u = \int_Q (u \circ \Phi) \det(D\Phi). \tag{28}$$

The one dimensional case

$$\int_{\phi(a)}^{\phi(b)} f = \int_a^b (f \circ \phi) \phi' = \int_a^b f(\phi(t)) \phi'(t) \mathrm{d}t, \tag{29}$$

can be interpreted as the line integral of f over the 1-dimensional oriented curve $\phi : [a, b] \to \mathbb{R}$. In this context, the function f must be interpreted as a 1-form, and the integrand $f(\phi(t))\phi'(t)$ should be understood as the "row vector" $f(\phi(t))$ applied to the velocity vector $\phi'(t)$. Hence if (28) is to be a special case of integration over manifolds in the same say (29) is a special case of line integrals, then it might be fruitful to think of $(u \circ \Phi) \det(D\Phi)$ as "something" applied to the Jacobian matrix $D\Phi$. Of course, we know what that "something" is. At each point x, it is the number $u(x)$, multiplied by the determinant as a function $\det : \mathbb{R}^{n \times n} \to \mathbb{R}$.

Definition 2.1. An *alternating k-linear form in* $\mathbb{R}^n$ is a multilinear function

$$\eta : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{k \text{ times}} \to \mathbb{R}, \tag{30}$$

that is linear in each of its k arguments, and satisfies

$$\eta(\ldots, V, \ldots, V, \ldots) = 0 \qquad \text{for any} \quad V \in \mathbb{R}^n. \tag{31}$$

The set of all alternating k-linear form in $\mathbb{R}^n$ is denoted by $\mathrm{Alt}^k\,\mathbb{R}^n$.

To remove any ambiguity, what we mean by (31) is the following: If any two of the arguments of η are equal to each other, then η returns 0.

Example 2.2. An example of an alternating 2-linear form (or bilinear form) in $\mathbb{R}^3$ is given by $\eta(V,W) = v_1w_3 - v_3w_1$ for $V = (v_1, v_2, v_3) \in \mathbb{R}^3$ and $W = (w_1, w_2, w_3) \in \mathbb{R}^3$.

Remark 2.3. An element $\eta \in \mathrm{Alt}^k\mathbb{R}^n$ can also be thought of as a function $\eta : \mathbb{R}^{n\times k} \to \mathbb{R}$, with the understanding that η takes the columns of $A \in \mathbb{R}^{n\times k}$ as arguments. For instance, $\eta \in \mathrm{Alt}^2\mathbb{R}^3$ from the preceding example can be written as $\eta(A) = a_{11}a_{32} - a_{31}a_{12}$ where a_{ik} are the elements of $A \in \mathbb{R}^{3\times 2}$.

Remark 2.4. The requirement (31) is vacuous when $k = 1$, and therefore any linear function $\eta : \mathbb{R}^n \to \mathbb{R}$ is an alternating 1-linear form (or simply linear form). These are exactly the elements of $\mathbb{R}^{n*}$, that is, $\mathrm{Alt}^1\mathbb{R}^n = \mathbb{R}^{n*}$. Furthermore, it is a standard result in linear algebra that any $\eta \in \mathrm{Alt}^n\mathbb{R}^n$ satisfies

$$\eta(A) = \eta(I)\det(A), \qquad A \in \mathbb{R}^{n\times n}, \tag{32}$$

meaning that a general $\eta \in \mathrm{Alt}^n\mathbb{R}^n$ can deviate from the determinant only by a constant factor. In particular, $\mathrm{Alt}^n\mathbb{R}^n$ is 1-dimensional.

With this preparation, we recognize that at each $\xi \in Q$, $(u\circ\Phi)\det(D\Phi)$ in (28) is an expression of the type (32). Thus we define $\omega(x) \in \mathrm{Alt}^n\mathbb{R}^n$ for each $x \in \mathbb{R}^n$ by

$$\omega(x)(A) = u(x)\det(A), \qquad x \in \mathbb{R}^n, \tag{33}$$

and (28) becomes

$$\int_U \omega(x)(I)\mathrm{d}^n x = \int_Q \omega(\Phi(\xi))(D\Phi(\xi))\mathrm{d}^n\xi. \tag{34}$$

Note that both sides are in the same form, since we can think of I as the Jacobian matrix of the the trivial parameterization $\Psi(x) = x$. Note also that ω is indeed a function $\omega : \mathbb{R}^n \to \mathrm{Alt}^n\mathbb{R}^n$, that is, at each $x \in \mathbb{R}^n$, the *function value* $\omega(x)$ is an alternating k-linear form. Such functions are called *(differential) n-forms*. For any n-form ω, if we let $u(x) = \omega(x)(I)$, then according to (32), we have (33). We *define* the integral of ω over the oriented domain U by

$$\int_U \omega = \int_Q \omega(\Phi(\xi))(D\Phi(\xi))\mathrm{d}^n\xi. \tag{35}$$

The change of variables formula guarantees that the right hans side does not depend on the map Φ, as long as the orientation of $\Phi(Q)$ agrees with that of U.

We are now ready to define integrals over manifolds.

Definition 2.5. Let $G \subset \mathbb{R}^n$ be an open set. Then a *(differential) k-form* on G is a function $\omega : G \to \mathrm{Alt}^k\mathbb{R}^n$, such that $\omega(x)(A)$ is continuously differentiable as a function of $x \in G$, for any matrix $A \in \mathbb{R}^{n\times k}$.

