FUNCTIONS OF A REAL VARIABLE

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Abstract. We review some of the important concepts of single variable calculus. The discussions are centred around building and establishing the main properties of the elementary functions such as x^a, $\exp x$, $\log x$, $\sin x$, and $\arctan x$. We start with an axiomatic treatment of real numbers. If the reader is willing to assume the basic properties of real numbers, then they can skip Section 1 in its entirety, and simply skim through Section 2.

Contents

1. Ordered fields

In this and the following sections we state (one version of) the real number axioms, and derive the most fundamental properties of real numbers from them.

The axioms can be thought of as the minimal requirements any logical system that claims itself to be the real number system must satisfy. It is possible to construct such systems by using other systems such as the natural number system, but we will not consider those here.

The real number axioms can be divided into *three groups*. The set of real numbers is denoted by $\mathbb{R}$, and for any two real numbers $a, b \in \mathbb{R}$, their sum $a + b \in \mathbb{R}$ and product $a \cdot b \in \mathbb{R}$ are well defined. In other words, we assume the existence of two binary operations. Then the first group of axioms requires that $\mathbb{R}$ be a *field* with respect to these operations. This basically means that the addition and multiplication satisfy commutativity, associativity, and distributivity laws, that the numbers 0 and 1 exist and are distinct, and finally, that subtraction and division (by any nonzero number) can be defined.

The second group of axioms adds more structure to the field, and demands that $\mathbb{R}$ be an *ordered field*. This means that any two real numbers $a, b \in \mathbb{R}$ satisfy one (and only one) of $a < b$, $a = b$, and $b < a$, that the binary relation $<$ is transitive, and that $a < b$ is preserved under addition of any number and under multiplication by a positive number. In this section,

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we will consider some of the consequences of the first two groups of axioms. The third group, that is an axiom on completeness of $\mathbb{R}$, will be considered in the next section.

Axiom 1 (Ordered field)**.** *(a) The addition operation satisfies the following properties.*

(i) $a, b \in \mathbb{R}$ then $a + b = b + a$.
(ii) $a, b, c \in \mathbb{R}$ then $(a + b) + c = a + (b + c)$.
(iii) There exists an element $0 \in \mathbb{R}$ such that $a + 0 = a$ for each $a \in \mathbb{R}$.
(iv) For any $a \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $x + a = 0$.

(b) The multiplication operation satisfies the following properties.

(i) $a, b \in \mathbb{R}$ then $a \cdot b = b \cdot a$.
(ii) $a, b, c \in \mathbb{R}$ then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
(iii) $a, b, c \in \mathbb{R}$ then $a \cdot (b + c) = a \cdot b + a \cdot c$.
(iv) There exists an element $1 \in \mathbb{R}$ with $1 \neq 0$ such that $a \cdot 1 = a$ for each $a \in \mathbb{R}$.
(v) For any $a \in \mathbb{R}$ not equal to 0, there exists $x \in \mathbb{R}$ such that $x \cdot a = 1$.

(c) The binary relation $<$ satisfies the following properties.

(i) $a, b \in \mathbb{R}$ then one and only one of the following is true: $a < b$, $a = b$, or $b < a$.
(ii) If $a, b, c \in \mathbb{R}$ satisfy $a < b$ and $b < c$ then $a < c$.
(iii) If $a, b, c \in \mathbb{R}$ and $a < b$ then $a + c < b + c$.
(iv) If $a, b \in \mathbb{R}$ satisfy $a > 0$ and $b > 0$ then $a \cdot b > 0$.

Remark 1.1. The relation $a > b$ is defined as $b < a$. Similarly, $a \leq b$ means $a < b$ or $a = b$, and $a \geq b$ means $a > b$ or $a = b$. The product $a \cdot b$ can be written simply as ab.

Definition 1.2. Given $a, b, c \in \mathbb{R}$ with $c \neq 0$, we define the *difference* $a - b \in \mathbb{R}$ and the *quotient* $\frac{a}{c} \in \mathbb{R}$ as the solutions to the equations $b + x = a$ and $cx = a$, respectively. Then b) and f) of the following theorem guarantee that these concepts are well defined. We also define the *opposite number* (or negation) $-a = 0 - a$ and the *reciprocal* (or inverse) $a^{-1} = \frac{1}{a}$.

Theorem 1.3 (Algebraic properties)**.** *For $a, b, c \in \mathbb{R}$, we have the following.*

a) $a + b = a$ implies $b = 0$ (uniqueness of 0).
b) $a + b = a + c$ implies $b = c$ (subtraction of a).
c) $0 \cdot a = 0$.
d) $ab = 0$ implies $a = 0$ or $b = 0$.
e) $ab = a$ and $a \neq 0$ imply $b = 1$ (uniqueness of 1).
f) $ab = ac$ and $a \neq 0$ imply $b = c$ (division by a).

Proof. a) By Axiom (a)(iii), we have $b = 0 + b$. Now let x be such that $x + a = 0$. Then the assumed property of b implies that

$$b = 0 + b = (x + a) + b = x + (a + b) = x + a = 0. \tag{1}$$

b) Let x be such that $x + a = 0$. Then we have

$$b = 0 + b = (x + a) + b = x + (a + b) = x + (a + c) = (x + a) + c = 0 + c = c. \tag{2}$$

c) Using the distributivity axiom, we observe that

$$a = a \cdot 1 = a \cdot (0 + 1) = a \cdot 0 + a \cdot 1 = a \cdot 0 + a, \tag{3}$$

and part a) of the current theorem (i.e., uniqueness of 0) finishes the job.
d) Suppose that $a \neq 0$, and let x be such that $xa = 1$. Then we infer

$$b = 1 \cdot b = (xa) \cdot b = x \cdot (ab) = x \cdot 0 = 0, \tag{4}$$

where in the last step we have used part c) of the current theorem. □

Exercise 1.4. Prove e) and f) of the preceding theorem.

Exercise 1.5. Prove the following.

(a) $a - b = a + (-b)$ for $a, b \in \mathbb{R}$.
(b) $-(ab) = (-a) \cdot b$ for $a, b \in \mathbb{R}$. In particular, $-a = (-1) \cdot a$ and $(-a) \cdot (-a) = a \cdot a$.
(c) $(ab)^{-1} = a^{-1}b^{-1}$ for $a, b \in \mathbb{R} \setminus \{0\}$.

Theorem 1.6 (Order properties)**.** *For $a, b, c, d \in \mathbb{R}$, we have the following.*
a) $a > 0$ is equivalent to $-a < 0$.
b) $a < b$ is equivalent to $b - a > 0$.
c) $b < c$ and $a > 0$ imply $ab < ac$.
d) $b < c$ and $a < 0$ imply $ab > ac$.
e) $a \neq 0$ implies $a \cdot a > 0$.
f) $0 < a < b$ and $ac = bd > 0$ imply $0 < d < c$.

Proof. a) If $a > 0$, then Axiom (c)(iii) yields $0 = a + (-a) > 0 + (-a) = -a$. Similarly, if $-a < 0$, then $0 = -a + a < 0 + a = a$.

b) By Axiom (c)(iii), $a < b$ implies $0 = a + (-a) < b + (-a) = b - a$. Similarly, in the other direction, $b - a > 0$ implies $b = b - a + a > 0 + a = a$.

c) By the preceding paragraph, $b < c$ is the same as $c - b > 0$, and then Axiom (c)(iv) implies that $ac - ab = a(c - b) > 0$, or $ac > ab$.

f) Suppose that $c < 0$. Then $-c > 0$, and hence $-(ac) = a \cdot (-c) > 0$, or $ac < 0$. Since $c = 0$ would imply that $ac = 0$, we conclude that $c > 0$, and similarly, that $d > 0$. Now assume $d \geq c$. Then $ac < bc \leq bd$, contradicting $ac = bd$. Hence $d < c$. □

Exercise 1.7. Prove d) and e) of the preceding theorem.

Definition 1.8. We introduce the following notations.

- $(a, \infty) = \{x \in \mathbb{R} : x > a\}$, $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$.
- $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$, $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$.
- $(a, b) = (a, \infty) \cap (-\infty, b)$.
- $[a, b] = [a, \infty) \cap (-\infty, b]$.
- $(a, b] = (a, \infty) \cap (-\infty, b]$, $[a, b) = [a, \infty) \cap (-\infty, b)$.

Exercise 1.9. Show that $\mathbb{R} \setminus (a, b) = (-\infty, a] \cup [b, \infty)$ and $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$.

Definition 1.10. For $a \in \mathbb{R}$, we define its *modulus* (or *absolute value*) by

$$|a| = \begin{cases} a, & \text{for } a \geq 0, \\ -a, & \text{for } a < 0. \end{cases} \tag{5}$$

Remark 1.11. One can think of $|a|$ as the distance between the points a and 0 on the real line. The distance between two points a and b is defined to be $|a - b|$.

Exercise 1.12. Prove the following.
(a) $|a| \geq 0$ for any $a \in \mathbb{R}$, and $|a| = 0$ if and only if $a = 0$.
(b) $|a^{-1}| = \frac{1}{|a|}$ for $a \neq 0$.
(c) $|ab| = |a||b|$ for $a, b \in \mathbb{R}$.
(d) $|a + b| \leq |a| + |b|$ for $a, b \in \mathbb{R}$.
(e) $||a| - |b|| \leq |a - b|$ for $a, b \in \mathbb{R}$.

Having established the basic algebraic properties of $\mathbb{R}$, we now turn to identifying some important subsets of $\mathbb{R}$, such as the integers and the rational numbers. We have $0 \in \mathbb{R}$ and $1 \in \mathbb{R}$ with $0 \neq 1$, thus obviously $-1 \neq 0$. Moreover, $1 < 0$ would imply $-1 > 0$, and hence $-1 = (-1) \cdot 1 < (-1) \cdot 0 = 0$ by Theorem 1.6(c). Since $-1 > 0$ and $-1 < 0$ cannot be simultaneously true, we conclude that $0 < 1$. Now, the number $2 = 1 + 1$ is distinct from each of -1, 0, and 1, because $1 + 1 > 1 + 0 = 1$. We continue this process as $3 = 2 + 1$, $4 = 3 + 1$,

..., constructing larger and larger numbers. The numbers obtained are of course the natural numbers. To make it more precise, we define an *inductive subset* of $\mathbb{R}$ as a subset $A \subset \mathbb{R}$ with the property that $1 \in A$ and that $x \in A$ implies $x+1 \in A$. Then we define the set of *natural numbers* $\mathbb{N}$ as the intersection of all inductive subsets of $\mathbb{R}$. Since inductivity is preserved under intersections (Exercise!), $\mathbb{N}$ is the smallest inductive subset of $\mathbb{R}$. This property is the basis of *proof by induction*, and of *definition by recurrence*. We illustrate these by a couple of examples.

Let $n \in \mathbb{N}$, and let $x_k \in \mathbb{R}$ for each $k \in \mathbb{N}$ with $k \leq n$. In other words, $x : \mathbb{N}_n \to \mathbb{R}$ is a function with $\mathbb{N}_n = \{k \in \mathbb{N} : k \leq n\}$ and $x(k) = x_k$. Such a function is called a *finite sequence* of real numbers, an *n-tuple*, or an *n-vector*, and denoted by $(x_1, x_2, \ldots, x_n)$. Sets of the form $\{x_k : k \in \mathbb{N}_n\}$ are called *finite sets*. For an arbitrary set $B \subset \mathbb{R}$, a number $m \in B$ is called a *maximum of* B if $b \leq m$ for all $b \in B$. If B admits a maximum, the maximum must be unique, because the existence of two maxima $m, m' \in B$ implies that $m \leq m'$ and $m' \leq m$.

Lemma 1.13. *Any nonempty finite set of real numbers admits a maximum.*

Proof. Let $A \subset \mathbb{N}$ be the set of $n \in \mathbb{N}$ with the property that any set of the form $\{x_k : k \in \mathbb{N}_n\}$ admits a maximum. Clearly, $1 \in A$ since given any set $\{x_1\}$, we can check that $m = x_1$ is a maximum of $\{x_1\}$. Suppose that $n \in A$, that is, suppose that any set of the form $\{x_k : k \in \mathbb{N}_n\}$ admits a maximum. Let $B = \{x_1, x_2, \ldots, x_{n+1}\}$ be given. Then $\{x_1, x_2, \ldots, x_n\}$ admits a maximum, which we denote by x_i. Now if $x_i > x_{n+1}$, we set $m = x_i$, and if $x_i \leq x_{n+1}$, we set $m = x_{n+1}$. Since $m \geq x_{n+1}$ and $m \geq x_i \geq x_k$ for any $k \leq n$, we see that m is the maximum of B. All this shows that $1 \in A$, and that $n \in A$ implies $n+1 \in A$. Thus A is an inductive set, meaning that $\mathbb{N} \subset A$. $\square$

Exercise 1.14. Introduce the concept of minimum, and show that any nonempty set $A \subset \mathbb{N}$ admits a minimum.

Definition 1.15. For $a \in \mathbb{R}$ and $n \in \mathbb{N}$, the *n-th power* of a is the real number $a^n \in \mathbb{R}$ defined by the recurrent formula

$$a^n = \begin{cases} a, & \text{for } n = 1, \\ a \cdot a^{n-1}, & \text{for } n > 1. \end{cases} \tag{6}$$

Fix $a \in \mathbb{R}$, and let $A \subset \mathbb{R}$ be the set of $x \in \mathbb{R}$ for which the power a^x is uniquely defined by the preceding definition. Since the definition restricts itself to $x \in \mathbb{N}$, we have $A \subset \mathbb{N}$. It is clear that $1 \in A$. Moreover, if $x \in A$ then $x+1 \in A$, because $x+1 \in \mathbb{N}$ and a^{x+1} is defined as $a \cdot a^x$. Hence A is inductive, meaning that $\mathbb{N} \subset A$.

Exercise 1.16. (a) Informally, the factorials are defined by $1! = 1$, $2! = 1 \cdot 2$, $3! = 1 \cdot 2 \cdot 3$, etc. Give a definition of $n!$ by using a recurrent formula.

(b) Given $x_1, x_2, \ldots, x_n$, informally, we have

$$\sum_{i=1}^{1} x_i = x_1, \qquad \sum_{i=1}^{2} x_i = x_1 + x_2, \qquad \sum_{i=1}^{3} x_i = x_1 + x_2 + x_3, \qquad \text{etc.} \tag{7}$$

Define $\sum_{i=1}^{n} x_i$ by using a recurrent formula.

