

SOLUTIONS TO SELECTED PROBLEMS

1. Let $M \in \mathbb{R}^{n \times n}$ be a symmetric matrix, and let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ be smooth functions satisfying

$$M\gamma''(t) = -(D\phi(\gamma(t)))^T, \quad t \in \mathbb{R}.$$

Show that the quantity

$$E(t) = \frac{1}{2}(\gamma'(t))^T M \gamma'(t) + \phi(\gamma(t)),$$

is independent of $t \in \mathbb{R}$, i.e., E is a *conserved* along $\gamma(t)$.

Solution: For any differentiable functions $A : \mathbb{R} \rightarrow \mathbb{R}^{n \times m}$ and $B : \mathbb{R} \rightarrow \mathbb{R}^{m \times \ell}$, we have $(AB)' = A'B + AB'$, because

$$(AB)'_{ij} = \sum_{k=1}^m (A_{ik}B_{kj})' = \sum_{k=1}^m (A'_{ik}B_{kj} + A_{ik}B'_{kj}) = (A'B)_{ij} + (AB')_{ij},$$

for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, \ell\}$. Now we apply this to differentiate E :

$$E' = \frac{1}{2}(\gamma'')^T M \gamma' + \frac{1}{2}(\gamma')^T M \gamma'' + D\phi(\gamma)\gamma',$$

where all functions of t are understood to be evaluated at $t \in \mathbb{R}$. Since M is symmetric, we have $(\gamma')^T M \gamma'' = (\gamma'')^T M \gamma'$ (exercise!), which yields

$$E' = (\gamma'')^T M \gamma' + D\phi(\gamma)\gamma' = ((\gamma'')^T M + D\phi(\gamma)) \gamma'.$$

Now, the equation $M\gamma''(t) = -(D\phi(\gamma(t)))^T$ becomes $(\gamma'')^T M = -D\phi(\gamma)$ upon transposing, which means that $E'(t) = 0$ for all $t \in \mathbb{R}$.

2. Let $A : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ be a smooth, matrix valued function of a single variable. Assuming that $A(t)$ is an invertible diagonal matrix for all $t \in \mathbb{R}$, show that

$$(\det A)' = \det(A) \operatorname{tr}(A^{-1}A').$$

Solution: Let us write

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & a_{nn} \end{pmatrix},$$

so that we have $\det A = a_{11}a_{22} \cdots a_{nn}$. Since A is invertible, $\det A \neq 0$, and hence $a_{kk} \neq 0$ for any k . Then we compute

$$\begin{aligned} (\det A)' &= (a_{11}a_{22} \cdots a_{nn})' = a'_{11}a_{22} \cdots a_{nn} + a_{11}a'_{22} \cdots a_{nn} + \dots + a_{11}a_{22} \cdots a'_{nn} \\ &= a_{11}a_{22} \cdots a_{nn} \left(\frac{a'_{11}}{a_{11}} + \frac{a'_{22}}{a_{22}} + \dots + \frac{a'_{nn}}{a_{nn}} \right) \end{aligned}$$

We can recognize that the expression between the brackets is the trace of some matrix. What is that matrix? If we recall

$$A' = \begin{pmatrix} a'_{11} & 0 & \dots & 0 \\ 0 & a'_{22} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & a'_{nn} \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1/a_{11} & 0 & \dots & 0 \\ 0 & 1/a_{22} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1/a_{nn} \end{pmatrix},$$

we immediately see that

$$A^{-1}A' = \begin{pmatrix} a'_{11}/a_{11} & 0 & \dots & 0 \\ 0 & a'_{22}/a_{22} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & a'_{nn}/a_{nn} \end{pmatrix}.$$

Therefore we conclude

$$\begin{aligned} (\det A)' &= a_{11}a_{22} \cdots a_{nn} \left(\frac{a'_{11}}{a_{11}} + \frac{a'_{22}}{a_{22}} + \dots + \frac{a'_{nn}}{a_{nn}} \right) \\ &= \det(A) \left(\frac{a'_{11}}{a_{11}} + \frac{a'_{22}}{a_{22}} + \dots + \frac{a'_{nn}}{a_{nn}} \right) \\ &= \det(A) \operatorname{tr}(A^{-1}A'). \end{aligned}$$

3. Show that the product rule $(fg)' = f'g + fg'$ for single variable functions is a special case of the multivariable chain rule.