Let M be a k-dimensional manifold in $\mathbb{R}^n$, and let $\Psi : \Omega \to M$ be a smooth parameterization, with $\Omega \subset \mathbb{R}^k$ open. Suppose that $G \subset \mathbb{R}^n$ is an open set, such that $M \subset G$, and let ω be a k-form on G. Then, we *define*

$$\int_{\Psi(\Omega)} \omega = \int_\Omega \omega(\Psi(\xi))(D\Psi(\xi))\mathrm{d}^k\xi, \tag{36}$$

provided the function $\omega(\Psi(\xi))(D\Psi(\xi))$ is integrable, and there exists a closed bounded set $K \subset \Omega$ such that $\omega(\Psi(\xi)) = 0$ whenever $\xi \in \Omega\setminus K$. We now show that (36) is preserved under reparameterization $\Xi = \Psi\circ\Phi$, as long as $\det(D\Phi) > 0$.

Lemma 2.6. *The right hand side in* (36) *is preserved under reparameterization* $\Xi = \Psi \circ \Phi$, *as long as* $\det(D\Phi) > 0$.

Proof. Let $\Phi : \Omega' \to \Omega$. The change of variables formula gives

$$\int_\Omega \omega(\Psi(\xi))(D\Psi(\xi))\mathrm{d}^k\xi = \int_{\Omega'} \omega(\Xi(\eta))(D\Psi(\Phi(\eta)))\det(D\Phi(\eta))\mathrm{d}^k\eta. \tag{37}$$

We compute $D\Psi = D(\Xi \circ \Phi^{-1})$ by the chain rule as

$$D\Psi = D(\Xi \circ \Phi^{-1}) = (D\Xi \circ \Phi^{-1})D(\Phi^{-1}) = (D\Xi \circ \Phi^{-1})(D\Phi \circ \Phi^{-1})^{-1}. \tag{38}$$

Finally, the property (32) yields

$$\omega(\Xi(\eta))(D\Psi(\Phi(\eta))) = \omega(\Xi(\eta))(D\Xi(\eta)(D\Phi)^{-1}) = \omega(\Xi(\eta))(D\Xi(\eta))\det((D\Phi)^{-1}), \tag{39}$$

completing the proof. ☐

To extend the definition (36) to the entire manifold M, we consider a collection of charts $\Psi_\alpha : \Omega_\alpha \to M$ covering M, and smooth functions $\phi_\alpha : M \to [0,1]$, such that

- $\sum_\alpha \phi_\alpha = 1$ in M.
- $\phi_\alpha(\Psi_\alpha(\xi)) = 0$ whenever $\xi \in \Omega_\alpha \setminus K_\alpha$, for some closed bounded set $K_\alpha \subset \Omega_\alpha$.

Such a collection $\{\phi_\alpha\}$ is called a smooth *partition of unity* subordinate to the covering $\{\Psi_\alpha(\Omega_\alpha)\}$ of M. Finally, we *define*

$$\int_M \omega = \sum_\alpha \int_{\Omega_\alpha} \phi_\alpha(\Psi_\alpha(\xi))\omega(\Psi_\alpha(\xi))(D\Psi_\alpha(\xi))\mathrm{d}^k\xi. \tag{40}$$

Note that the product $\phi_\alpha\omega$ vanishes outside $\Psi_\alpha(\Omega_\alpha)$ for each index α, and $\sum_\alpha \phi_\alpha\omega = \omega$.

We shall study how (40) depends on the charts $\Psi_\alpha : \Omega_\alpha \to M$ and the partition of unity $\{\phi_\alpha\}$. Consider another collection of charts $\Xi_\beta : \Omega'_\beta \to M$, with an associated partition of unity $\{\varphi_\beta\}$. For each α and β, let $\Phi_{\alpha,\beta} = \Psi_\alpha^{-1} \circ \Xi_\beta$ be the coordinate transformation, defined on an appropriate domain, so that $\Xi_\beta = \Psi_\alpha \circ \Phi_{\alpha,\beta}$. Then (40) can be written as

$$\begin{aligned}\sum_\alpha \int_{\Omega_\alpha} (\phi_\alpha \circ \Psi_\alpha)(\omega \circ \Psi_\alpha)(D\Psi_\alpha) &= \sum_{\alpha,\beta} \int_{\Omega_\alpha} (\varphi_\beta \circ \Psi_\alpha)(\phi_\alpha \circ \Psi_\alpha)(\omega \circ \Psi_\alpha)(D\Psi_\alpha)\\ &= \sum_{\alpha,\beta} \int_{\Omega_\beta} (\phi_\alpha \circ \Xi_\beta)(\varphi_\beta \circ \Xi_\beta)(\omega \circ \Xi_\beta)(D\Xi_\beta)\\ &= \sum_{\beta} \int_{\Omega_\beta} (\varphi_\beta \circ \Xi_\beta)(\omega \circ \Xi_\beta)(D\Xi_\beta),\end{aligned} \tag{41}$$