(c) Prove the *binomial formula*

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}, \tag{8}$$

where $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, and $\binom{n}{i} = \frac{n!}{i!(n-i)!}$, with the conventions $x^0 = 1$ and $0! = 1$.

(d) Prove the formula

$$a^n - b^n = (a-b) \cdot \sum_{i=0}^{n-1} a^i b^{n-1-i}, \tag{9}$$

for $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$.

The natural numbers are closed under addition and multiplication, but the equations $a+x = b$ and $ax = b$ are not always solvable. By adjoining the negative integers and 0 to $\mathbb{N}$, the equation $a + x = b$ can be solved. We let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and define the set of *integers* by $\mathbb{Z} = \mathbb{N}_0 \cup \{-n : n \in \mathbb{N}\}$. Then $\mathbb{Z}$ is closed under addition, subtraction, and multiplication, i.e., $\mathbb{Z}$ is a ring. We extend the power function to integers as follows.

$$a^n = \begin{cases} a^n, & \text{for } n \in \mathbb{N}, \\ 1, & \text{for } n = 0, \\ \frac{1}{a^{-n}}, & \text{for } n < 0, \end{cases} \tag{10}$$

where $a \in \mathbb{R}$ and $n \in \mathbb{Z}$. Note that negative powers are not defined for $a = 0$.

Exercise 1.17. Prove the following.

(a) $a^n a^m = a^{n+m}$, $(a^n)^m = a^{nm}$, $(ab)^n = a^n b^n$.
(b) $|a^n| = |a|^n$
(c) If $0 < a < b$ then $a^n < b^n$ for $n > 0$ and $a^n > b^n$ for $n < 0$.
(d) If $m < n$ then $a^m < a^n$ for $a > 1$ and $a^m > a^n$ for $0 < a < 1$.

Exercise 1.18. Let $A \subset \mathbb{Z}$ be bounded above, in the sense that there is $b \in \mathbb{Z}$ such that $a < b$ for all $a \in A$. Show that A admits a maximum.

Going further, the *rational numbers* are $\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$, which is closed under addition, subtraction, multiplication, and division by nonzero numbers. It is almost immediate that $\mathbb{Q}$ satisfies the ordered field axioms (Axiom 1) considered in this section. Therefore Axiom 1 is not stringent enough to differentiate $\mathbb{R}$ from $\mathbb{Q}$. What we need is an additional axiom, that will be considered in the next section.

2. The real number continuum

The following notions will be used in the statement of the anticipated axiom.

Definition 2.1. Given a set $A \subset \mathbb{R}$, a number $s \in \mathbb{R}$ is called the *least upper bound* or the *supremum* of A, and written as

$$\sup A = s, \tag{11}$$

if $a \le s$ for all $a \in A$, and for any $c < s$ there is $a \in A$ with $a > c$.

Example 2.2. For $A = (0, 1]$ and $B = (0, 1)$, we have $\sup A = \sup B = 1$. The set $(1, \infty)$ does not have a supremum.

Definition 2.3. A subset $A \subset \mathbb{R}$ is called *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a < b$ for all $a \in A$.

Example 2.4. The set $\{x - x^2 : x \in \mathbb{R}\}$ is bounded above, while $\{n^2 : n \in \mathbb{N}\}$ is not.

Now we are ready to state the only remaining axiom for real numbers.

Axiom 2 (Continuum property, the least upper bound property)**.** *If $A \subset \mathbb{R}$ is nonempty and bounded above, then there exists $s \in \mathbb{R}$ such that $s = \sup A$.*

Remark 2.5. Axiom 1 and Axiom 2 together pinpoint the real numbers completely, in the sense that if $\mathbb{R}$ and $\mathbb{R}'$ both satisfy the aforementioned axioms, then there exists an order preserving field isomorphism between $\mathbb{R}$ and $\mathbb{R}'$. The basic reason behind this result is that each of $\mathbb{R}$ and $\mathbb{R}'$ contains a copy of $\mathbb{Q}$, and these two copies of $\mathbb{Q}$ can be naturally identified. Furthermore, the only field isomorphism from $\mathbb{R}$ onto itself is the identity map.

Remark 2.6. Foreshadowing the detailed study that will occupy the rest of this section, here we want to include an informal discussion on the continuum property. In a certain sense, the continuum property combines two important characteristics of the real numbers. Firstly, it states that no single real number is larger than all the natural numbers, or equivalently, that there is no real number between 0 and all positive rational numbers. Loosely speaking, how large a real number can be is comparable to how large a natural number can be. This property is called the *Archimedean property* of the real numbers. The second characteristic of the real numbers that is embedded in the continuum property is basically the requirement that the real numbers have "no gaps" between them. This property is called the *Cauchy completeness* or the *Cauchy property*. Recall that by the ordered field axioms, $\mathbb{R}$ must contain $\mathbb{Q}$ as a subset. Then, the Cauchy property asserts that real numbers are used to fill in the gaps in $\mathbb{Q}$, and the Archimedean property states that this "filling in" process is done efficiently, without adding any unnecessary elements.

Theorem 2.7 (Archimedean property)**.** *a) Given any $x \in \mathbb{R}$, there is $n \in \mathbb{N}$ such that $x < n$. b) Moreover, given any $a \in \mathbb{R}$ and $b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$.*

Proof. a) Suppose that there is $x \in \mathbb{R}$ satisfying $n < x$ for all $n \in \mathbb{N}$. Then by the least upper bound property, the supremum of $\mathbb{N}$ exists, i.e., $s = \sup \mathbb{N} \in \mathbb{R}$. Now, by definition of supremum, there is $m \in \mathbb{N}$ such that $m > s - \frac{1}{2}$. This leads to contradiction, since $m+1 \in \mathbb{N}$ and $m + 1 > s + \frac{1}{2}$.

b) Let $\delta = b - a$, and let $n \in \mathbb{N}$ be such that $n > \frac{1}{\delta}$. The set $G = \{k \in \mathbb{Z} : k < bn\}$ is by construction bounded above, and hence $m = \max G$ exists (Exercise 1.18). Let $q = \frac{m}{n}$. Since $m \in G$, we have $m < bn$, meaning that $q < b$. Anticipating a contradiction, suppose that $q \leq a$. This would mean that $m \leq an$, or $m + 1 \leq an + 1 = (a + \frac{1}{n})n < (a + \delta)n = bn$. Therefore, we have $m + 1 \in G$, which contradicts the maximality of m. □

In the remainder of this section, we explore further consequences of the continuum property. We start by introducing the notion of infinite sequences.

Definition 2.8. A *real number sequence* is a function $x : \mathbb{N} \to \mathbb{R}$, which is usually written as $\{x_n\} = \{x_1, x_2, \ldots\}$, with $x_n = x(n)$. We say that a sequence $\{x_n\}$ *converges* to $x \in \mathbb{R}$, if for any given $\varepsilon > 0$, there exists an index N such that

$$|x_n - x| \leq \varepsilon \qquad \text{for all } n \geq N. \tag{12}$$

If $\{x_n\}$ converges to x, we write

$$\lim_{n\to\infty} x_n = x, \qquad \text{or} \qquad \lim x_n = x, \qquad \text{or} \qquad x_n \to x \quad \text{as} \quad n \to \infty. \tag{13}$$

In some contexts, the sequence $\{x_n\}$ is identified with the set $\{x_n : n \in \mathbb{N}\}$. For instance, $\{x_n\} \subset \Omega$ with some $\Omega \subset \mathbb{R}$ means that $x_n \in \Omega$ for all n. Moreover, often times one considers sequences such as $\{a_0, a_1, \ldots\}$, whose indices start with $n = 0$.

Definition 2.9. If a sequence does not converge to any number, then the sequence is said to *diverge*. A special type of divergence occurs when the divergence is caused by growth, rather than oscillation. More precisely, we say that $\{a_n\}$ *diverges to* ∞, and write

$$\lim_{n\to\infty} a_n = \infty, \qquad \text{or} \qquad \lim a_n = \infty, \qquad \text{or} \qquad a_n \to \infty \quad \text{as} \quad n \to \infty, \tag{14}$$

if for any given number M, the sequence is eventually larger than M, that is, only finitely many terms of $\{a_n\}$ stay in $(-\infty, M)$. Divergence to $-\infty$ can be defined in an obvious manner.

Example 2.10. (a) A *constant sequence* is a sequence whose terms are all equal to each other. An example is the sequence $\{1, 1, 1, \ldots\}$ whose n-th term is $a_n = 1$.

(b) An *arithmetic progression* is a sequence whose n-th term satisfies the formula $a_n = kn+b$, where k and b are constants. For example, $\{5, 8, 11, 14, \ldots\}$ is an arithmetic progression, with $k = 3$ and $b = 2$. Note that we have chosen $b = 2$ so that the first term "5" corresponds to $n = 1$ in the formula $a_n = kn + b$. However, this is not necessary. We could have taken $b = 5$, and specified that we start the sequence at the index $n = 0$.

(c) Similarly, a *geometric progression* is a sequence with the general term $a_n = bq^n$, where q and b are constants. For example, $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ is a geometric progression, with $q = \frac{1}{2}$ and $b = 1$, and with the understanding that the first term of the sequence corresponds to $n = 0$ in the formula $a_n = bq^n$.

(d) The sequence $\{1, 4, 9, 16, \ldots\}$, with $a_n = n^2$, is called the sequence of *square numbers*.

(e) The *Fibonacci sequence* is $\{0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$, where each term (except the first two) is the sum of the two terms immediately preceding it. In other words, we have the recurrent formula $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

Example 2.11. (a) We want to show that if $x > 1$ then $\lim x^n = \infty$. To this end, we start with the binomial formula

$$(a + b)^n = a^n + na^{n-1}b + \ldots + nab^{n-1} + b^n, \tag{15}$$

and substitute $a = 1$ and $b = x - 1 > 0$, which gives

$$x^n = (1 + b)^n = 1 + nb + \ldots + nb^{n-1} + b^n \geq 1 + nb, \tag{16}$$

since all the terms of the sum are positive. Now, given a large number M, choose N so that $1 + Nb > M$. For instance, $N > M/b$ would be sufficient. Then for all $n > N$, we would have

$$x^n \geq 1 + nb > 1 + Nb > M. \tag{17}$$

This means, by definition, that $\lim x^n = \infty$.

(b) Let us show that if $0 < x < 1$ then $\lim x^n = 0$. So let $\varepsilon > 0$ be given. We know that $\lim y^n = \infty$, where $y = \frac{1}{x}$. By definition, for any given M, there exists an index N such that $y^n > M$ for all $n > N$. Let N be such an index that corresponds to the choice $M = \frac{1}{\varepsilon}$. Then, for all $n > N$, we have $|x^n| = x^n = \frac{1}{y^n} < \frac{1}{M} = \varepsilon$. This shows that $\lim x^n = 0$.

Remark 2.12. The limit of a sequence depends only on "behaviour at $n = \infty$", in the sense that if $\lim a_n = a$, then after an arbitrary modification (or removal) of finitely many terms of $\{a_n\}$, we would still have $\lim a_n = a$. To change the limit behaviour one would have to modify infinitely many terms.

Exercise 2.13. Prove the following.

(a) If $-1 < x < 1$ then $\lim x^n = 0$.

(b) If $x > 1$ then $\lim \frac{x^n}{n} = \infty$.

(c) If $x > 1$ and $a \in \mathbb{N}$ then $\lim \frac{x^n}{n^a} = \infty$ and $\lim n^a x^{-n} = 0$.

(d) $\lim \frac{x^n}{n!} = 0$ for any $x \in \mathbb{R}$.

Exercise 2.14. Prove the following.

(a) $\lim a_n = a$ if and only if $\lim |a_n - a| = 0$.

(b) If $\lim a_n = \infty$ and $b_n \geq a_n$ for all n then $\lim b_n = \infty$.

(c) If $\{a_n\}$ converges then it is bounded, i.e., there is $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all n.

Theorem 2.15 (Monotone convergence)**.** *Let $\{x_n\} \subset \mathbb{R}$ be a sequence that is nondecreasing and bounded above, in the sense that*

$$x_n \leq x_{n+1} \leq M \qquad \text{for each } n, \tag{18}$$

and with some constant $M \in \mathbb{R}$. Then there is $x \leq M$ such that $x_n \to x$ as $n \to \infty$.

Proof. Let $x = \sup\{x_n\}$, and let $\varepsilon > 0$. Then there is N such that $x - \varepsilon < x_N$. Since $\{x_n\}$ is nondecreasing, we have $x - \varepsilon < x_n \leq x$ for all $n \geq N$. This means that $\{x_n\}$ converges to x. The inequality $x \leq M$ is obvious because x is the *least* upper bound of $\{x_n\}$. □

Exercise 2.16. Show that every nonincreasing, bounded below sequence converges.

Theorem 2.17 (Nested intervals principle)**.** *Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that*

$$[a_0, b_0] \supset [a_1, b_1] \supset \ldots \supset [a_n, b_n] \supset \ldots, \tag{19}$$

with $a_n < b_n$ for each $n \in \mathbb{N}$. Then the intersection $\bigcap_{n\in\mathbb{N}}[a_n, b_n]$ is nonempty. In addition, if $b_n - a_n \to 0$ as $n \to \infty$, then $\bigcap_{n\in\mathbb{N}}[a_n, b_n]$ consists of a single point.