Solution: Let the function $\Psi : \mathbb{R} \rightarrow \mathbb{R}^2$ given by $\Psi(t) = (f(t), g(t))$, and let $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $u(x, y) = xy$. Then we see that $f(t)g(t) = u(\Psi(t)) = (u \circ \Psi)(t)$, and thus

$$[f(t)g(t)]' = Du(\Psi(t))\Psi'(t).$$

We have

$$Du(x, y) = (\partial_x u \quad \partial_y u) = (y \quad x), \quad \text{and} \quad \Psi'(t) = \begin{pmatrix} f'(t) \\ g'(t) \end{pmatrix},$$

leading to

$$[f(t)g(t)]' = (g(t) \quad f(t)) \begin{pmatrix} f'(t) \\ g'(t) \end{pmatrix} = g(t)f'(t) + f(t)g'(t),$$

which is of course the product rule for $(fg)'$.

4. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function satisfying $f(tx) = tf(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$. Show that if f is differentiable at 0, then f is linear, i.e., $f(x) = b^T x$ for some $b \in \mathbb{R}^n$.

Solution: By Caratheodory's criterion, there exists a function $F : \mathbb{R}^n \rightarrow \mathbb{R}^{1 \times n}$, continuous at 0, such that

$$f(x) = f(0) + F(x)x, \quad x \in \mathbb{R}^n.$$

Since $f(0) = tf(0)$ for all $t > 0$, for instance taking $t = 2$, we infer that $f(0) = 0$. Thus we have $f(x) = F(x)x$ for all $x \in \mathbb{R}^n$. Now the property $f(tx) = tf(x)$ becomes $tF(tx)x = tf(x)$, that is,

$$f(x) = F(tx)x, \quad t > 0, \quad x \in \mathbb{R}^n.$$

As F is continuous at 0, $F(tx)x$ tends to $F(0)x$ as $t \rightarrow 0$. With $b = F(0)^T$, this yields

$$f(x) = F(0)x = b^T x, \quad x \in \mathbb{R}^n.$$

5. Let $\phi : (0, \infty) \rightarrow \mathbb{R}$ be a smooth function, and let $u(x) = \phi(|x|)$ for $x \in \mathbb{R}^n \setminus \{0\}$. Show that

$$\Delta u(x) = \partial_1^2 u(x) + \dots + \partial_n^2 u(x) = \phi''(|x|) + \frac{n-1}{|x|} \phi'(|x|),$$

for $x \in \mathbb{R}^n \setminus \{0\}$.

Solution: For $x \in \mathbb{R}^n \setminus \{0\}$, we have

$$\partial_k u(x) = \phi'(|x|) \partial_k |x|, \quad \text{and} \quad \partial_k^2 u(x) = \phi''(|x|) (\partial_k |x|)^2 + \phi'(|x|) \partial_k^2 |x|.$$

To compute $\partial_k |x|$, write $|x| = \sqrt{x_1^2 + \dots + x_n^2}$, and use the chain rule, as

$$\partial_k |x| = \partial_k \sqrt{x_1^2 + \dots + x_n^2} = \frac{2x_k}{2\sqrt{x_1^2 + \dots + x_n^2}} = \frac{x_k}{|x|}.$$

Furthermore, we infer

$$\partial_k^2 |x| = \partial_k \frac{x_k}{|x|} = \frac{|x| - x_k \partial_k |x|}{|x|^2} = \frac{|x|^2 - x_k^2}{|x|^3},$$

and substituting these into the expression for $\partial_k^2 u$, we get

$$\partial_k^2 u(x) = \frac{x_k^2}{|x|^2} \phi''(|x|) + \frac{|x|^2 - x_k^2}{|x|^3} \phi'(|x|).$$

Finally, summing over k , we conclude that

$$\begin{aligned} \Delta u(x) &= \sum_{k=1}^n \partial_k^2 u(x) = \frac{x_1^2 + \dots + x_n^2}{|x|^2} \phi''(|x|) + \frac{n|x|^2 - x_1^2 - \dots - x_n^2}{|x|^3} \phi'(|x|) \\ &= \phi''(|x|) + \frac{n-1}{|x|} \phi'(|x|). \end{aligned}$$

6. Show that the equation $z^3 + ze^{xy} - xy = 0$ defines z as a function of $(x, y) \in \mathbb{R}^2$. Is $z = z(x, y)$ differentiable?