where we have assumed that $\det(D\Phi_{\alpha,\beta}) > 0$ for all α and β, and have invoked Lemma 2.6. Hence the idea would be to start with some initial collection of charts $\Psi_\alpha : \Omega_\alpha \to M$, and then allow all and only those reparameterizations $\{\Xi_\beta\}$ satisfying $\det(D(\Psi_\alpha^{-1} \circ \Xi_\beta)) > 0$ for all α and β. This is clearly an extension of the idea of oriented curves, and we would call such a structure an *oriented manifold*. However, there is an issue we need to deal with, which is not present in the 1-dimensional case. Namely, if $U = \Psi_\alpha(\Omega_\alpha) \cap \Psi_{\alpha'}(\Omega_{\alpha'})$ is nonempty, then we must expect

$$\int_{\Omega_\alpha} (\omega \circ \Psi_\alpha)(D\Psi_\alpha) = \int_{\Omega_{\alpha'}} (\omega \circ \Psi_{\alpha'})(D\Psi_{\alpha'}), \tag{42}$$

for any k-form ω that vanishes outside U, but we know from Lemma 2.6 that this is only true if $\det(D(\Psi_\alpha^{-1} \circ \Psi_{\alpha'})) > 0$. Moreover, there exist surfaces, let alone manifolds, such as the Möbius band and the Klein bottle, which do not admit any atlas $\{\Psi_\alpha\}$ satisfying $\det(D(\Psi_\alpha^{-1} \circ \Psi_{\alpha'})) > 0$ for all α and α'. Such surfaces are called *non-orientable* surfaces. Thus in order to integrate k-forms over M, we need to require that the manifold M be *orientable*, in the sense that there exists an atlas $\{\Psi_\alpha\}$ of M satisfying $\det(D(\Psi_\alpha^{-1}\circ\Psi_{\alpha'}))>0$ for all α and α'. An orientable manifold, equipped with such an atlas, is called an *oriented manifold*, with the understanding that we allow any reparameterization $\{\Xi_\beta\}$ satisfying $\det(D(\Psi_\alpha^{-1}\circ\Xi_\beta))>0$ for all α and β. As long as M is oriented, and the atlas $\{\Psi_\alpha\}$ is in the class specified by the orientation, the quantity in the right hand side of the definition (40) is independent of the atlas $\{\Psi_\alpha\}$ and of the partition of unity $\{\phi_\alpha\}$ used.

3. Exterior algebra

Alternating 1-linear forms are simply row vectors, that is, $\mathrm{Alt}^1\mathbb{R}^n=\mathbb{R}^{n*}$. An alternating n-linear form is the determinant, scaled by some numerical factor, hence in particular, $\dim(\mathrm{Alt}^n\mathbb{R}^n)=1$. Moreover, $\mathrm{Alt}^0\mathbb{R}^n=\mathbb{R}$ by convention. In this section, we want to shed some light on what $\mathrm{Alt}^k\mathbb{R}^n$ is for general k.

We will discuss $\mathrm{Alt}^2\mathbb{R}^n$ as a paradigmatic example. An alternating 2-linear form $\eta\in\mathrm{Alt}^2\mathbb{R}^n$ is by definition a map $\eta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, that is linear in each of the 2 arguments, satisfying $\eta(V,V)=0$ for all $V\in\mathbb{R}^n$. We have

$$\begin{aligned}\eta(V+W,V+W)&=\eta(V,V)+\eta(V,W)+\eta(W,V)+\eta(W,W)\\&=\eta(V,V)+\eta(W,W),\end{aligned}\tag{43}$$

implying that η is *antisymmetric*:

$$\eta(V,W)+\eta(W,V)=0\qquad\text{for any}\quad V,W\in\mathbb{R}^n.\tag{44}$$

Let us define the coefficients

$$\eta_{ij}=\eta(e_i,e_j),\qquad i,j=1,\ldots,n,\tag{45}$$

where $e_j\in\mathbb{R}^n$ are the standard basis vectors. If $V=v_1e_1+\ldots+v_ne_n$ and $W=w_1e_1+\ldots+w_ne_n$, then

$$\eta(V,W)=\eta\Big(\sum_{i=1}^n v_ie_i,\sum_{j=1}^n w_je_j\Big)=\sum_{i,j=1}^n\eta_{ij}v_iw_j,\tag{46}$$

meaning that the coefficients $\{\eta_{ij}\}$ determine η completely. Furthermore, the antisymmetry condition (44) implies

$$\eta_{ij}=\eta(e_i,e_j),=-\eta(e_j,e_i)=-\eta_{ji},\tag{47}$$

that is, η_{ij} is antisymmetric. In particular, we have $\eta_{ii}=0$ for $i=1,\ldots,n$.

Now suppose that $\{\eta_{ij}:i,j=1,\ldots,n\}$ is a collection of coefficients satisfying

$$\eta_{ij}=-\eta_{ji}.\tag{48}$$

Then $\eta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ defined by

$$\eta(V,W)=\sum_{i,j=1}^n\eta_{ij}v_iw_j,\tag{49}$$

is an alternating 2-linear form. Thus we have established a one-to-one correspondence between alternating 2-linear forms and the coefficients $\{\eta_{ij}:i,j=1,\ldots,n\}$ satisfying (48). At this point, an interesting question is if and "how many" nontrivial alternating 2-linear forms exist. The set of all possible coefficients $\{\eta_{ij}:i,j=1,\ldots,n\}$ is the same as $\mathbb{R}^{n^2}$, but the conditions (48) significantly reduce the possibilities. Namely, by permuting indices and using (48), all coefficients can be expressed only in terms of the coefficients η_{ij} with $i<j$. For example, all alternating 2-linear forms in $\mathbb{R}^3$ can be generated by specifying the three coefficients η_{12}, η_{13}, and η_{23}, which yields

$$\eta(V,W)=\sum_{i,j=1}^n\eta_{ij}v_iw_j=\eta_{12}(v_1w_2-v_2w_1)+\eta_{13}(v_1w_3-v_3w_1)+\eta_{23}(v_2w_3-v_3w_2).\tag{50}$$

In general, the dimension of the space $\{\eta_{ij} \in \mathbb{R} : 1 \le i < j \le n\}$ is $\binom{n}{2} = \frac{n(n-1)}{2}$, and hence $\dim(\mathrm{Alt}^2\mathbb{R}^n) = \frac{n(n-1)}{2}$.