Proof. We have

$$a_0 \leq a_1 \leq \ldots \leq a_n < b_n \leq \ldots \leq b_1 \leq b_0, \tag{20}$$

which makes it clear that $\{a_n\}$ is nondecreasing and $\{b_n\}$ is nonincreasing. Since both of these sequences are bounded, by the monotone convergence theorem (Theorem 2.15), there exist a and b such that $a_m \to a$ and $b_m \to b$ as $m \to \infty$. Given any m, we have $a_m \leq a_n < b_n \leq b_m$ whenever $n \geq m$. This implies that a and b are both in the interval $[a_m, b_m]$ for any m. In addition, if $b_n - a_n \to 0$, we must have $a = b$. □

Theorem 2.18 (Bolzano-Weierstrass)**.** *Let $\{x_n\} \subset [a, b]$. Then there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ that converges to some point $x \in [a, b]$.*

Proof. Let us subdivide the interval $[a, b]$ into two subintervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$. Then at least one of these subintervals must contain infinitely many terms from the sequence $\{x_n\}$. Pick one such subinterval, and call it $[a_1, b_1]$. Obviously, we have $b_1 - a_1 = \frac{b-a}{2}$. Now we subdivide $[a_1, b_1]$ into two halves $[a_1, \frac{a_1+b_1}{2}]$ and $[\frac{a_1+b_1}{2}, b_1]$, one of which must contain infinitely many terms from $\{x_n\}$. Recall that interval $[a_2, b_2]$. Of course, we have $b_2 - a_2 = \frac{b-a}{4}$. We continue this process indefinitely, and obtain a sequence of intervals

$$[a, b] \supset [a_1, b_1] \supset \ldots \supset [a_m, b_m] \supset \ldots, \tag{21}$$

with each $[a_m, b_m]$ containing infinitely many terms from the sequence $\{x_n\}$, and satisfying $b_m - a_m = 2^{-m}(b - a)$. Now the nested intervals principle (Theorem 2.17) implies that there exists $a \in \mathbb{R}$ such that $a \in [a_n, b_n]$ for all $n \in \mathbb{N}$. Hence $|a - a_n| \leq 2^{-n}(b - a)$ for all n.

Let $n_0 = 1$, and for $k \in \mathbb{N}$, let n_k be an index such that $n_k > n_{k-1}$ and that $x_{n_k} \in [a_k, b_k]$. Such n_k exists since $[a_k, b_k]$ contains infinitely many terms from $\{x_n\}$. Then we have

$$|x_{n_k} - a| \leq |x_{n_k} - a_k| + |a_k - a| \leq 2^{-k}(b - a) + 2^{-k}(b - a), \tag{22}$$

which shows that the sequence $\{x_{n_k}\}$ converges to a. □

Theorem 2.19 (Cauchy's criterion)**.** *Let $\{x_n\} \subset \mathbb{R}$ be a Cauchy sequence, in the sense that*

$$|x_n - x_m| \to 0, \qquad \text{as} \quad \min\{n, m\} \to \infty. \tag{23}$$

Then $\{x_n\}$ is convergent.

Proof. Let N be such that $|x_n - x_N| \leq 1$ for all $n \geq N$. Then we have

$$|x_n| \leq |x_N| + 1 \qquad \text{for all} \quad n \geq N, \tag{24}$$

and therefore

$$|x_n| \leq \max\{|x_1|, \ldots, |x_{N-1}|, |x_N| + 1\} \qquad \text{for all} \quad n, \tag{25}$$

meaning that $\{x_n\}$ is bounded. By the Bolzano-Weierstrass theorem (Theorem 2.18), there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ that converges to some point $x \in \mathbb{R}$.

So far we only have shown that a subsequence of $\{x_n\}$ converges to x. Now we will show that the whole sequence $\{x_n\}$ indeed converges to x. To this end, let $\varepsilon > 0$, and let N be such that $|x_n - x_m| \leq \varepsilon$ for all $n \geq N$ and $m \geq N$. Moreover, let $k \geq N$ be large enough that $|x_{n_k} - x| \leq \varepsilon$. Then for $m \geq N$, we have

$$|x_m - x| \leq |x_m - x_{n_k}| + |x_{n_k} - x| \leq 2\varepsilon, \tag{26}$$

which shows that the entire sequence $\{x_n\}$ converges to x. □

3. Limits and continuity

In this section, we will study continuous functions. Intuitively, a continuous function f sends nearby points to nearby points, i.e, if x is close to y then $f(x)$ is close to $f(y)$. This intuition can be made precise by saying that if a sequence x_k converges to x, then $f(x_k)$ converges to $f(x)$. Before doing that, we prove a preliminary result to handle limits. Recall from Definition 2.8 that a sequence $\{x_n\}$ converges to $x \in \mathbb{R}$, if for any given $\varepsilon > 0$, there exists an index N such that

$$|x_n - x| \leq \varepsilon \qquad \text{for all } n \geq N. \tag{27}$$

If $\{x_n\}$ converges to x, we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Theorem 3.1. *Let $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Then the following are true.*

a) $a_n \pm b_n \to a \pm b$ as $n \to \infty$.
b) $a_n b_n \to ab$ as $n \to \infty$.
c) If $a \neq 0$, then $a_n = 0$ for only finitely many indices n, and after the removal of those zero terms from the sequence $\{a_n\}$, we have $\lim \frac{1}{a_n} = \frac{1}{a}$.
d) $a_n \leq b_n$ implies $a \leq b$.
e) If $\{x_n\}$ is a sequence satisfying $a_n \leq x_n \leq b_n$ for all n, and if $a = b$, then $\lim x_n = a$.

Proof. b) We have $a_n b_n - ab = a_n(b_n - b) + (a_n - a)b$, and so

$$|a_n b_n - ab| \leq |a_n||b_n - b| + |a_n - a||b| \qquad \text{for all } n. \tag{28}$$

By choosing n large enough, we can make $|b_n - b|$ and $|a_n - a||b|$ as small as we want. The question is if can do the same for the product $|a_n||b_n - b|$. We claim that $\{a_n\}$ is bounded, i.e., there is $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all n. Indeed, since $\{a_n\}$ converges to a, taking $\varepsilon = 1$ in the definition of convergence, there exists an index N such that $|a_n - a| \leq 1$ for all $n \geq N$. Hence $|a_n| = |a_n - a + a| \leq |a_n - a| + |a| \leq 1 + |a|$ for all $n \geq N$, and if we take $M = \max\{|a| + 1, |a_1|, |a_2|, \ldots, |a_{N-1}|\}$, then $|a_n| \leq M$ for all n. Now (28) yields

$$|a_n b_n - ab| \leq M|b_n - b| + |b||a_n - a| \qquad \text{for all } n. \tag{29}$$

Let $\varepsilon > 0$ be given. Let N' be such that $M|b_n - b| \leq \frac{\varepsilon}{2}$ for all $n \geq N'$, and let N'' be such that $|b||a_n - a| \leq \frac{\varepsilon}{2}$ for all $n \geq N''$. This is possible since $b_n \to b$ and $a_n \to a$ as $n \to \infty$. Now we set $N = \max\{N', N''\}$. Then we have

$$|a_n b_n - ab| \leq M|b_n - b| + |b||a_n - a| \leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon \qquad \text{for all } n \geq N, \tag{30}$$

which means that $a_n b_n \to ab$ as $n \to \infty$.

e) Let $\varepsilon > 0$ be given. Let N' be such that $|a_n - a| \leq \varepsilon$ for all $n \geq N'$, and let N'' be such that $|b_n - b| \leq \varepsilon$ for all $n \geq N''$. We set $N = \max\{N', N''\}$. Then we have

$$x_n - a = x_n - b \leq b_n - b \leq |b_n - b| \leq \varepsilon \qquad \text{for all } n \geq N, \tag{31}$$

and

$$a - x_n \leq a - a_n \leq |a_n - a| \leq \varepsilon \qquad \text{for all } n \geq N, \tag{32}$$

which imply

$$|x_n - a| = \max\{x_n - a, a - x_n\} \le \varepsilon \qquad \text{for all } n \ge N. \tag{33}$$

By definition, this means that $x_n \to a$ as $n \to \infty$. $\square$

Exercise 3.2. Prove a), c) and d) of the preceding theorem.

Example 3.3. Let us try to compute the limit of $\frac{3n+1}{2n+5}$ as $n \to \infty$. If we approach naively, we get

$$\lim_{n\to\infty} \frac{3n+1}{2n+5} = \lim_{n\to\infty} (3n+1) \cdot \lim_{n\to\infty} \frac{1}{2n+5} = \infty \cdot 0, \tag{34}$$

which is nonsense. The error is in the first step, where we attempt to apply Theorem 3.1b). This is not justified, because $\lim(3n+1)$ does not exist. A correct way to proceed is to write

$$\frac{3n+1}{2n+5} = \frac{3+\frac{1}{n}}{2+\frac{5}{n}} = (3+\frac{1}{n}) \cdot \frac{1}{2+\frac{5}{n}}, \tag{35}$$

and to note that

- $\lim(3+\frac{1}{n}) = 3$ and $\lim(2+\frac{5}{n}) = 2$ by Theorem 3.1a).
- Hence $\lim \frac{1}{2+\frac{5}{n}} = \frac{1}{2}$ by Theorem 3.1c).
- Therefore $\lim(3+\frac{1}{n}) \cdot \frac{1}{2+\frac{5}{n}} = 3 \cdot \frac{1}{2} = \frac{3}{2}$ by Theorem 3.1b).

This process is usually written as

$$\lim \frac{3n+1}{2n+5} = \lim \frac{3+\frac{1}{n}}{2+\frac{5}{n}} = \frac{3+\lim\frac{1}{n}}{2+\lim\frac{5}{n}} = \frac{3}{2}. \tag{36}$$

Exercise 3.4. Prove the following.

(a) If $\lim a_n = \infty$ and $\lim \frac{b_n}{a_n} = 0$, then $\lim(a_n \pm b_n) = \infty$.
(b) $\lim a_n = \infty$ if and only if $\lim \frac{1}{a_n} = 0$ and $\{a_n\}$ is eventually positive.
(c) If $\lim a_n = 0$ and $\{b_n\}$ is bounded, then $\lim(a_n b_n) = 0$. Recall that a sequence $\{a_n\}$ is *bounded* if there exists a number M such that $|a_n| \le M$ for all n.

We define continuous functions as the ones that send convergent sequences to convergent sequences. This is sometimes called the *sequential criterion of continuity.*

Definition 3.5. Let $K \subset \mathbb{R}$ be a set. A function $f : K \to \mathbb{R}$ is called *continuous at* $x \in K$ if $f(x_n) \to f(x)$ as $n \to \infty$ for every sequence $\{x_n\} \subset K$ converging to x.

Example 3.6. (a) Let $c \in \mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(y) = c$ for $y \in \mathbb{R}$. Then f is continuous at every point $x \in \mathbb{R}$, since for any sequence $\{x_n\} \subset \mathbb{R}$ converging to x, we have $f(x_n) = c \to c = f(x)$ as $n \to \infty$.
(b) Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(y) = y$ for $y \in \mathbb{R}$. Then f is continuous at every point $x \in \mathbb{R}$, because given any sequence $\{x_n\} \subset \mathbb{R}$ converging to x, we have $f(x_n) = x_n \to x = f(x)$ as $n \to \infty$.

We now confirm an intuitive property of continuous functions, namely that if f is continuous at x then for all points y close to x the value $f(y)$ is close to $f(x)$.

Lemma 3.7. *Let $K \subset \mathbb{R}$ be a set. Then $f : K \to \mathbb{R}$ is continuous at $x \in K$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $y \in (x-\delta, x+\delta) \cap K$ implies $f(x)-\varepsilon < f(y) < f(x)+\varepsilon$.*

Proof. Let f be continuous at $x \in K$, and let $\varepsilon > 0$. Suppose that no such δ exists, i.e., that there is a sequence $\{x_n\} \subset K$ converging to x, with $|f(x_n) - f(x)| \ge \varepsilon$ for all n. Since f is continuous at x, we have $f(x_n) \to f(x)$ as $n \to \infty$. In particular, there is an index N such that $|f(x) - f(x_N)| < \varepsilon$, which is a contradiction.

In the other direction, assume that $x \in K$ and that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $y \in (x - \delta, x + \delta) \cap K$ implies $f(x) - \varepsilon < f(y) < f(x) + \varepsilon$. In particular, there is a positive sequence $\{\delta_n\}$ such that $y \in (x - \delta_n, x + \delta_n) \cap K$ implies $f(x) - \frac{1}{n} < f(y) < f(x) + \frac{1}{n}$. Let $\{x_m\} \subset K$ be a sequence converging to x. Then we can choose a sequence of indices $m_1, m_2, \ldots$, such that $x_m \in (x-\delta_n, x+\delta_n)$ for all $m \geq m_n$, that is, $f(x)-\frac{1}{n} < f(x_m) < f(x)+\frac{1}{n}$ for all $m \geq m_n$. This shows that $f(x_m) \to f(x)$ as $m \to \infty$. □

The following result shows that continuity is a local property.

Lemma 3.8. *Let $f : K \to \mathbb{R}$ with $K \subset \mathbb{R}$, and let $g = f|_{(a,b)\cap K}$ for some (a, b). Then f is continuous at $x \in (a, b) \cap K$ if and only if g is continuous at x.*

Proof. Suppose that f is continuous at $x \in (a, b) \cap K$. Then by definition, $f(x_n) \to f(x)$ as $n \to \infty$ for every sequence $\{x_n\} \subset K$ converging to x. In particular, this is true for every sequence $\{x_n\} \subset (a, b) \cap K$ converging to x. Since $g = f$ on $(a, b) \cap K$, g is continuous at x.

Now suppose that g is continuous at $x \in (a, b) \cap K$, i.e., that $f(x_n) = g(x_n) \to g(x) = f(x)$ as $n \to \infty$ for every sequence $\{x_n\} \subset (a, b) \cap K$ converging to x. Let $\{x_n\} \subset K$ be a sequence converging to x. Then there exists N such that $x_n \in (a, b) \cap K$ for all $n \geq N$, and hence the sequence $\{f(x_N), f(x_{N+1}), \ldots\}$ converges to $f(x)$. This is the same as saying that the full sequence $\{f(x_n)\}$ converges to $f(x)$, meaning that f is continuous at x. □

Our next step is to combine known continuous functions to create new continuous functions.

Definition 3.9. Given two functions $f, g : \Omega \to \mathbb{R}$, with $\Omega \subset \mathbb{R}$, we define their *sum*, *difference*, *product*, and *quotient* by

$$(f \pm g)(x) = f(x) \pm g(x), \qquad (fg)(x) = f(x)g(x), \qquad \text{and} \qquad (\frac{f}{g})(x) = \frac{f(x)}{g(x)}, \tag{37}$$

for $x \in \Omega$, where for the quotient definition we assume that g does not vanish anywhere in Ω. Furthermore, we define the function $|f|$ by

$$|f|(z) = |f(z)|, \qquad \text{for} \quad x \in \Omega. \tag{38}$$

Theorem 3.10. *Let $\Omega \subset \mathbb{R}$, and let $f, g : \Omega \to \mathbb{R}$ be functions continuous at $x \in \Omega$. Then the following are true.*

a) The sum and difference $f \pm g$, the product fg, and the modulus $|f|$ are all continuous at x.
b) The function $\frac{1}{f}$ is continuous at x, provided that $f(x) \neq 0$.
c) Suppose that $U \subset \mathbb{R}$ is a set satisfying $g(\Omega) \subset U$, the latter meaning that $y \in \Omega$ implies $g(y) \in U$. Let $F : U \to \mathbb{R}$ be a function continuous at $g(x)$. Then the composition $F \circ g : \Omega \to \mathbb{R}$, defined by $(F \circ g)(y) = F(g(y))$, is continuous at x.