Solution: First of all, we observe that x and y appear only in combination xy , so that upon substitution $t = xy$, the equation becomes $z^3 + ze^t - t = 0$. We introduce the notation $f(z, t) = z^3 + ze^t - t$, and formulate our plan as follows.

- (i) For any $t \in \mathbb{R}$, there exists $z \in \mathbb{R}$ such that $f(t, z) = 0$.
- (ii) The solution z of $f(t, z) = 0$ is unique, i.e., $f(t, z) = f(t, w) = 0$ implies $z = w$.
- (iii) The dependence $z = z(t)$ is differentiable.

In what follows, we will carry out this program.

(i) Note that the implicit function theorem would be able to give only a solution $z = z(t)$ for t in some small interval at a time. It also does not guarantee that $z = z(t)$ is the only possible solution even when we consider t in that small interval. Since we need to show that $f(z, t) = 0$ defines a function of $t \in \mathbb{R}$, we need to use a different method. Thus fix an arbitrary $t \in \mathbb{R}$, and let $g(z) = f(z, t)$. We want to show that $g(a) \leq 0$ and $g(b) \geq 0$, for $a = \min\{0, -1, t\}$ and $b = \max\{0, t\}$. The intermediate value theorem would then guarantee that there exists $z \in [a, b]$ such that $g(z) = 0$. We have

$$g(b) = \max\{0, t^3\} + \max\{0, t\}e^t - t = \begin{cases} t(t^2 + e^t - 1) & \text{for } t \geq 0 \\ -t & \text{for } t < 0, \end{cases}$$

and so $g(b) \geq 0$ either way. On the other hand, we have

$$g(a) = \min\{0, -1, t^3\} + \min\{0, -1, t\}e^t - t = \begin{cases} -t & \text{for } t \geq 0 \\ -1 - e^t - t & \text{for } -1 \leq t < 0 \\ t(t^2 + e^t - 1) & \text{for } t < -1, \end{cases}$$

As $-1 - t \leq 0$ in the second case, and $t^2 - 1 > 0$ in the third case, we conclude that $g(a) \leq 0$ in all cases. By the intermediate value theorem, there exists $z \in [a, b]$ such that $g(z) = 0$, that is, for any $t \in \mathbb{R}$, there exists $z \in \mathbb{R}$ such that $f(z, t) = 0$.

(ii) Now we need to show that for any fixed $t \in \mathbb{R}$, the equation $f(z, t) = 0$ has a unique solution $z \in \mathbb{R}$. We shall use here a monotonicity argument. Thus fix an arbitrary $t \in \mathbb{R}$, and let $g(z) = f(z, t)$ as before. We have

$$g'(z) = 3z^2 + e^t > 0, \quad z \in \mathbb{R}.$$

Suppose that $g(z) = g(w) = 0$ for some $z < w$. Then by the mean value theorem (or by Rolle's theorem), there exists $\xi \in (z, w)$ such that $g'(\xi) = 0$. But we have just proven that $g' \neq 0$ everywhere. This means that $g(z) = g(w) = 0$ cannot hold for different values $z \neq w$, and hence $g(z) = 0$ has a unique solution. To conclude, the relation $f(z, t) = 0$ defines a function $z = z(t)$ of $t \in \mathbb{R}$.

(iii) Finally, for each $t_* \in \mathbb{R}$, we apply the implicit function theorem to $f(z, t) = 0$ around the point $(z_*, t_*) = (z(t_*), t_*)$, to infer that there is a continuously differentiable

function $z = h(t)$ for t varying in a small interval I centred at t_* , such that $f(h(t), t) = 0$ for all $t \in I$. However, as $f(z, t) = 0$ is uniquely solvable for z , this function $h(t)$ must agree with our previously defined function $z(t)$ for $t \in I$. Therefore $z(t)$ is continuously differentiable at $t = t_*$. Since $t_* \in \mathbb{R}$ is arbitrary, we conclude that $z(t)$ is continuously differentiable everywhere. Note that the application of the implicit function theorem is justified by the fact that $f(z, t)$ is continuously differentiable function of $(z, t) \in \mathbb{R}^2$, and that $\partial_z f(z, t) = 3z^2 + e^t \neq 0$ everywhere.

7. Show that the equation $x + y + z + \cos(xyz) = 0$ can be solved for $z = z(x, y)$ in an open set containing the origin. Find the plane tangent to $z = z(x, y)$ at the origin.