Exercise 3.1. Generalize the above discussion to $\mathrm{Alt}^k\mathbb{R}^n$, and show that $\dim(\mathrm{Alt}^k\mathbb{R}^n) = \binom{n}{k}$.

So far, we have regarded $\{\eta_{ij}\}$ simply as a collection of coefficients, because this generalizes better to the cases $k > 2$, but for $k = 2$ it is convenient to think of the coefficients $\{\eta_{ij}\}$ as the elements of a matrix. Thus, letting $H = (\eta_{ij}) \in \mathbb{R}^{n\times n}$, from (46) we have

$$\eta(V,W) = \sum_{i,j=1}^{n} \eta_{ij} v_i w_j = V^{\mathsf{T}} H W, \tag{51}$$

and the condition (48) is equivalent to saying that the matrix H is antisymmetric: $H^{\mathsf{T}} + H = 0$. Consequently, $\mathrm{Alt}^2\mathbb{R}^n$ can be identified with the space of antisymmetric $n \times n$ matrices.

Given $\alpha, \beta \in \mathbb{R}^{n*}$, we can generate an $n \times n$ matrix by taking their outer product as

$$\alpha^{\mathsf{T}}\beta = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 & \dots & \alpha_1\beta_n \\ \alpha_2\beta_1 & \alpha_2\beta_2 & \dots & \alpha_2\beta_n \\ \dots & \dots & \dots & \dots \\ \alpha_n\beta_1 & \alpha_n\beta_2 & \dots & \alpha_n\beta_n \end{pmatrix}, \tag{52}$$

and then "antisymmetrize" it to get

$$\alpha^{\mathsf{T}}\beta - \beta^{\mathsf{T}}\alpha = \begin{pmatrix} 0 & \alpha_1\beta_2 - \alpha_2\beta_1 & \dots & \alpha_1\beta_n - \alpha_n\beta_1 \\ \alpha_2\beta_1 - \alpha_1\beta_2 & 0 & \dots & \alpha_2\beta_n - \alpha_1\beta_n \\ \dots & \dots & \dots & \dots \\ \alpha_n\beta_1 - \alpha_1\beta_n & \alpha_n\beta_2 - \alpha_2\beta_n & \dots & 0 \end{pmatrix}. \tag{53}$$

The alternating 2-linear form represented by the matrix $\alpha^{\mathsf{T}}\beta - \beta^{\mathsf{T}}\alpha$ is called the exterior product of α and β.

Definition 3.2. Given $\alpha, \beta \in \mathrm{Alt}^1\mathbb{R}^n$, their *exterior product* (or *wedge product*) $\alpha \wedge \beta \in \mathrm{Alt}^2\mathbb{R}^n$ is defined by

$$(\alpha \wedge \beta)(V,W) = V^{\mathsf{T}}(\alpha^{\mathsf{T}}\beta - \beta^{\mathsf{T}}\alpha)W = \alpha(V)\beta(W) - \beta(V)\alpha(W). \tag{54}$$

Example 3.3. (a) For $\alpha, \beta \in \mathbb{R}^{2*}$, the matrix of $\alpha \wedge \beta \in \mathrm{Alt}^2\mathbb{R}^2$ is

$$\alpha^{\mathsf{T}}\beta - \beta^{\mathsf{T}}\alpha = \begin{pmatrix} 0 & \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_2\beta_1 - \alpha_1\beta_2 & 0 \end{pmatrix}, \tag{55}$$

and hence $(\alpha \wedge \beta)(V,W) = (\alpha_1\beta_2 - \alpha_2\beta_1)(v_1w_2 - v_2w_1)$.

(b) For $\alpha, \beta \in \mathbb{R}^{3*}$, the matrix of $\alpha \wedge \beta \in \mathrm{Alt}^2\mathbb{R}^3$ is

$$\alpha^{\mathsf{T}}\beta - \beta^{\mathsf{T}}\alpha = \begin{pmatrix} 0 & \alpha_1\beta_2 - \alpha_2\beta_1 & \alpha_1\beta_3 - \alpha_3\beta_1 \\ \alpha_2\beta_1 - \alpha_1\beta_2 & 0 & \alpha_2\beta_3 - \alpha_1\beta_3 \\ \alpha_3\beta_1 - \alpha_1\beta_3 & \alpha_3\beta_2 - \alpha_2\beta_3 & 0 \end{pmatrix}. \tag{56}$$

Remark 3.4. The exterior product can be extended to $\alpha \in \mathrm{Alt}^p\mathbb{R}^n$ and $\beta \in \mathrm{Alt}^q\mathbb{R}^n$, yielding $\alpha \wedge \beta \in \mathrm{Alt}^{p+q}\mathbb{R}^n$, but we will not discuss that here.