Proof. The results are immediate from the definition of continuity. For instance, let us prove that fg is continuous at x. Thus let $\{x_n\} \subset \Omega$ be an arbitrary sequence converging to x. Then $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$ as $n \to \infty$, and Theorem 3.1 gives $f(x_n)g(x_n) \to f(x)g(x)$ as $n \to \infty$. Hence fg is continuous at x. □

Exercise 3.11. Prove b) and c) of the preceding theorem.

Definition 3.12. A function $f : \Omega \to \mathbb{R}$ is called *continuous in* Ω, if f is continuous at each point of Ω. The set of all continuous functions in Ω is denoted by $\mathscr{C}(\Omega)$.

Exercise 3.13. Show that if $f, g \in \mathscr{C}(\Omega)$, then $f \pm g, fg, |f| \in \mathscr{C}(\Omega)$.

Example 3.14. (a) Recall from Example 3.6 that the constant function $f(x) = c$ (where $c \in \mathbb{R}$) and the identity map $f(x) = x$ are continuous in $\mathbb{R}$. Then by Theorem 3.10a), any monomial $f(x) = ax^n$ with a constant $a \in \mathbb{R}$, is continuous in $\mathbb{R}$, since we can write $ax^n = a \cdot x \cdots x$. Applying Theorem 3.10a) again, we conclude that any *polynomial*

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n, \tag{39}$$

where $a_0, \ldots a_n \in \mathbb{R}$ are the coefficients, as a function $p : \mathbb{R} \to \mathbb{R}$, is continuous in $\mathbb{R}$.

(b) Let p and q be polynomials, and let $Z = \{x \in \mathbb{R} : q(x) = 0\}$ be the set of real roots of q. Then by Theorem 3.10b), the function $r : \mathbb{R} \setminus Z \to \mathbb{R}$ given by $r(x) = \frac{p(x)}{q(x)}$ is continuous in $\mathbb{R} \setminus Z$. The functions of this form are called *rational functions*. For instance, $f(x) = \frac{x^2+1}{x-1}$ is a rational function defined for $x \in \mathbb{R} \setminus \{1\}$.

Exercise 3.15. Show that functions of the form $\frac{r_1(x)+r_2(|x|)}{r_3(x)+r_4(|x|)}$ are continuous in an appropriate subset of $\mathbb{R}$, where r_1, r_2, r_3, and r_4 are all rational functions.

The following theorem was proved by Bernard Bolzano in 1817.

Theorem 3.16 (Intermediate value)**.** *Let $a, b \in \mathbb{R}$ satisfy $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$. Then for any $y \in \mathbb{R}$ satisfying $\min\{f(a), f(b)\} \le y \le \max\{f(a), f(b)\}$, there exists $x \in [a, b]$ such that $f(x) = y$.*

Proof. We apply what is known as the *bisection method*. We define two sequences $\{a_n\}$ and $\{b_n\}$ as follows. First, set $a_0 = a$ and $b_0 = b$. Obviously, the value y lies between $f(a_0)$ and $f(b_0)$, that is, $y \in [f(a_0), f(b_0)] \cup [f(b_0), f(a_0)]$. Let $c_0 = \frac{1}{2}(a_0 + b_0)$. Then at least one of $y \in [f(a_0), f(c_0)] \cup [f(c_0), f(a_0)]$ and $y \in [f(c_0), f(b_0)] \cup [f(b_0), f(c_0)]$ must hold. If the former holds, we set $a_1 = a_0$ and $b_1 = c_0$. Otherwise, we set $a_1 = c_0$ and $b_1 = b_0$. In any case, we have $y \in [f(a_1), f(b_1)] \cup [f(b_1), f(a_1)]$, and $b_1 - a_1 = \frac{1}{2}(b_0 - a_0)$. By repeating this process, we get $\{a_n\}$ and $\{b_n\}$ such that

$$[a_0, b_0] \supset [a_1, b_1] \supset \ldots \supset [a_n, b_n] \supset \ldots, \tag{40}$$

with $y \in [f(a_n), f(b_n)] \cup [f(b_n), f(a_n)]$ and $b_n - a_n = 2^{-n}(b - a)$ for each $n \in \mathbb{N}$. Then by the nested intervals principle (Theorem 2.17), $\bigcap_{n\in\mathbb{N}}[a_n, b_n]$ is nonempty and consists of a single point. Let us denote this point by x. Since $a_n \to x$ and $b_n \to x$ as $n \to \infty$, by continuity of f we have $f(a_n) \to f(x)$ and $f(b_n) \to f(x)$ as $n \to \infty$. In particular, for any given $\varepsilon > 0$, there exists N such that $|f(a_n) - f(x)| \le \varepsilon$ and $|f(b_n) - f(x)| \le \varepsilon$ for all $n \ge N$. This means that $|f(b_n) - f(a_n)| \le 2\varepsilon$ for all $n \ge N$, and hence $|y - f(a_n)| \le 2\varepsilon$ for all $n \ge N$. Finally, by the triangle inequality we have $|f(x) - y| \le |f(x) - f(a_N)| + |f(a_N) - y| \le 3\varepsilon$. In other words, for any $\varepsilon > 0$ we have $|f(x) - y| \le 3\varepsilon$. This shows that $f(x) = y$, since $f(x) \ne y$ would imply that $|f(x) - y| > 0$. ☐

Example 3.17. Given $y \ge 0$, let us try to solve the equation $x^2 = y$. Consider the function $f(x) = x^2$ in the interval $[0, b]$, where $b = \max\{1, y\}$. Obviously, f is continuous in $[0, b]$, and since $y \le \max\{1, y^2\}$, we have $f(0) \le y \le f(y)$. By the intermediate value theorem (Theorem 3.16), there exists $x \in [0, b]$ such that $f(x) = y$, that is, the equation $x^2 = y$ has a solution in the interval $[0, b]$. This solution is in fact the unique solution of $x^2 = y$ in $[0, \infty)$. Indeed, $0 \le x < z$ implies $x^2 < z^2$, and hence $x^2 = z^2$ with $x, z \ge 0$ is possible only when $x = z$. The function $y \mapsto x$ is called the *arithmetic (or principal) square root* function, and denoted by $x = \sqrt{y}$. It is clear that the arithmetic square root function is the inverse of $f : [0, \infty) \to [0, \infty)$ given by $f(x) = x^2$. More generally, let $F : \mathbb{R} \to [0, \infty)$ be given by $F(x) = x^2$. Then we have $F|_{[0,\infty)} = f$, and $F(x) = f(|x|)$ for all $x \in \mathbb{R}$. This means that $x^2 = z^2$ implies $|x| = |z|$, and hence $x^2 = z^2$ if and only if $x = z$ or $x = -z$. To conclude, the equation $x^2 = y$ has no solution for $y < 0$, exactly one solution for $y = 0$, and exactly two solutions $x = \sqrt{y}$ and $x = -\sqrt{y}$ for $y > 0$.

Exercise 3.18. Investigate the equation $x^n = y$ where $n \in \mathbb{N}$.

Corollary 3.19. *Let $f \in \mathscr{C}([a,b])$ be a strictly increasing function in the sense that $x < y$ implies $f(x) < f(y)$. Then f is injective, the image of f is $f([a,b]) = [f(a), f(b)]$, and the inverse $f^{-1} : f([a,b]) \to [a,b]$ is continuous and strictly increasing.*

Proof. Considered as a function $f : [a,b] \to f([a,b])$, f is obviously surjective. Since $x \neq y$ implies $f(x) \neq f(y)$, f is injective. Hence there exists the inverse $f^{-1} : f([a,b]) \to [a,b]$. Moreover, $x \geq y$ implies $f(x) \geq f(y)$, and so by contrapositive, f^{-1} is strictly increasing. We have $[f(a), f(b)] \subset f([a,b])$, since by the intermediate value theorem (Theorem 3.16), the equation $f(x) = y$ has a solution for every $y \in [f(a), f(b)]$. On the other hand, $a \leq x \leq b$ implies $f(a) \leq f(x) \leq f(b)$, meaning that $f([a,b]) = [f(a), f(b)]$.

It remains to show that f^{-1} is continuous. To this end, let $\{y_n\} \subset [f(a), f(b)]$ be such that $y_n \to y$ for some $y \in [f(a), f(b)]$. Let $\varepsilon > 0$ be given, and let $x = f^{-1}(y)$. Then with $\alpha_\varepsilon = f(\max\{a, x - \varepsilon\})$ and $\beta_\varepsilon = f(\min\{b, x + \varepsilon\})$, we have $f^{-1}(y_n) \in [x - \varepsilon, x + \varepsilon]$ whenever $y_n \in [\alpha_\varepsilon, \beta_\varepsilon]$. Note that $\alpha_\varepsilon < y$ unless $y = f(a)$, and $\beta_\varepsilon > y$ unless $y = f(b)$. Thus there exists N such that $y_n \in [\alpha_\varepsilon, \beta_\varepsilon]$ for all $n \geq N$, which shows that $f^{-1}(y_n) \to x$ as $n \to \infty$. □

Exercise 3.20. Let $f \in \mathscr{C}([a,b])$ be a *strictly decreasing* function in the sense that $x < y$ implies $f(x) > f(y)$. Show that f is injective, and that the inverse $f^{-1} : f([a,b]) \to [a,b]$ is continuous and strictly decreasing.

Example 3.21. (a) Fix $n \in \mathbb{N}$, and consider the power function $f(x) = x^n$. This is a continuous and strictly increasing function in the range $[0, \infty)$. Given any $b > 0$, the restriction $f|_{[0,b]}$ is in particular strictly increasing, and so its inverse $g_b : [0, b^n] \to [0, b]$ is also strictly increasing and continuous. Now if we consider g_c with $c > b$, then g_c must agree with g_b on their common domain, that is, $g_c(y) = g_b(y)$ for $y \in [0, b^n]$, since $g_b(f(x)) = x$ and $g_c(f(x)) = x$ for $x \in [0, b]$. Therefore we can define the function $g : [0, \infty) \to [0, \infty)$ by $g(y) = g_b(y)$ with $b > \max\{1, y\}$ for $y \in [0, \infty)$, and this function satisfies $g(f(x)) = x$ for $x \in [0, \infty)$, i.e., g is the inverse of $f|_{[0,\infty)}$. Moreover, g is continuous and strictly increasing. Of course, g is the *arithmetic n-th root* function, denoted by $\sqrt[n]{y} \equiv g(y)$.

(b) With the help of the n-th root function, we also define the *rational power* as

$$x^{\frac{m}{n}} = (\sqrt[n]{x}\,)^m \qquad \text{for } x \geq 0,\ m \in \mathbb{Z},\ n \in \mathbb{N}. \tag{41}$$

That this definition is unambiguous can be seen as follows.

- We have $((\sqrt[n]{x})^m)^n = (\sqrt[n]{x})^{mn} = ((\sqrt[n]{x})^n)^m = x^m$, and hence $(\sqrt[n]{x})^m = \sqrt[n]{x^m}$.
- By writing $\frac{m}{n} = \frac{mk}{nk}$ with some $k \in \mathbb{N}$, we get $x^{\frac{mk}{nk}} = (\sqrt[nk]{x})^{mk}$. We need to show that this is equal to $(\sqrt[n]{x})^m$. Since $((\sqrt[nk]{x})^k)^n = (\sqrt[nk]{x})^{nk} = x$, we have $(\sqrt[nk]{x})^k = \sqrt[n]{x}$. Therefore we conclude that $(\sqrt[nk]{x})^{mk} = ((\sqrt[nk]{x})^k)^m = (\sqrt[n]{x})^m$.

As the composition of two continuous functions, the rational power function $w(x) = x^{\frac{m}{n}}$ is a continuous function of $x \geq 0$. Moreover, w is *strictly increasing*, provided $\frac{m}{n} > 0$, since it is the composition of two strictly increasing functions. Another important property is for $a, b \in \mathbb{Q}$ with $a < b$, we have

$$x > 1 \quad \Longrightarrow \quad x^a < x^b \qquad \text{and} \qquad 0 < x < 1 \quad \Longrightarrow \quad x^a > x^b. \tag{42}$$

Indeed, by writing $a = \frac{m}{n}$ and $b = \frac{k}{n}$, the question is reduced to comparing the integer powers $(\sqrt[n]{x})^m$ and $(\sqrt[n]{x})^k$.

(c) We claim that for any $x > 0$, $\sqrt[n]{x} \to 1$ as $n \to \infty$. This result will be used later to define the power x^a for any $a \in \mathbb{R}$. Let $x > 1$. Then $\sqrt[n]{x} > 1$ for all n. Suppose that there exists some $\varepsilon > 0$ such that $\sqrt[n]{x} > 1 + \varepsilon$ for all n. This would imply that $x = (1 + \varepsilon)^n \geq 1 + \varepsilon n$ for all n, which is impossible by the Archimedean property. Hence $\sqrt[n]{x} \to 1$ as $n \to \infty$, for $x > 1$. The case $0 < x < 1$ is given as an exercise below.

Exercise 3.22. Prove the following.

(a) For odd n, the inverse of $f(x) = x^n$ can be defined on all of $\mathbb{R}$.
(b) The function $f(x) = x^q$ with rational $q < 0$ is a strictly decreasing function of $x \geq 0$.
(c) For $x, y \geq 0$ and $p, q \in \mathbb{Q}$, there hold that

$$x^p x^q = x^{p+q}, \qquad (x^p)^q = x^{pq}, \qquad (xy)^q = x^q y^q. \tag{43}$$

Exercise 3.23. Prove the following.

(a) If $0 < x < 1$ then $\sqrt[n]{x} \to 1$ as $n \to \infty$.
(b) $\sqrt[n]{n} \to 1$ as $n \to \infty$.
(c) If $a \in \mathbb{N}$ then $\sqrt[n]{n^a} \to 1$ as $n \to \infty$.