Solution: Define $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$\phi(x, y, z) = x + y + z + \cos(xyz).$$

It is easy to check that ϕ is continuously differentiable in \mathbb{R}^3 . For $x = y = 0$, we have $\phi(0, 0, z) = z + \cos(0 \cdot z) = z + 1$, and so we need to take $p = (0, 0, -1)$ in order for $\phi(p) = 0$ to hold. For the implicit function theorem to be applicable to $\phi = 0$ at p and to be able to give us a function $z = z(x, y)$, what remains is to verify that the z -derivative of ϕ is nonzero at p . We have

$$\partial_z \phi(x, y, z) = 1 - xy \sin(xyz),$$

and hence

$$\partial_z \phi(0, 0, -1) = 1 \neq 0.$$

Therefore, with the purpose of writing z as a function of (x, y) , the implicit function theorem can be applied to $\phi(x, y, z) = 0$ at p , which yields the existence of $\delta > 0$, and a continuously differentiable function $h : (-\delta, \delta)^2 \rightarrow \mathbb{R}$, such that $\phi(x, y, h(x, y)) = 0$ for all $(x, y) \in (-\delta, \delta)^2$. Note that $h(0, 0) = -1$.

To find the equation of the tangent plane, we need to compute the partial derivatives of h at the origin. From the implicit function theorem, we infer

$$\partial_x h(0, 0) = -\frac{\partial_x \phi(p)}{\partial_z \phi(p)} = -1, \quad \partial_y h(0, 0) = -\frac{\partial_y \phi(p)}{\partial_z \phi(p)} = -1.$$

Recall that

$$h(x, y) = h(0, 0) + \partial_x h(0, 0)x + \partial_y h(0, 0)y + o(\max\{|x|, |y|\}),$$

and recognize that the linear part of this is the equation of the tangent plane of the surface at p . By substituting the relevant values to the equation, we conclude that the equation of the tangent plane is

$$z = -1 - x - y, \quad \text{or} \quad x + y + z + 1 = 0.$$

8. The point $p = (1, -1, 1)$ lies on the surfaces

$$x^3(y^3 + z^3) = 0, \quad (x - y)^3 - z^2 = 7.$$

Show that, in an open set containing p , the curve of intersection of the surfaces can be parameterized by x , that is, the curve can be described by a system of equations of the form $\{y = f(x), z = g(x)\}$.

Solution: Define $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by

$$\phi(x, y, z) = \begin{pmatrix} x^3(y^3 + z^3) \\ (x - y)^3 - z^2 - 7 \end{pmatrix}.$$

It is easy to check that $\phi(p) = 0$, and also that ϕ is continuously differentiable in \mathbb{R}^3 . In order to apply the implicit function theorem, we now need to verify that a certain derivative of ϕ at p is invertible. Thus let us call $\beta = (y, z)$, and compute

$$D_{\beta}\phi(x, y, z) = \begin{pmatrix} 3x^3y^2 & 3x^3z^2 \\ -3(x - y)^2 & -2z^2 \end{pmatrix},$$

and hence

$$\det D_{\beta}\phi(p) = \det \begin{pmatrix} 3 & 3 \\ -12 & -2 \end{pmatrix} = 3 \cdot (-2) - 3 \cdot (-12) = 30 \neq 0.$$

Therefore, with the purpose of writing (y, z) as a function of x , the implicit function theorem can be applied to $\phi(x, y, z) = 0$ at p , which yields the existence of $\delta > 0$, and a continuously differentiable function $\gamma : (1 - \delta, 1 + \delta) \rightarrow \mathbb{R}^2$, such that $\phi(x, \gamma(x)) = 0$ for all $x \in (1 - \delta, 1 + \delta)$. Note that the notation $\phi(x, \gamma(x))$ simply means $\phi(x, \gamma_1(x), \gamma_2(x))$, where γ_1 and γ_2 are the components of the function γ .

Aside: The velocity vector γ' can be computed from

$$\gamma' = -(D_{\beta}\phi)^{-1}\partial_x\phi,$$

where

$$[D_{\beta}\phi(x, y, z)]^{-1} = \begin{pmatrix} -2z^2 & -3x^3z^2 \\ 3(x - y)^2 & 3x^3y^2 \end{pmatrix},$$

and

$$\partial_x\phi(x, y, z) = \begin{pmatrix} 3x^2(y^3 + z^3) \\ 3(x - y)^2 \end{pmatrix}.$$

Note that in all function evaluations we must have $x \in (1 - \delta, 1 + \delta)$ and $(y, z) = \gamma(x)$.