In $\mathbb{R}^3$, alternating 2-linear forms have 3 degrees of freedom, i.e., $\dim(\mathrm{Alt}^2\mathbb{R}^3) = 3$. This suggests that there may be some relation between $\mathbb{R}^3$ and $\mathrm{Alt}^2\mathbb{R}^3$. Indeed, in the context of (50), introducing the vector $F = (f_1, f_2, f_3) \in \mathbb{R}^3$ whose components are given by $f_1 = \eta_{23}$, $f_2 = -\eta_{13}$, and $f_3 = \eta_{12}$, we get

$$\begin{aligned} \eta(V,W) &= \eta_{12}(v_1w_2 - v_2w_1) + \eta_{13}(v_1w_3 - v_3w_1) + \eta_{23}(v_2w_3 - v_3w_2) \\ &= f_3(v_1w_2 - v_2w_1) - f_2(v_1w_3 - v_3w_1) + f_1(v_2w_3 - v_3w_2) \\ &= \det(F,V,W), \end{aligned} \tag{57}$$

meaning that in $\mathbb{R}^3$, there is the identification $F \mapsto \det(F,\cdot,\cdot)$ between $\mathbb{R}^3$ and $\mathrm{Alt}^2\mathbb{R}^3$.

Remark 3.5. The identification (57) is not canonical, since for example, we could have chosen $f_1 = -\eta_{23}$, $f_2 = \eta_{13}$, and $f_3 = -\eta_{12}$, ending up with

$$\eta(V,W) = -\det(F,V,W). \tag{58}$$

More generally, any $\mu \in \mathrm{Alt}^3\mathbb{R}^3$ induces the identification between $F \in \mathbb{R}^3$ and $\eta \in \mathrm{Alt}^2\mathbb{R}^3$ given by $\eta(V,W) = \mu(F,V,W)$. Another way to see this is to notice that the expression (57) is not preserved under change of coordinates. For example, under the simple scaling $V = \lambda\tilde{V}$, $W = \lambda\tilde{W}$, $F = \lambda\tilde{F}$, we have

$$\det(F,V,W) = \lambda^3 \det(\tilde{F},\tilde{V},\tilde{W}). \tag{59}$$

In other words, if we are using (57) to go between $\mathbb{R}^3$ and $\mathrm{Alt}^2\mathbb{R}^3$, then we are effectively declaring that the current coordinate system is a special one. For example, if we flip all coordinate axes, then the preceding formula applies with $\lambda = -1$, and hence in the flipped coordinate system, we need to use the identification $\tilde{F} \mapsto -\det(\tilde{F},\cdot,\cdot)$ in order to be consistent with the identification $F \mapsto \det(F,\cdot,\cdot)$ in the original coordinate system.

Exercise 3.6. Show that there is a correspondence between $\mathbb{R}^n$ and $\mathrm{Alt}^{n-1}\mathbb{R}^n$.

Remark 3.7 (Cross product)**.** In the context of Example 3.3, the vector associated to $\alpha \wedge \beta$ under the identification (57) is

$$F = (\alpha_2\beta_3 - \alpha_1\beta_3, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1). \tag{60}$$

Recall that the Euclidean inner product $V \cdot W$ induces the correspondence $V \mapsto V^{\mathsf{T}}$ between $\mathbb{R}^n$ and $\mathbb{R}^{n*}$. Given $V, W \in \mathbb{R}^3$, the vector associated to $V^{\mathsf{T}} \wedge W^{\mathsf{T}} \in \mathrm{Alt}^2\mathbb{R}^3$ is called the *cross product* of V and W, and denoted by

$$F = V \times W = (v_2w_3 - v_1w_3, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1). \tag{61}$$

Note that to define this operation, we have made two choices. First, we used the Euclidean inner product to turn vectors into row vectors. Then we used the determinant to identify $\mathrm{Alt}^2\mathbb{R}^3$ with $\mathbb{R}^3$. From (57), we observe that

$$\det(E,V,W) = E \cdot (V \times W), \tag{62}$$

which will be useful later.

4. Exterior differentiation

Let $\Omega \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ be open sets, and let $\Psi : \Omega \to U$ be a smooth map. Recall that the push-forward of a vector field $V : \Omega \to \mathbb{R}^n$ under Ψ is the vector field $\tilde{V} : U \to \mathbb{R}^m$ whose components are given by

$$\tilde{V}_i = \sum_{k=1}^n \frac{\partial \tilde{x}_i}{\partial x_k} V_k, \qquad (i = 1,\ldots,m), \tag{63}$$

where $\tilde{x} = \Psi(x)$. Recall also that the pull-back of a 1-form $\tilde{\alpha} : U \to \mathbb{R}^{m*}$ under the mapping Ψ is a 1-form $\alpha : \Omega \to \mathbb{R}^{n*}$ with components

$$\alpha_k = \sum_{i=1}^m \frac{\partial \tilde{x}_i}{\partial x_k} \tilde{\alpha}_i, \qquad (k = 1,\ldots,n). \tag{64}$$

Now let $\tilde{\eta} : U \to \mathrm{Alt}^2\mathbb{R}^m$ be a 2-form on U. Then we define a 2-form η on Ω by requiring

$$\eta(x)(V(x),W(x)) = \tilde{\eta}(\tilde{x})(\tilde{V}(\tilde{x}),\tilde{W}(\tilde{x})), \qquad (x \in \Omega), \tag{65}$$

for all vector fields $V, W : \Omega \to \mathbb{R}^n$. By choosing V and W to be the standard basis vectors of $\mathbb{R}^n$, we get the transformation law

$$\eta_{k\ell} = \eta(e_k, e_\ell) = \sum_{i,j=1}^{m} \frac{\partial \tilde{x}_i}{\partial x_k} \frac{\partial \tilde{x}_j}{\partial x_\ell} \tilde{\eta}_{ij}, \qquad (k, \ell = 1, \ldots, n), \tag{66}$$

The 2-form η is called the *pull-back* of $\tilde{\eta}$ under the mapping Ψ, and denoted by $\eta = \Psi^* \tilde{\eta}$.