Remark 3.24. Let $s = \sqrt{2}$. We claim that $s \notin \mathbb{Q}$, which would mean that $\mathbb{Q}$ is *strictly* contained in $\mathbb{R}$. To show this, suppose that $s \in \mathbb{Q}$, i.e., that $s = \frac{m}{n}$ with $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Thus we have $(\frac{m}{n})^2 = 2$ or $m^2 = 2n^2$. This implies that m is divisible by 2, i.e., $m = 2m_1$ for some $m_1 \in \mathbb{N}$, which in turn yields $n^2 = 2m_1^2$. Hence $n = 2n_1$ for some $n_1 \in \mathbb{N}$, and this leads to $m_1 = 2m_2$ for some $m_2 \in \mathbb{N}$. We can repeat this process indefinitely, arriving at the conclusion that for any $k \in \mathbb{N}$ there exists $a \in \mathbb{N}$ such that $m = 2^k a$. However, since $a \geq 1$, by choosing k large enough we can ensure that $2^k a \geq 2^k > m$, which is a contradiction. The numbers in $\mathbb{R} \setminus \mathbb{Q}$, such as $\sqrt{2}$, are called *irrational numbers.*

The following fundamental theorem was established by Weierstrass in 1861.

Theorem 3.25 (Extreme value)**.** *Let $f \in \mathscr{C}([a, b])$. Then there exists $c \in [a, b]$ such that $f(x) \leq f(c)$ for all $x \in [a, b]$.*

Proof. For $n \in \mathbb{N}$, let $Q_n = \{\frac{k}{m} : k \in \mathbb{Z}, m \in \mathbb{N}, m \leq n\} \cap [a, b]$. It is clear that Q_n is a finite set, and therefore f takes its maximum over Q_n, i.e., there exists $x_n \in Q_n$ such that $f(q) \leq f(x_n)$ for all $q \in Q_n$. Since $\{x_n\} \subset [a, b]$, the Bolzano-Weierstrass theorem (Theorem 2.18) guarantees the existence of a subsequence $\{x_{n_k}\} \subset \{x_n\}$ that converges to some point $c \in [a, b]$. The sequence $\{f(x_n)\}$ is nondecreasing, in the sense that $f(x_n) \leq f(x_{n+1})$ for all n. Thus $f(c) \geq f(x_n)$ for all n, and hence $f(c) \geq f(q)$ for all $q \in \mathbb{Q} \cap [a, b]$. We claim that $f(x) \leq f(c)$ for all $x \in [a, b]$. Suppose that there is $x \in [a, b]$ with $f(x) > f(c)$. Then by continuity there exists $\varepsilon > 0$ small enough, such that $y \in (x - \varepsilon, x + \varepsilon)$ implies $f(y) > f(c)$. However, there exists $q \in \mathbb{Q} \cap (x - \varepsilon, x + \varepsilon)$, contradicting the fact that $f(c) \geq f(q)$ for all $q \in \mathbb{Q} \cap [a, b]$. □

4. Differentiation

Let us consider the function $f(x) = x^2$ when x is very close to some given point $x_* \in \mathbb{R}$. Putting $h = x - x_*$, which is assumed to be small, we can write

$$x^2 = (x_* + h)^2 = x_*^2 + 2x_*h + h^2, \tag{44}$$

that is,

$$f(x) - f(x_*) = x^2 - x_*^2 = 2x_*h + h^2 = (2x_* + h)h. \tag{45}$$

Intuitively speaking, this means that when $|h|$ small, $f(x_* + h) - f(x_*)$ is basically equal to the linear function $\ell(h) = 2x_*h$. This leads to the concept of derivative.

Given a function $f : (a, b) \to \mathbb{R}$ with $a < b$, and a point $x_* \in (a, b)$, the idea is to require

$$f(x_* + h) - f(x_*) = (\lambda + e(h))h, \tag{46}$$

where $\lambda \in \mathbb{R}$ is a constant, and $e(h)$ is a function of h that can be made arbitrarily close to 0 by choosing $|h|$ small enough. In other words, e is continuous at $h = 0$ with $e(0) = 0$. An example of such a function is $e(h) = h$ as in (45). If (46) holds, then for $h \in \mathbb{R}$ with $|h|$ small, $f(x_* + h) - f(x_*)$ is equal to the linear function $\ell(h) = \lambda h$, up to the error $e(h)h$. The following definition was introduced by Carathéodory in 1950, and is a refined version of the definition given by Weierstrass in 1861.

Definition 4.1. A function $f : (a, b) \to \mathbb{R}$ is said to be *differentiable at* $x \in (a, b)$, if there exists a function $g : (a, b) \to \mathbb{R}$, which is continuous at x, such that

$$f(y) = f(x) + g(y)(y - x), \qquad y \in (a, b). \tag{47}$$

We call the value $g(x)$ the *derivative of* f *at* x, and write

$$f'(x) \equiv \frac{\mathrm{d}f}{\mathrm{d}x}(x) := g(x). \tag{48}$$

If f is differentiable at each $x \in K$ for some $K \subset (a, b)$, then we say that f is *differentiable in* K, and consider the derivative as a function $f' : K \to \mathbb{R}$ sending x to $f'(x)$.

It is immediate from (47) that if f is differentiable at x then f is continuous at x. The following lemma gives a *sequential criterion* of differentiability. This criterion was used as a definition by Cauchy in 1821.

Lemma 4.2. *A function* $f : (a, b) \to \mathbb{R}$ *is differentiable at* $x \in (a, b)$ *if and only if there exists a number* $\lambda \in \mathbb{R}$ *such that*

$$\frac{f(x_n) - f(x)}{x_n - x} \to \lambda \qquad \text{as} \quad n \to \infty, \tag{49}$$

for every sequence $\{x_n\} \subset (a, b) \setminus \{x\}$ *converging to* x.

Proof. Let f be differentiable at x. Then by (47), we have

$$g(y) = \frac{f(y) - f(x)}{y - x} \qquad \text{for} \quad y \in (a, b) \setminus \{x\}. \tag{50}$$

Since g is continuous at x, for any sequence $\{x_n\} \subset (a, b) \setminus \{x\}$ converging to x, we have

$$g(x_n) = \frac{f(x_n) - f(x)}{x_n - x} \to \lambda := g(x) \qquad \text{as} \quad n \to \infty. \tag{51}$$

This establishes the "only if" part of the lemma.

Now suppose that there exists $\lambda \in \mathbb{R}$ such that (49) holds for every sequence $\{x_n\} \subset (a, b) \setminus \{x\}$ converging to x. Then we define a function $g : (a, b) \to \mathbb{R}$ by

$$g(y) = \frac{f(y) - f(x)}{y - x} \quad \text{for} \quad y \in (a, b) \setminus \{x\}, \qquad \text{and} \qquad g(x) = \lambda. \tag{52}$$

This function satisfies (47) by construction. It remains to show that g is continuous at x. Let $\{x_n\} \subset (a, b)$ be a sequence converging to x. Suppose that we created a new sequence $\{x'_m\} \subset (a, b) \setminus \{x\}$ by removing every occurrence of x from $\{x_n\}$. There are two possibilities. The first possibility is that $\{x'_m\}$ is a finite sequence. In this case, there exists some N such that $x_n = x$ for all $n \geq N$, and hence it is trivial to observe that $g(x_n) \to g(x)$ as $n \to \infty$. The second possibility is that $\{x'_m\}$ is an infinite sequence. In this case, by (49) we have $g(x'_m) \to g(x)$ as $m \to \infty$, that is, for any $\varepsilon > 0$, there exists M such that $|g(x'_m) - g(x)| \leq \varepsilon$ for all $m \geq M$. Now if we let N to be the index of $\{x_n\}$ corresponding to the index M in $\{x'_m\}$, then it is clear that $|g(x_n) - g(x)| \leq \varepsilon$ for all $n \geq N$ because for $n \geq N$ we have either $x_n = x'_m$ for some $m \geq M$ or $x_n = x$. □

Example 4.3. (a) Let $c \in \mathbb{R}$, and let $f(x) = c$ be a constant function. Then since $f(y) = f(x) + 0 \cdot (y - x)$ for all x, y, we get $f'(x) = 0$ for all x.

(b) Let $a, c \in \mathbb{R}$, and let $f(x) = ax + c$ be a linear (also known as affine) function. Since $f(y) = f(x) + a(y - x)$ for all x, y, we get $f'(x) = a$ for all x.

(c) Let $f(x) = \frac{1}{x}$, and for $y \in \mathbb{R} \setminus \{0, x\}$ define

$$g(y) = \frac{\frac{1}{y} - \frac{1}{x}}{y - x} = -\frac{1}{xy}. \tag{53}$$

As long as $x \neq 0$, upon defining $g(x) = -\frac{1}{x^2}$, the function $g(y) = -\frac{1}{x} \cdot \frac{1}{y}$ becomes continuous at $y = x$, and therefore f is differentiable at x with

$$f'(x) = \big(\frac{1}{x}\big)' = -\frac{1}{x^2} \qquad (x \neq 0). \tag{54}$$

(d) Let us try to differentiate $f(x) = |x|$ at $x = 0$. With $x_n = \frac{1}{n}$ for $n \in \mathbb{N}$, we have $\{x_n\} \subset \mathbb{R} \setminus \{0\}$ and $x_n \to 0$ as $n \to \infty$. On one hand, we get

$$\lim_{n\to\infty} \frac{f(x_n) - f(0)}{x_n - 0} = \lim_{n\to\infty} \frac{|x_n|}{x_n} = 1, \tag{55}$$

but on the other hand, with $y_n = -x_n$, we infer

$$\lim_{n\to\infty} \frac{f(y_n) - f(0)}{y_n - 0} = \lim_{n\to\infty} \frac{|y_n|}{y_n} = -\lim_{n\to\infty} \frac{x_n}{x_n} = -1. \tag{56}$$

The definition of derivative requires these two limits to be the same, and thus we conclude that $f(x) = |x|$ is *not* differentiable at $x = 0$.

(e) Consider the differentiability of $f(x) = \sqrt[3]{x}$ at $x = 0$. Let $x_n = \frac{1}{n^3}$. It is obvious that $x_n \neq 0$ and $x_n \to 0$. We have

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{\sqrt[3]{x_n}}{x_n} = n^2, \tag{57}$$

which diverges as $n \to \infty$. Hence $f(x) = \sqrt[3]{x}$ is *not* differentiable at $x = 0$.

Exercise 4.4. Show that $f(x) = x^n$ is differentiable in $\mathbb{R}$, for $n \in \mathbb{N}$, with $f'(x) = nx^{n-1}$.

The following result shows that differentiability is a local property.

Lemma 4.5. *Let $f : (a, b) \to \mathbb{R}$, and let $g = f|_{(c,d)}$ for some $(c, d) \subset (a, b)$. Then f is differentiable at $x \in (c, d)$ if and only of g is differentiable at x. Moreover, if f is differentiable at $x \in (c, d)$, then $f'(x) = g'(x)$.*

Proof. Suppose that f is differentiable at $x \in (c, d)$. Then by definition, there is a function $\tilde{f} : (a, b) \to \mathbb{R}$, continuous at x, with $f'(x) = \tilde{f}(x)$, such that

$$f(y) = f(x) + \tilde{f}(y)(y - x), \qquad y \in (a, b). \tag{58}$$

Since $g(y) = f(y)$ for $y \in (c, d)$, we have

$$g(y) = g(x) + \tilde{f}(y)(y - x), \qquad y \in (c, d). \tag{59}$$

This shows that g is differentiable at x with $g'(x) = \tilde{f}(x) = f'(x)$.

Now suppose that g is differentiable at $x \in (c, d)$. Then there is a function $\tilde{g} : (c, d) \to \mathbb{R}$, continuous at x, with $g'(x) = \tilde{g}(x)$, such that

$$g(y) = g(x) + \tilde{g}(y)(y - x), \qquad y \in (c, d). \tag{60}$$

If we extend $\tilde{g}$ as

$$\tilde{f}(y) = \begin{cases} \tilde{g}(y), & \text{for } y \in (c, d), \\ \frac{f(y)-f(x)}{y-x}, & \text{for } y \in (a, b) \setminus (c, d), \end{cases} \tag{61}$$

then since f and g agree on (c, d), we have

$$f(y) = f(x) + \tilde{f}(y)(y - x), \qquad y \in (a, b). \tag{62}$$

Since $\tilde{f} = \tilde{g}$ on (c, d), by locality of continuity $\tilde{f}$ is continuous at x. Hence f is differentiable at x with $f'(x) = \tilde{f}(x) = \tilde{g}(x) = g'(x)$. ☐

We now investigate differentiability of various combinations of differentiable functions.