Next, let us differentiate (64), to see how derivatives change under Ψ. We have

$$\frac{\partial \alpha_k}{\partial x_\ell} = \sum_{i,j=1}^{m} \frac{\partial \tilde{x}_i}{\partial x_k} \frac{\partial \tilde{x}_j}{\partial x_\ell} \frac{\partial \tilde{\alpha}_i}{\partial \tilde{x}_j} + \sum_{i=1}^{m} \frac{\partial^2 \tilde{x}_i}{\partial x_k \partial x_\ell} \tilde{\alpha}_i. \tag{67}$$

Comparing this with (66), we notice the following differences.

- The transformation law (67) has an extra term that depends on the second derivatives or $\tilde{x} = \tilde{x}(x)$, which makes it very different from (66).
- The derivative $\frac{\partial \alpha_k}{\partial x_\ell}$ is in general not antisymmetric in the indices k and ℓ.

However, if we antisymmetrize (67), then we get

$$\begin{aligned} \frac{\partial \alpha_k}{\partial x_\ell} - \frac{\partial \alpha_\ell}{\partial x_k} &= \sum_{i,j=1}^{m} \frac{\partial \tilde{x}_i}{\partial x_k} \frac{\partial \tilde{x}_j}{\partial x_\ell} \frac{\partial \tilde{\alpha}_i}{\partial \tilde{x}_j} - \sum_{i,j=1}^{m} \frac{\partial \tilde{x}_i}{\partial x_\ell} \frac{\partial \tilde{x}_j}{\partial x_k} \frac{\partial \tilde{\alpha}_i}{\partial \tilde{x}_j} \\ &= \sum_{i,j=1}^{m} \frac{\partial \tilde{x}_i}{\partial x_k} \frac{\partial \tilde{x}_j}{\partial x_\ell} \frac{\partial \tilde{\alpha}_i}{\partial \tilde{x}_j} - \sum_{i,j=1}^{m} \frac{\partial \tilde{x}_j}{\partial x_\ell} \frac{\partial \tilde{x}_i}{\partial x_k} \frac{\partial \tilde{\alpha}_j}{\partial \tilde{x}_i} \\ &= \sum_{i,j=1}^{m} \frac{\partial \tilde{x}_i}{\partial x_k} \frac{\partial \tilde{x}_j}{\partial x_\ell} \Big(\frac{\partial \tilde{\alpha}_i}{\partial \tilde{x}_j} - \frac{\partial \tilde{\alpha}_j}{\partial \tilde{x}_i} \Big), \end{aligned} \tag{68}$$

which shows that $\frac{\partial \alpha_k}{\partial x_\ell} - \frac{\partial \alpha_\ell}{\partial x_k}$ transforms like a 2-form.

Definition 4.1. Given a 1-form α, its *exterior derivative* $\mathrm{d}\alpha$ is the 2-form whose components are defined by

$$(\mathrm{d}\alpha)_{\ell k} = \frac{\partial \alpha_k}{\partial x_\ell} - \frac{\partial \alpha_\ell}{\partial x_k}. \tag{69}$$

For a 2-form ω, we define

$$(\mathrm{d}\omega)_{ijk} = \frac{\partial \omega_{jk}}{\partial x_i} - \frac{\partial \omega_{ik}}{\partial x_j} + \frac{\partial \omega_{ij}}{\partial x_k}. \tag{70}$$

Remark 4.2. Note that (68) implies that $\mathrm{d}\Psi_* \alpha = \Psi_* \mathrm{d}\alpha$. In particular, the definition (69) is coordinate invariant.

Exercise 4.3. (a) Show that (70) is a coordinate invariant definition.
(b) Extend Definition 4.1 to k-forms with $k \geq 3$.

Remark 4.4. (a) If α is a 1-form in 2-dimensions, then $\mathrm{d}\alpha$ has only one independent degree of freedom, which is

$$(\mathrm{d}\alpha)_{12} = \frac{\partial \alpha_2}{\partial x_1} - \frac{\partial \alpha_1}{\partial x_2}. \tag{71}$$

(b) Similarly, if ω is a 2-form in 3-dimensions, then $\mathrm{d}\omega$ has only one independent degree of freedom, which is

$$(\mathrm{d}\omega)_{123} = \frac{\partial \omega_{23}}{\partial x_1} - \frac{\partial \omega_{13}}{\partial x_2} + \frac{\partial \omega_{12}}{\partial x_3}. \tag{72}$$

Under the usual identification $F = (\omega_{23}, -\omega_{13}, \omega_{12})$, cf. (57), we get

$$\mathrm{div} F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}, \tag{73}$$

which is called the *divergence* of the vector field F.