Theorem 4.6. *Let $f, g : (a, b) \to \mathbb{R}$ be functions differentiable at $x \in (a, b)$. Then the following are true.*

a) The sum and difference $f \pm g$ are differentiable at x, with

$$(f \pm g)'(x) = f'(x) \pm g'(x). \tag{63}$$

These are called the sum and difference rules*.*

b) The product fg is differentiable at x, with

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x). \tag{64}$$

This is called the product rule*.*

c) If $F : (c, d) \to \mathbb{R}$ is a function differentiable at $g(x)$, with $g((a, b)) \subset (c, d)$, then the composition $F \circ g : (a, b) \to \mathbb{R}$ is differentiable at x, with

$$(F \circ g)'(x) = F'(g(x))g'(x). \tag{65}$$

This is called the chain rule*.*

d) If $f : (a, b) \to f((a, b))$ is bijective and $f'(x) \neq 0$, then the inverse $f^{-1} : f((a, b)) \to (a, b)$ is differentiable at $y = f(x)$, with

$$(f^{-1})'(y) = \frac{1}{f'(x)}. \tag{66}$$

Proof. b) By definition, there is a function $\tilde{f} : (a, b) \to \mathbb{R}$, continuous at x, satisfying

$$f(y) = f(x) + \tilde{f}(y)(y - x), \qquad y \in (a, b), \tag{67}$$

and $f'(x) = \tilde{f}(x)$. Similarly, there is a function $\tilde{g} : (a, b) \to \mathbb{R}$, continuous at x, and with $g'(x) = \tilde{g}(x)$, such that

$$g(y) = g(x) + \tilde{g}(y)(y - x), \qquad y \in (a, b). \tag{68}$$

By multiplying (67) and (68), we get

$$\begin{aligned} f(y)g(y) &= f(x)g(x) + g(x)\tilde{f}(y)(y - x) + f(x)\tilde{g}(y)(y - x) + \tilde{f}(y)\tilde{g}(y)(y - x)^2 \\ &= f(x)g(x) + [g(x)\tilde{f}(y) + f(x)\tilde{g}(y) + \tilde{f}(y)\tilde{g}(y)(y - x)](y - x). \end{aligned} \tag{69}$$

The expression in the square brackets, as a function of y, is continuous at $y = x$, with

$$\begin{aligned} [g(x)\tilde{f}(y) + f(x)\tilde{g}(y) + \tilde{f}(y)\tilde{g}(y)(y - x)]\big|_{y=x} &= g(x)\tilde{f}(x) + f(x)\tilde{g}(x) \\ &= g(x)f'(x) + f(x)g'(x), \end{aligned} \tag{70}$$

which shows that fg is differentiable at x, and that (64) holds.

c) Since F is differentiable at $g(x)$, by definition, there is a function $\tilde{F} : (c, d) \to \mathbb{R}$, continuous at $g(x)$, and with $F'(g(x)) = \tilde{F}(g(x))$, such that

$$F(z) = F(g(x)) + \tilde{F}(z)(z - g(x)), \qquad z \in (c, d). \tag{71}$$

Plugging $z = g(y)$ into (71), we get

$$F(g(y)) = F(g(x)) + \tilde{F}(g(y))(g(y) - g(x)) = F(g(x)) + \tilde{F}(g(y))\tilde{g}(y)(y - x), \tag{72}$$

where in the last step we have used (68). The function $y \mapsto \tilde{F}(g(y))\tilde{g}(y)$ is continuous at $y = x$, with $\tilde{F}(g(x))\tilde{g}(x) = F'(g(x))g'(x)$, which confirms that $F \circ g$ is differentiable at x, and that (65) holds.

d) By definition, there is $g:(a,b)\to\mathbb{R}$, continuous at x, with $g(x)=f'(x)\neq 0$, such that

$$f(z)=f(x)+g(z)(z-x) \qquad \text{for } z\in(a,b). \tag{73}$$

Since g is continuous at x, we infer the existence if an open interval $(c,d)\ni x$ such that $g(z)\neq 0$ for all $z\in(c,d)$. For $t\in f((c,d))$, we have $z=f^{-1}(t)\in(c,d)$, and

$$f^{-1}(t)-f^{-1}(y)=z-x=\frac{f(z)-f(x)}{g(z)}=\frac{t-y}{g(f^{-1}(t))}. \tag{74}$$

The function $\frac{1}{g(f^{-1}(t))}$ is continuous at $t=y$, meaning that f^{-1} is differentiable at y, and that (66) holds. □

Exercise 4.7. Prove a) of the preceding theorem.

Example 4.8. (a) By the product rule, we have

$$\begin{aligned}
(x^2)' &= 1\cdot x + x\cdot 1 = 2x,\\
(x^3)' &= (x^2\cdot x)' = 2x\cdot x + x^2\cdot 1 = 3x^2,\ldots\\
(x^n)' &= nx^{n-1} \qquad (n\in\mathbb{N}).
\end{aligned} \tag{75}$$

(b) By the sum and product rules, all polynomials are differentiable in $\mathbb{R}$, and the derivative of a polynomial is again a polynomial.

(c) Given a function $f:(a,b)\to\mathbb{R}$ that does not vanish anywhere in (a,b), we can write the reciprocal function $\frac{1}{f}$ as $F\circ f$ with $F(z)=\frac{1}{z}$. If f is differentiable at $x\in(a,b)$, then by the chain rule, $\frac{1}{f}$ is differentiable at x and

$$\big(\frac{1}{f}\big)'(x)=(F\circ f)'(x)=F'(f(x))f'(x)=-\frac{f'(x)}{[f(x)]^2}. \tag{76}$$

In particular, we have

$$(x^{-n})'=-\frac{nx^{n-1}}{x^{2n}}=-nx^{-n-1} \qquad (n\in\mathbb{N}). \tag{77}$$

(d) Let $f(x)=x^n$ for $x\in[0,\infty)$, where $n\in\mathbb{N}$. We have $f'(x)=nx^{n-1}$ at $x>0$, and the inverse function is the arithmetic n-the root $f^{-1}(y)=\sqrt[n]{y}$ $(y\geq 0)$. Since $f'(x)>0$ for $x>0$, the inverse f^{-1} is differentiable at each $y>0$, with

$$(f^{-1})'(y)=\frac{1}{f'(f^{-1}(y))}=\frac{1}{n(\sqrt[n]{y})^{n-1}}=\frac{1}{n}y^{\frac{1-n}{n}}. \tag{78}$$

Moreover, by the chain rule, for $m\in\mathbb{Z}$ and $n\in\mathbb{N}$, we infer

$$(x^{\frac{m}{n}})'=((\sqrt[n]{x})^m)'=m(\sqrt[n]{x})^{m-1}\cdot\frac{1}{n}x^{\frac{1-n}{n}}=\frac{m}{n}x^{\frac{m-1}{n}+\frac{1-n}{n}}=\frac{m}{n}x^{\frac{m}{n}-1}, \tag{79}$$

that is

$$(x^a)'=ax^{a-1} \qquad \text{at each } x>0, \quad \text{for } a\in\mathbb{Q}. \tag{80}$$

Exercise 4.9. Let $f,g:(a,b)\to\mathbb{R}$ be functions differentiable at $x\in(a,b)$, with $g(x)\neq 0$. Show that the quotient f/g is differentiable at x, and the following *quotient rule* holds.

$$\big(\frac{f}{g}\big)'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}. \tag{81}$$

Compute the derivative of $q(x)=\frac{3x^3}{x^2+1}$.

5. Applications of differentiation

The following result is sometimes called the *first derivative test*.

Lemma 5.1. *Let $f:(a,b)\to\mathbb{R}$ be a function, and let $c\in(a,b)$ be a maximum of f, in the sense that $f(x)\leq f(c)$ for all $x\in(a,b)$. Suppose that f is differentiable at c. Then $f'(c)=0$.*

Proof. Suppose that $f'(c)\neq 0$. By definition of differentiability, there exists a function $g:(a,b)\to\mathbb{R}$, continuous at c, with $g(c)=f'(c)$, such that

$$f(x)=f(c)+g(x)(x-c),\qquad x\in(a,b). \tag{82}$$

If $f'(c)>0$, we let $x_n=c+\frac{1}{n}$, and if $f'(c)<0$, we let $x_n=c-\frac{1}{n}$, for $n\in\mathbb{N}$. Then by continuity, for sufficiently large n, we have $|g(x_n)-g(c)|\leq\frac{1}{2}|g(c)|$. This means that $g(x_n)(x_n-c)\geq\frac{1}{2n}|f'(c)|$ or $f(x)\geq f(c)+\frac{1}{2n}|f'(c)|$, contradicting the maximality of $f(c)$. □

Remark 5.2. At least in principle, the first derivative test gives a way to find the maximums and minimums of a differentiable function. Namely, let $f\in\mathscr{C}([a,b])$ be given. Then the extreme value theorem (Theorem 3.25) guarantees the existence of a maximum $\xi\in[a,b]$. If $\xi\in(a,b)$ and if f is differentiable in (a,b), then $f'(\xi)=0$. In other words, all maximums located in the interior (a,b) can be found by comparing the values $f(c)$ at the *critical points*, which are by definition the solutions $c\in(a,b)$ of the equation $f'(c)=0$.

The consideration of critical points leads to the following fundamental result, known as *Rolle's theroem*. It was proved by Michel Rolle in 1690.

Theorem 5.3 (Rolle)**.** *Let $f\in\mathscr{C}([a,b])$ be differentiable at each $x\in(a,b)$, with $f(a)=f(b)$. Then there exists $\xi\in(a,b)$ such that $f'(\xi)=0$.*

Proof. The extreme value theorem (Theorem 3.25) yields the existence of $\xi\in[a,b]$ with the property that $f(x)\leq f(\xi)$ for all $x\in[a,b]$. Without loss of generality, we can assume that f is not constant, and that $\xi\in(a,b)$, because if $f(x)\leq f(a)$ for all $x\in[a,b]$, then we can replace f by $-f$. Then the first derivative test (Lemma 5.1) implies that $f'(\xi)=0$. □

The following important consequence was proved by Lagrange in 1796.

Theorem 5.4 (Mean value)**.** *Let $f\in\mathscr{C}([a,b])$ be differentiable at each $x\in(a,b)$. Then there exists $\xi\in(a,b)$ such that $f(b)-f(a)=f'(\xi)(b-a)$.*

Proof. Define the function $F:[a,b]\to\mathbb{R}$ by

$$F(x)=f(x)-\frac{f(b)-f(a)}{b-a}(x-a). \tag{83}$$

We have $F(a)=F(b)=f(a)$, $F\in\mathscr{C}([a,b])$, and F is differentiable at each $x\in(a,b)$, with

$$F'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}. \tag{84}$$

By Rolle's theorem (Theorem 5.3), there exists $\xi\in(a,b)$ such that $F'(\xi)=0$. □

Remark 5.5. If f is continuous at c, then for y close to c, we have $f(y)=f(c)+e$, with $e\to 0$ as $|y-c|\to 0$. Thus we can use $f(c)$ to approximate $f(y)$, but there is a very little information on the size of the error e. If in addition, f is differentiable at c, then we have

$$f(y)=f(c)+f'(c)(y-c)+e_1, \tag{85}$$

with e_1 vanishing faster than $|y-c|$ as $|y-c|\to 0$. This shows that $e=f'(c)(y-c)+e_1$, but we still do not have a precise quantitative information on e_1. The mean value theorem (Theorem 5.4) reveals a quantitative bound on the error e, provided that f is differentiable in a region (not only at the point c), even when c and y are at a finite distance from each other.

For example, we have $(x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}}$, which implies that for $y > c > 0$ there exists $\xi \in (c, y)$ such that

$$\sqrt{y} = \sqrt{c} + \frac{y-c}{2\sqrt{\xi}}. \tag{86}$$

Taking into account that $\sqrt{\xi}$ is strictly increasing in ξ, we infer

$$\sqrt{c} < \sqrt{y} < \sqrt{c} + \frac{y-c}{2\sqrt{c}} \qquad (y > c > 0). \tag{87}$$

Corollary 5.6. *Let $f : (a, b) \to \mathbb{R}$ be differentiable in (a, b), and suppose that $f'(x) > 0$ for all $x \in (a, b)$. Then f is strictly increasing in (a, b).*

Proof. Suppose that there exist $x, y \in (a, b)$ with $x < y$, such that $f(x) \geq f(y)$. Then by the mean value theorem (Theorem 5.4), there exists $\xi \in (x, y)$ such that $f'(\xi) = \frac{f(y)-f(x)}{y-x} \leq 0$, leading to contradiction. □

Definition 5.7. A function $f : (a, b) \to \mathbb{R}$ is called *continuously differentiable* in $K \subset (a, b)$, and written $f \in \mathscr{C}^1(K)$, if it is differentiable in K, with f' is continuous in K.

Corollary 5.8 (Inverse functions)**.** *Let $f \in \mathscr{C}^1((a, b))$ satisfy $f'(x) \neq 0$ for some $x \in (a, b)$. Then there exists an interval $(c, d) \ni x$, such that $f|_{(c,d)} : (c, d) \to f((c, d))$ is bijective.*

Proof. Without loss of generality, we shall only consider the case $f'(x) > 0$. By continuity of f', there is an open interval $I \ni x$ such that $f'(\xi) > 0$ for all $\xi \in I$. Hence by the preceding corollary, f is strictly increasing in I, and by Corollary 3.19, the function $f : [c, d] \to \mathbb{R}$ on $[c, d] \subset I$ admits the inverse $f^{-1} : f([c, d]) \to [c, d]$, which is continuous and strictly increasing. □

Corollary 5.9 (L'Hôpital's rule)**.** *Let $f, g \in \mathscr{C}([a, b))$ be differentiable in (a, b), satisfying $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for $x \in (a, b)$. Suppose that there exists $q \in \mathbb{R}$ such that*

$$\frac{f'(x_n)}{g'(x_n)} \to q \qquad \text{as } n \to \infty, \tag{88}$$

for every sequence $\{x_n\} \subset (a, b)$ converging to a. Then $g(x) \neq 0$ for $x \in (a, b)$, and

$$\frac{f(x_n)}{g(x_n)} \to q \qquad \text{as } n \to \infty, \tag{89}$$

for every sequence $\{x_n\} \subset (a, b)$ converging to a.

Proof. Suppose that $g(x) = 0$ for some $x \in (a, b)$. Then by Rolle's theorem (Theorem 5.3), there exists $\xi \in (a, x)$ such that $g'(\xi) = 0$. This contradicts the assumption that g' does not vanish in (a, b), which means that g does not vanish in (a, b).

Let $\{x_n\} \subset (a, b)$ be a sequence converging to a. For each $n \in \mathbb{N}$ and $x \in [a, x_n]$, we define $F_n(x) = g(x_n)f(x) - f(x_n)g(x)$. Then $F_n \in \mathscr{C}([a, x_n])$ with $F_n(a) = F_n(x_n) = 0$, and moreover F_n is differentiable in (a, x_n) with $F_n'(x) = g(x_n)f'(x) - f(x_n)g'(x)$. Hence by Rolle's theorem, there exists $\xi_n \in (a, x_n)$ such that $F_n'(\xi_n) = 0$, i.e., that $g(x_n)f'(\xi_n) = f(x_n)g'(\xi_n)$. Since $\{x_n\}$ converges to a, it is obvious that $\{\xi_n\}$ converges to a. Let $\varepsilon > 0$, and let N be such that $|\frac{f'(\xi_n)}{g'(\xi_n)} - q| < \varepsilon$ for all $n \geq N$. Then we have $|\frac{f(x_n)}{g(x_n)} - q| = |\frac{f'(\xi_n)}{g'(\xi_n)} - q| < \varepsilon$ for all $n \geq N$, and hence the sequence $\{\frac{f(x_n)}{g(x_n)}\}$ converges to q. □

6. Higher order derivatives

If $f : (a, b) \to \mathbb{R}$ is a function differentiable in (a, b), then the derivative $g = f'$ is a function $g : (a, b) \to \mathbb{R}$. Hence it makes sense to talk about differentiability of f', which leads to the notion of higher order derivatives.

Definition 6.1. We say that $f : (a, b) \to \mathbb{R}$ is *twice differentiable* at $x \in (a, b)$, if there exists $\varepsilon > 0$ such that f is differentiable in $(x - \varepsilon, x + \varepsilon)$, and if f' is differentiable at x. We call

$$f''(x) \equiv \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) = g'(x), \tag{90}$$

the *second order derivative* of f at x.

Example 6.2. For $f(x) = x^3$, we have $f'(x) = 3x^2$ and $f''(x) = 6x$. Hence f is twice differentiable at each $x \in \mathbb{R}$, i.e., it is twice differentiable in $\mathbb{R}$.

Remark 6.3. Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$. Then by definition, there is a function $\phi : (a, b) \to \mathbb{R}$ that is continuous at c with $\phi(c) = 0$, such that

$$f(x) = f(c) + f'(c)(x - c) + \phi(x)(x - c), \qquad x \in (a, b). \tag{91}$$

Now, assume that f is twice differentiable at c. By definition, this means that f is differentiable in $(c - \varepsilon, c + \varepsilon)$ for some $\varepsilon > 0$, and that there is a function $g : (a, b) \to \mathbb{R}$ that is continuous at c with $g(c) = f''(c)$, such that

$$f'(x) = f'(c) + g(x)(x - c), \qquad x \in (c - \varepsilon, c + \varepsilon). \tag{92}$$

In other words, for any sequence $\{x_n\} \subset (c - \varepsilon, c) \cup (c, c + \varepsilon)$, we have

$$\frac{f'(x_n) - f'(c)}{x_n - c} \to f''(c) \qquad \text{as } n \to \infty. \tag{93}$$

Since $[f(x) - f(c) - f'(c)(x - c)]' = f'(x) - f'(c)$ and $[\frac{1}{2}(x - c)^2]' = x - c$, by L'Hôpital's rule, for any sequence $\{x_n\} \subset (c - \varepsilon, c) \cup (c, c + \varepsilon)$, this implies that

$$\frac{f(x_n) - f(c) - f'(c)(x_n - c)}{\frac{1}{2}(x_n - c)^2} \to f''(c) \qquad \text{as } n \to \infty. \tag{94}$$

Note that the function

$$F(x) = \frac{f(x) - f(c) - f'(c)(x - c)}{\frac{1}{2}(x - c)^2}, \tag{95}$$

is well defined in $(a, c) \cup (c, b)$. Then upon setting $F(c) = f''(c)$, by (94) we have F continuous at c. In other words, there is a function $\psi : (a, b) \to \mathbb{R}$ that is continuous at c with $\psi(c) = 0$, such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2}(x - c)^2 + \psi(x)(x - c)^2, \qquad x \in (a, b). \tag{96}$$

Thus in a certain sense, the existence of the second derivative guarantees that the function can be approximated by a quadratic polynomial well.

The following result is known as the *second derivative test.*

Lemma 6.4. *Let $f : (a, b) \to \mathbb{R}$ be a function, and suppose that f is twice differentiable at $c \in (a, b)$. Then the following are true.*

a) If c is a maximum of f over the interval (a, b), in the sense that $f(x) \le f(c)$ for all $x \in (a, b)$, then $f'(c) = 0$ and $f''(c) \le 0$.
b) If $f''(c) < 0$, then there exists $\varepsilon > 0$ such that c is a strict maximum of f over $(c - \varepsilon, c + \varepsilon)$.

Proof. a) Let c be a maximum of f over the interval (a, b). Then the assertion $f'(c)$ simply follows from the first derivative test (Lemma 5.1). Suppose that $f''(c) > 0$. By Remark 6.3, there exists a function $\psi : (a, b) \to \mathbb{R}$ that is continuous at c with $\psi(c) = 0$, such that (96) holds. Let $x \in (c, b)$ be such that $|\psi(x)| < \frac{1}{2} f''(c)$. Then (96) implies that $f(x) > f(c)$, contradicting the maximality of $f(c)$. □

Exercise 6.5. Prove b) of the preceding lemma.

Remark 6.6. Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$. Then by definition, there is a function $\phi : (a, b) \to \mathbb{R}$ that is continuous at c with $\phi(c) = 0$, such that

$$f(x) = f(c) + f'(c)(x - c) + \phi(x)(x - c), \qquad x \in (a, b). \tag{97}$$

On the other hand, if f is continuous in $[c, y]$ and f is differentiable in (c, y) where $y \in (c, b)$, then the mean value theorem tells us that there exists $\xi \in (c, y)$ such that

$$f(y) = f(c) + f'(\xi)(y - c). \tag{98}$$

The analogue of this for second order derivatives can be obtained as follows. In the proof of the mean value theorem, we constructed a function of the form $F(x) = f(x) + A(x - c)$ with $F(c) = F(y) = f(c)$. Here we look for a function $F(x) = f(x) + A(x - c) + B(x - c)^2$ with $F(c) = F(y) = f(c)$ and $F'(c) = 0$, and easily find such a function as

$$F(x) = f(x) - f'(c)(x - c) - \big[f(y) - f(c) - f'(c)(y - c)\big] \frac{(x - c)^2}{(y - c)^2}. \tag{99}$$

We assume that $f : (a, b) \to \mathbb{R}$ is twice differentiable in (c, y), and continuously differentiable in $[c, y)$, where $y \in (c, b)$. Then F is twice differentiable in (c, y), with

$$F'(x) = f'(x) - f'(c) - \big[f(y) - f(c) - f'(c)(y - c)\big] \frac{2(x - c)}{(y - c)^2}, \tag{100}$$

and

$$F''(x) = f''(x) - \frac{2[f(y) - f(c) - f'(c)(y - c)]}{(y - c)^2}. \tag{101}$$

Moreover, $F'(c)$ exists and $F' \in \mathscr{C}([c, y))$. Since $F(c) = F(y)$, by Rolle's theorem, there is $\xi \in (c, y)$ such that $F'(\xi) = 0$. Now recalling that $F'(c) = 0$ and $F' \in \mathscr{C}([c, y))$, another application of Rolle's theorem gives the existence of $\eta \in (c, \xi)$ such that $F''(\eta) = 0$. In other words, we have

$$f(y) = f(c) + f'(c)(y - c) + \frac{1}{2} f''(\eta)(y - c)^2, \tag{102}$$

for some $\eta \in (c, y)$.

Remark 6.7. We give here an application of (102). Let $f : (a, b) \to \mathbb{R}$ be twice differentiable in (a, b), satisfying $f''(x) \geq 0$ for $x \in (a, b)$. Of course, this implies that $f \in \mathscr{C}^1((a, b))$. Let $x, y, z \in (a, b)$ be such that $x < y < z$. Then by the preceding remark, there exists $\xi \in (y, z)$ such that

$$f(z) = f(y) + f'(y)(z - y) + \frac{1}{2} f''(\xi)(z - y)^2. \tag{103}$$

Since $f''(\xi) \geq 0$, we infer that

$$f(z) \geq f(y) + f'(y)(z - y). \tag{104}$$

Similarly, we get

$$f(x) \geq f(y) + f'(y)(x - y). \tag{105}$$

By multiplying the last two inequalities by positive constants α and β, respectively, and summing them, we have

$$\alpha f(x) + \beta f(z) \geq (\alpha + \beta) f(y) + [\alpha(x - y) + \beta(z - y)] f'(y). \tag{106}$$

Now we impose the conditions $\alpha + \beta = 1$ and $\alpha(x - y) + \beta(z - y) = 0$, or what is the same, pick $\alpha + \beta = 1$ and then set $y = \alpha x + \beta z$. This gives

$$\alpha f(x) + (1 - \alpha) f(z) \geq f(\alpha x + (1 - \alpha) z) \qquad x, z \in (a, b),\ \alpha \in (0, 1). \tag{107}$$

The assertion (107) is precisely the definition of *convexity of f in (a, b)*. Therefore, functions with nonnegative second derivatives are convex. Note that if $f'' > 0$ in (a, b) then we would have (107) with strict inequality, meaning that f would be *strictly convex*.

Example 6.8. Let $f(x) = x^q$ for some $q \in \mathbb{Q}$. Then $f''(x) = q(q-1)x^{q-2}$ for $x > 0$. So if $q > 1$ or $q < 0$ then f is strictly convex, i.e.,

$$(\lambda x + (1-\lambda)y)^q < \lambda x^q + (1-\lambda)y^q \qquad \text{for } x > y > 0,\ \lambda \in (0,1), \tag{108}$$

with $y = 0$ allowed when $q > 1$. In particular, taking $\lambda = \frac{1}{2}$, we infer

$$(x+y)^q < 2^{q-1}(x^q + y^q) \qquad \text{for } x > y > 0. \tag{109}$$

On the other hand, if $0 < q < 1$, then $-f$ is strictly convex, that is, f is *strictly concave*. This simply means that the inequalities will be reversed, i.e., we have

$$(\lambda x + (1-\lambda)y)^q > \lambda x^q + (1-\lambda)y^q \qquad \text{for } x > y \geq 0,\ \lambda \in (0,1). \tag{110}$$

Let $y = 0$ and $x = a + b$ for some positive numbers a and b. Moreover, put $\lambda = \frac{a}{a+b}$ so that $\lambda x = a$. Then we get

$$\frac{a}{a+b}(a+b)^q < a^q. \tag{111}$$

Now put $\lambda = \frac{b}{a+b}$ so that $\lambda x = b$. This yields

$$\frac{b}{a+b}(a+b)^q < b^q, \tag{112}$$

and by summing the last two inequalities, we conclude

$$(a+b)^q < a^q + b^q \qquad \text{for } a > 0,\ b > 0. \tag{113}$$

Definition 6.9. Let $f : (a,b) \to \mathbb{R}$. Then for $n = 0, 1, \ldots$, we define the *n-th order derivative* $f^{(n)} : K_n \to \mathbb{R}$ with the domain $K_n \subset (a,b)$, as follows.

- We set $f^{(0)}(x) = f(x)$ for $x \in (a,b)$ and $K_0 = (a,b)$. So any function is zero times differentiable in its domain.
- For $n = 1, 2, \ldots$, we say that f is *n times differentiable* at $x \in (a,b)$, if there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset K_{n-1}$, and if $f^{(n-1)}$ is differentiable at x. We set $f^{(n)}(x) \equiv \frac{\mathrm{d}^n f}{\mathrm{d}x^n}(x) = [f^{(n-1)}]'(x)$, and define K_n to be the set of all $x \in (a,b)$ at which f is n times differentiable.

Example 6.10. (a) Let $f(x) = x^k$ with $k \in \mathbb{N}_0$ and take its domain to be $\mathbb{R}$. Then we have

$$f^{(0)}(x) = x^k, \qquad f^{(1)}(x) = kx^{k-1}, \qquad f^{(2)}(x) = k(k-1)x^{k-2}, \tag{114}$$

and in general,

$$\begin{aligned} f^{(n)}(x) &= k(k-1)\cdots(k-n+1)x^{k-n} = \frac{k!}{(k-n)!}x^{k-n} \qquad &&\text{for } n \leq k, \\ f^{(n)}(x) &= 0 &&\text{for } n > k. \end{aligned} \tag{115}$$

Obviously, $K_n = \mathbb{R}$ for all n.

(b) Let $f(x) = x^q$ with $q \in \mathbb{Q} \setminus \mathbb{N}_0$ and take its domain to be $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. We have

$$f^{(n)}(x) = q(q-1)\cdots(q-n+1)x^{q-n} \qquad \text{for } n \in \mathbb{N},\ x > 0. \tag{116}$$

The following theorem extends Remark 6.3 and Remark 6.6 to higher order derivatives. A version of this theorem was stated by Brook Taylor in 1712, but the first rigorous proof was given by Joseph-Louis Lagrange in 1796.

Theorem 6.11. *Let $f : (a,b) \to \mathbb{R}$ and let $c \in (a,b)$.*

a) If f is n times differentiable at $c \in (a,b)$, then there is a function $\psi : (a,b) \to \mathbb{R}$ that is continuous at c with $\psi(c) = 0$, such that

$$f(x) = f(c) + f'(c)(x-c) + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \psi(x)(x-c)^n, \qquad x \in (a,b). \tag{117}$$

b) *We assume that $f:(a,b)\to\mathbb{R}$ is n times differentiable in (c,x), and that $f^{(n-1)}$ exists and continuous in $[c,x)$, where $x\in(c,b)$. Then there exists $\xi\in(c,x)$, such that*

$$f(x)=f(c)+f'(c)(x-c)+\ldots+\frac{f^{(n-1)}(c)}{(n-1)!}(x-c)^{n-1}+\frac{f^{(n)}(\xi)}{n!}(x-c)^n. \tag{118}$$

Exercise 6.12. Prove this theorem. *Hint*: (117) extends (96), and (118) extends (102).

Example 6.13. (a) For $f(x)=x^m$ with $m\in\mathbb{N}_0$, we have $f^{(k)}\equiv 0$ for $k>m$. Thus Theorem 6.11b) with $n=m+1$ yields

$$x^m=c^m+mc^{m-1}(x-c)+\ldots+\frac{m!}{n!}(x-c)^m=\sum_{k=0}^m\frac{m!}{(m-k)!k!}c^{m-k}(x-c)^k, \tag{119}$$

which is of course the binomial formula, cf. Exercise 1.16. Putting $c=1$ and replacing $x-c$ with x, we can bring it into the convenient form

$$(1+x)^m=\sum_{k=0}^m\binom{m}{k}x^k,\qquad x\in\mathbb{R}, \tag{120}$$

where $\binom{m}{k}=\frac{m(m-1)\cdots(m-k+1)}{m!}=\frac{m!}{k!(m-k)!}$.