(c) If α is a 1-form in 3-dimensions, then $\mathrm{d}\alpha$ has only three independent degrees of freedom, which are

$$(\mathrm{d}\alpha)_{23} = \frac{\partial \alpha_3}{\partial x_2} - \frac{\partial \alpha_2}{\partial x_3}, \qquad (\mathrm{d}\alpha)_{13} = \frac{\partial \alpha_3}{\partial x_1} - \frac{\partial \alpha_1}{\partial x_3}, \qquad (\mathrm{d}\alpha)_{12} = \frac{\partial \alpha_2}{\partial x_1} - \frac{\partial \alpha_1}{\partial x_2}. \tag{74}$$

Given a vector field E, the vector field associated to $\mathrm{d}(E^{\mathsf{T}})$ under (57) is called the *curl* of E. More explicitly, we can write

$$\mathrm{curl} E = \Big(\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3}, \frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}, \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}\Big). \tag{75}$$

Remark 4.5. It is not accidental that we have been using the notation $\mathrm{d}u$ for the derivative Du of a scalar function u. By convention, scalar functions are called 0*-forms*, and the exterior derivative on 0-forms is simply the derivative of the function, so we can reinterpret $\mathrm{d}u$ as the exterior derivative of the 0-form u. Under the identification between $\mathbb{R}^n$ and $\mathbb{R}^{n*}$ induced by the Euclidean inner product, the exterior derivative of a 0-form u is also called the *gradient* of the function u. Namely, we define the *gradient* of a differentiable function $u : \Omega \to \mathbb{R}$ to be

$$\mathrm{grad}\, u = (\mathrm{d}u)^{\mathsf{T}} = \Big(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\Big), \tag{76}$$

which is a vector field in Ω.

5. Stokes' theorem

Let $\gamma : [a, b] \to \mathbb{R}^n$ be an oriented curve, and let $u : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Then by the fundamental theorem of calculus, we have

$$\int_\gamma \mathrm{d}u = \int_a^b \mathrm{d}u(\gamma(t))\gamma'(t)\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\big(u(\gamma(t))\big)\mathrm{d}t = u(\gamma(b)) - u(\gamma(a)). \tag{77}$$

It turns out that this result admits the following higher dimensional generalization

$$\int_M \mathrm{d}\omega = (-1)^\kappa \int_{\partial M} \omega, \tag{78}$$

where ω is a $(k-1)$-form in $\mathbb{R}^n$, M is a k-dimensional oriented manifold, and ∂M denotes the boundary of M, with the sign $(-1)^\kappa$ depending on how we defined the orientation of ∂M. By flipping the orientation of ∂M if necessary, we can always ensure that $(-1)^\kappa = 1$. This result is known as *Stokes' theorem*, and contains as special cases several classical theorems of vector calculus. Note that in order to fit (77) into (78), the right hand side of (77) needs to be interpreted (or defined) as the "integral" of the 0-form u over the boundary of the oriented curve γ, which consists of 2 points $\gamma(a)$ and $\gamma(b)$, with $\gamma(a)$ having a negative orientation.

Remark 5.1. At this point, we need to clarify what ∂M is and in particular how to specify its orientation. Let $\tilde{M}$ be a k-dimensional oriented manifold in $\mathbb{R}^n$, and let $\Psi_\alpha : \Omega_\alpha \to \tilde{M}$ be coordinate charts covering $\tilde{M}$. Without loss of generality, we assume that all Ω_α are contained in the rectangle $[0,1]^{k-1} \times [-1, 1]$. Suppose that $M \subset \tilde{M}$ is a subset, satisfying

$$M = \bigcup_\alpha \Psi_\alpha(\Omega_\alpha \cap Q), \tag{79}$$

where $Q = [0, 1]^k$. We call M a k-dimensional *oriented manifold with boundary*, and define

$$\partial M = \bigcup_\alpha \Psi_\alpha(\Omega_\alpha \cap Q'), \tag{80}$$

as its boundary, with $Q' = [0,1]^{k-1} \times \{0\}$. The boundary ∂M itself is a $(k-1)$-dimensional oriented manifold, where the charts $\psi_\alpha : \Sigma_\alpha \to \partial M$ are given by $\psi_\alpha(\xi) = \Psi_\alpha(\xi, 0)$ and $\Sigma_\alpha \subset [0,1]^{k-1}$ satisfies $\Omega_\alpha \cap Q' = \Sigma_\alpha \times \{0\}$.

The basic cases of the Stokes theorem that illustrates all the main ideas are the cases $k = 2$ and $k = 3$. With the help of a partition of unity, we can reduce (78) to the case where there exist an index α and a closed bounded set $K \subset \Omega_\alpha$, such that $\omega(x) = 0$ whenever $x \in M \setminus \Psi_\alpha(K)$. By using the coordinate invariance, we have

$$\int_M \mathrm{d}\omega = \int_Q \Psi_\alpha^* \mathrm{d}\omega = \int_Q \mathrm{d}\Psi_\alpha^* \omega. \tag{81}$$

Putting $\tilde\omega = \Psi_\alpha^* \omega$, for $k = 2$, we compute

$$\begin{aligned}\int_Q \mathrm{d}\tilde\omega &= \int_0^1 \int_0^1 \Big(\frac{\partial \tilde\omega_2}{\partial x_1} - \frac{\partial \tilde\omega_1}{\partial x_2}\Big)\mathrm{d}x_1 \mathrm{d}x_2 = \int_0^1 \int_0^1 \frac{\partial \tilde\omega_2}{\partial x_1}\mathrm{d}x_1 \mathrm{d}x_2 - \int_0^1 \int_0^1 \frac{\partial \tilde\omega_1}{\partial x_2}\mathrm{d}x_1 \mathrm{d}x_2 \\ &= \int_0^1 \big(\tilde\omega_2(1, x_2) - \tilde\omega_2(0, x_2)\big)\mathrm{d}x_2 - \int_0^1 \big(\tilde\omega_1(x_1, 1) - \tilde\omega_1(x_1, 0)\big)\mathrm{d}x_1 \\ &= \int_0^1 \tilde\omega_1(x_1, 0)\mathrm{d}x_1 = \int_{Q'} \tilde\omega,\end{aligned} \tag{82}$$

yielding

$$\int_M \mathrm{d}\omega = \int_{Q'} \Psi_\alpha^* \omega = \int_{\partial M} \omega. \tag{83}$$

Hence for $k = 2$, we get the positive sign $(-1)^\kappa = 1$ in the Stokes theorem (78), when the orientation of ∂M is as in Remark 5.1. On the other hand, for $k = 3$, we have

$$\begin{aligned}\int_Q \mathrm{d}\tilde\omega &= \iiint_Q \Big(\frac{\partial \tilde\omega_{23}}{\partial x_1} - \frac{\partial \tilde\omega_{13}}{\partial x_2} + \frac{\partial \tilde\omega_{12}}{\partial x_3}\Big)\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \\ &= -\int_0^1 \int_0^1 \tilde\omega_{12}(x_1, x_2, 0)\mathrm{d}x_1 \mathrm{d}x_2 \\ &= -\int_{Q'} \tilde\omega,\end{aligned} \tag{84}$$

yielding

$$\int_M \mathrm{d}\omega = -\int_{Q'} \Psi_\alpha^* \omega = -\int_{\partial M} \omega. \tag{85}$$

This means that for $k = 3$, we need to equip ∂M with the orientation opposite of that given in Remark 5.1, in order to get the positive sign $(-1)^\kappa = 1$ in the Stokes theorem (78).

Exercise 5.2. Prove the Stokes theorem (78) for general k.

Remark 5.3 (Green's theorem)**.** Applying the Stokes theorem to a 1-form $\alpha = E\mathrm{d}x + F\mathrm{d}y$ in 2-dimensions and a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, we get *Green's theorem*

$$\iint_\Omega \Big(\frac{\partial F}{\partial x} - \frac{\partial E}{\partial y}\Big)\mathrm{d}x\mathrm{d}y = \int_{\partial\Omega} E\mathrm{d}x + F\mathrm{d}y, \tag{86}$$

where $\partial\Omega$ is positively oriented, meaning that the domain Ω is "in the left hand side" of $\partial\Omega$. Here $\iint$ is a traditional (if not a bit old fashioned) notation for double integrals.

Remark 5.4 (Divergence theorem)**.** Similarly, if ω is a 2-form in 3-dimensions, represented by the vector field $F = (\omega_{23}, -\omega_{13}, \omega_{12})$, cf. (57), and if $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, then

$$\int_\Omega \mathrm{d}\omega = \iiint_\Omega \mathrm{div} F \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{87}$$

On the other hand, if $\Psi : U \to \partial\Omega$ is a local parameterization of $\partial\Omega$, then

$$\begin{aligned}\int_{\Psi(U)} \omega &= \int_U \omega(\Psi(\xi))(D\Psi(\xi))\mathrm{d}^2\xi = \int_U \det(F(\xi), \partial_1\Psi(\xi), \partial_2\Psi(\xi))\mathrm{d}^2\xi \\ &= \int_U F(\xi) \cdot \big(\partial_1\Psi(\xi) \times \partial_2\Psi(\xi)\big)\mathrm{d}^2\xi.\end{aligned} \tag{88}$$

Note that since we have used the Stokes theorem (78) with the positive sign $(-1)^\kappa = 1$, the parameterization Ψ must be oriented so that the vector $\partial_1\Psi(\xi) \times \partial_2\Psi(\xi)$ is pointing *outward* with respect to the domain Ω. This suggests the notation

$$\iint_{\partial\Omega} F \cdot \mathrm{d}S := \int_{\partial\Omega} \omega, \tag{89}$$

where $\mathrm{d}S$ symbolizes an outward normal vector to the surface $\partial\Omega$. Hence we recover the divergence (or Gauss-Ostrogradsky) theorem

$$\iiint_\Omega \mathrm{div} F \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{\partial\Omega} F \cdot \mathrm{d}S. \tag{90}$$

Remark 5.5 (Kelvin-Stokes theorem)**.** If α is a 1-form in 3-dimensions, represented by the vector field $F = \alpha^{\mathsf{T}}$, and $S \subset \mathbb{R}^3$ is an oriented surface with smooth boundary, then we have

$$\iint_S \mathrm{curl} F \cdot \mathrm{d}S = \int_S \mathrm{d}\alpha = \int_{\partial S} \alpha = \int_{\partial S} F \cdot \mathrm{d}\ell. \tag{91}$$

Here the orientation of ∂S should be so that, seen from the tip of the normal vector $\mathrm{d}S$, the surface S is in the left side of ∂S.