(b) For $f(x)=x^q$ with $q\in\mathbb{Q}\setminus\mathbb{N}_0$, the derivative $f^{(n)}$ is not trivial for any n, so Theorem 6.11 would not yield a finite formula for $(1+x)^q$. What we get is that given any $x>-1$ and any $n\in\mathbb{N}$, there exists $\xi_n\in\mathbb{R}$ with $|\xi_n|<|x|$ such that

$$(1+x)^q=\sum_{k=0}^{n-1}\frac{q(q-1)\cdots(q-k+1)}{k!}x^k+\frac{q(q-1)\cdots(q-n+1)}{n!}\xi_n^n. \tag{121}$$

We write it as

$$(1+x)^q=T_{n-1}(x)+\binom{q}{n}\xi_n^n, \tag{122}$$

where

$$T_n(x)=\sum_{k=0}^n\binom{q}{k}x^k\qquad\text{and}\qquad\binom{q}{k}=\frac{q(q-1)\cdots(q-k+1)}{k!}. \tag{123}$$

It is in fact true that $T_n(x)\to(1+x)^q$ as $n\to\infty$, whenever $|x|<1$, leading to *Newton's binomial theorem* (also known as the *binomial series*)

$$(1+x)^q=\sum_{k=0}^\infty\binom{q}{k}x^k\qquad\text{for } |x|<1. \tag{124}$$

To establish convergence, we start with the observation

$$|(1+x)^q-T_{n-1}(x)|=\Big|\binom{q}{n}\xi_n^n\Big|\leq\Big|\binom{q}{n}\Big||x|^n. \tag{125}$$

From now on, we assume that $|x|<1$. Let $\rho\in(|x|,1)$, and let N be so large that

$$\Big|\frac{q-n+1}{n}\Big||x|\leq\rho\qquad\text{for all } n\geq N. \tag{126}$$

This is possible, because

$$\Big|\frac{q-n+1}{n}\Big|-1\leq\frac{|q|+1}{n}\to 0\qquad\text{as } n\to\infty. \tag{127}$$

Then for $n\geq N$, we have

$$\Big|\binom{q}{n}\Big||x|^n\leq\Big|\binom{q}{N}\Big||x|^N\cdot\Big|\frac{q-N}{N+1}\Big||x|\cdots\Big|\frac{q-n+1}{n}\Big||x|\leq\Big|\binom{q}{N}\Big||x|^N\cdot\rho^{n-N} \tag{128}$$

Since N is fixed, we conclude that

$$|(1+x)^q - T_{n-1}(x)| \leq \left|\binom{q}{n}\right||x|^n \leq \left|\binom{q}{N}\right||x|^N \cdot \rho^{n-N} \to 0 \qquad \text{as } n \to \infty. \tag{129}$$

7. Uniform convergence

The class of functions we have considered so far has been generated by starting with the rational powers (x^q, $q \in \mathbb{Q}$), and by combining them by using finitely many addition, subtraction, multiplication, quotient, and composition operations. Then a natural question is if we can extend the rational powers to real powers, i.e., x^a for any $a \in \mathbb{R}$. A simple idea would be

- to construct a sequence $\{a_n\} \subset \mathbb{Q}$ converging to $a \in \mathbb{R}$,
- and to define x^a as the limit of x^{a_n} as $n \to \infty$.

In order to successfully carry out this program, we need to address the following questions.

- Given any $a \in \mathbb{R}$, can we construct a sequence $\{a_n\} \subset \mathbb{Q}$ converging to a?
- Supposing that we have such a sequence $\{a_n\}$, does the sequence $\{x^{a_n}\}$ converge?
- Would the limit of $\{x^{a_n}\}$ depend on the particular sequence $\{a_n\}$?

The first question can be answered easily. Namely, given $a \in \mathbb{R}$, let

$$a_n = \max\left\{\frac{k}{m} : k \in \mathbb{Z},\, m \in \mathbb{N},\, m \leq n,\, \frac{k}{m} \leq a\right\} \qquad \text{for } n \in \mathbb{N}. \tag{130}$$

By construction, we have $a_n \leq a_{n+1} \leq a$ for all n. Moreover, for any $\varepsilon > 0$, there is $q \in \mathbb{Q}$ such that $a - \varepsilon < q < a$ by Theorem 2.7. If we write $q = \frac{k}{n}$, then $q \leq a_n \leq a$, meaning that $a_n \to a$ as $n \to \infty$.

The affirmative answers to the remaining questions are given in the following lemma.

Lemma 7.1. *Let $x > 0$, and let $a \in \mathbb{R}$. Then there exists $y \in \mathbb{R}$ such that $x^{a_n} \to y$ as $n \to \infty$, whenever $\{a_n\} \subset \mathbb{Q}$ is a sequence converging to a.*

Proof. Let $\{a_n\} \subset \mathbb{Q}$ be a sequence converging to a. We want to show that $\{x^{a_n}\}$ is a Cauchy sequence, and then invoke Theorem 2.19 to establish the convergence of $\{x^{a_n}\}$. Assume that $x > 1$. Since $\{a_n\}$ is convergent, it is bounded, i.e., there is M such that $|a_n| \leq M$ for all n. Hence for any $m, n \in \mathbb{N}$, we have

$$|x^{a_n} - x^{a_m}| = x^{\min\{a_m,a_n\}}(x^{|a_m-a_n|} - 1) \leq x^M(x^{|a_m-a_n|} - 1) \tag{131}$$

Let $\varepsilon > 0$, and let $k \in \mathbb{N}$ be so large that $x^M(\sqrt[k]{x} - 1) < \varepsilon$. This is possible since $\sqrt[k]{x} \to 1$ as $k \to \infty$. Now, let N be such that $|a_m - a_n| < \frac{1}{k}$ for all $m \geq N$ and $n \geq N$. Then we have

$$|x^{a_n} - x^{a_m}| \leq x^M(x^{|a_m-a_n|} - 1) \leq x^M(x^{1/k} - 1) \leq \varepsilon, \tag{132}$$

for all $m \geq N$ and $n \geq N$, which means that $\{x^{a_n}\}$ is a Cauchy sequence. By Theorem 2.19, there exists $y \in \mathbb{R}$ such that $x^{a_n} \to y$ as $n \to \infty$.

The case $0 < x < 1$ can be reduced to the case $x > 1$ as follows. Let $0 < x < 1$. Then $\frac{1}{x} > 0$, and so $x^{a_n} = (\frac{1}{x})^{-a_n}$ form a Cauchy sequence by the preceding paragraph, as the sequence $\{-a_n\}$ is convergent.

Now let $\{a_n\}$ and $\{b_n\}$ be sequences both converging to a. By the preceding discussion, there exist $y \in \mathbb{R}$ and $z \in \mathbb{R}$ such that $x^{a_n} \to y$ and $x^{b_n} \to z$ as $n \to \infty$. In order to show that $y = z$, we start with the inequality

$$|y - z| \leq |y - x^{a_n}| + |x^{a_n} - x^{b_n}| + |x^{b_n} - z|. \tag{133}$$

Let $\varepsilon > 0$. Then there exists N such that $|y - x^{a_n}| \leq \varepsilon$ and $|x^{b_n} - z| \leq \varepsilon$ for all $n \geq N$. Assuming that $x > 1$, similarly to (131), we infer

$$|x^{a_n} - x^{b_n}| = x^{\min\{a_n,b_n\}}(x^{|a_n-b_n|} - 1) \leq x^M(x^{|a_n-b_n|} - 1), \tag{134}$$

where M is an upper bound on both $\{a_n\}$ and $\{b_n\}$. This makes it clear that as in (132), we can choose $n \geq N$ so large that $|x^{a_n} - x^{b_n}| \leq \varepsilon$, i.e., $|y - z| \leq 3\varepsilon$. As $\varepsilon > 0$ is arbitrary, we conclude that $y = z$. The remaining case $0 < x < 1$ can be dealt with as in the preceding paragraph. □

This result makes the following definition possible.

Definition 7.2. Given $x > 0$ and $a \in \mathbb{R}$, the *power function* x^a is defined as the limit of x^{a_n}, where $\{a_n\} \subset \mathbb{Q}$ is a sequence converging to a.

Now we derive some of the important algebraic properties of the power function.

Lemma 7.3. *a) For $x > 0$ and $a, b \in \mathbb{R}$, we have $x^a x^b = x^{a+b}$ and $(x^a)^b = x^{ab}$.*
b) For $x, y > 0$ and $a \in \mathbb{R}$, we have $(xy)^a = x^a y^a$.
c) If $0 < x < y$ then $x^a < y^a$ for $a > 0$ and $x^a > y^a$ for $a < 0$.
d) If $a < b$ then $x^a < x^b$ for $x > 1$ and $x^a > x^b$ for $0 < x < 1$.

Proof. These properties follow from their rational power equivalents by continuity.

b) Let $\{a_n\} \subset \mathbb{Q}$ be a sequence converging to a. Then we have $(xy)^{a_n} = x^{a_n} y^{a_n}$. Since $x^{a_n} \to x^a$ and $y^{a_n} \to y^a$ as $n \to \infty$, by Theorem 3.1b) we have $x^{a_n} y^{a_n} \to x^a y^a$ as $n \to \infty$. Therefore for any $\varepsilon > 0$, there exists N such that $|x^{a_n} y^{a_n} - x^a y^a| \leq \varepsilon$ for all $n \geq N$. On the other hand, there exists M such that $|(xy)^{a_n} - (xy)^a| \leq \varepsilon$ for all $n \geq M$. Thus we have

$$|(xy)^a - x^a y^a| \leq |(xy)^a - (xy)^{a_n}| + |(xy)^{a_n} - x^{a_n} y^{a_n}| + |x^{a_n} y^{a_n} - x^a y^a| \leq 2\varepsilon, \tag{135}$$

for all $n \geq \max\{M, N\}$. As $\varepsilon > 0$ is arbitrary, we conclude that $(xy)^a = x^a y^a$.

d) Let $\{a_n\} \subset \mathbb{Q}$ be a non-increasing sequence ($a_{n+1} \leq a_n$) that converges to a, and let $\{b_n\} \subset \mathbb{Q}$ be a non-decreasing sequence ($b_{n+1} \geq b_n$) that converges to b, cf. (130). Assume that $x > 1$. Then since $x^{b_n} \leq x^{b_{n+1}}$, we have $x^{b_n} \leq x^b$ for all n. Similarly, we have $x^{a_n} \geq x^a$ for all n. Hence for n so large that $a_n < b_n$, it holds that $x^a \leq x^{a_n} < x^{b_n} \leq x^b$. The case $0 < x < 1$ can be dealt with in an analogous manner. □

Exercise 7.4. Prove a) and c) of the preceding lemma.

Now we turn to continuity and differentiability of the power function $f(x) = x^a$ with fixed $a \in \mathbb{R}$. For rational powers $a \in \mathbb{Q}$, we know that this function is not only continuous, but also differentiable at each $x > 0$, with $f'(x) = ax^{a-1}$. As x^a with $a \in \mathbb{R} \setminus \mathbb{Q}$ can be approximated by x^q with $q \in \mathbb{Q}$ arbitrarily well, it is reasonable to expect that f be continuous. However, the question is not trivial, as the following example shows.

Example 7.5. For each $n \in \mathbb{N}$, let $\theta_n : \mathbb{R} \to \mathbb{R}$ be given by

$$\theta_n(x) = \begin{cases} 1 & \text{for } x \geq \frac{1}{n}, \\ nx & \text{for } 0 < x < \frac{1}{n}, \\ 0 & \text{for } x \leq 0. \end{cases} \tag{136}$$

Each θ_n is a continuous function in $\mathbb{R}$. Given any $x > 0$, we have $\theta_n(x) = 1$ for all large n, and hence for each $x \in \mathbb{R}$, we have $\theta_n(x) \to \theta(x)$ as $n \to \infty$, where

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases} \tag{137}$$

Clearly, θ is not continuous at 0. Thus just because each θ_n is continuous, and $\theta_n(x)$ converges to $\theta(x)$ at each fixed x, does not mean that the limit θ is continuous.

Let $K \subset \mathbb{R}$ be a nonempty set. By a *sequence of functions* (with the domain K) we mean simply an assignment of a function $f_n : K \to \mathbb{R}$ to each index n, with the latter usually having positive integers as values. We denote this sequence by $\{f_1, f_2, \ldots\}$ or $\{f_n\}$, and consider it also as a collection of functions. So for example, $\{f_n\} \subset \mathscr{C}(K)$ would mean that every function in the sequence is continuous.

Definition 7.6. A sequence $\{f_n\}$ is said to *converge pointwise in* K to a function $f : K \to \mathbb{R}$, if for each $x \in K$, $f_n(x) \to f(x)$ as $n \to \infty$.

Pointwise convergence is a very weak kind of convergence. For instance, as we have seen in Example 7.5, the pointwise limit of a sequence of continuous functions is *not* necessarily continuous. The notion of uniform convergence is a stronger type of convergence that remedies this deficiency.

Definition 7.7. A sequence $\{f_n\}$ is said to *converge uniformly in* K to a function $f : K \to \mathbb{R}$, if for any $\varepsilon > 0$, there exists N such that $|f_n(x) - f(x)| \leq \varepsilon$ for any $x \in K$ and all $n \geq N$.

Example 7.8. Let $\alpha \in \mathbb{R}$, and let $\{\alpha_n\} \subset \mathbb{Q}$ be a sequence converging to α. We consider the sequence $f_n(x) = x^{\alpha_n}$, $n \in \mathbb{N}$, with the domain $[a, b]$ for some $0 < a < b$. We want to show that $\{f_n\}$ converges uniformly in $[a, b]$ to the function f given by $f(x) = x^\alpha$. Note that

$$|x^{\alpha_n} - x^\alpha| = x^\alpha |x^{\alpha_n - \alpha} - 1| \leq \max\{a^\alpha, b^\alpha\} \cdot \max\{|a^{\alpha_n - \alpha} - 1|, |b^{\alpha_n - \alpha} - 1|\}, \tag{138}$$

for $x \in [a, b]$. Given any $\varepsilon > 0$, there exists N such that $M|a^{\alpha_n - \alpha} - 1| < \varepsilon$ and $M|b^{\alpha_n - \alpha} - 1| < \varepsilon$ for all $n \geq N$, where $M = \max\{a^\alpha, b^\alpha\}$. Hence $\{f_n\}$ converges uniformly in $[a, b]$ to f.

Exercise 7.9. Show that uniform convergence implies pointwise convergence.

Theorem 7.10 (Weierstrass 1861)**.** *Suppose that $\{f_n\} \subset \mathscr{C}(K)$ converges uniformly in K to a function $f : K \to \mathbb{R}$. Then $f \in \mathscr{C}(K)$.*

Proof. We will use the sequential criterion of continuity. Let $x \in K$, and let $\{x_k\} \subset K$ be a sequence converging to x. Then we have

$$f(x_k) - f(x) = f(x_k) - f_n(x_k) + f_n(x_k) - f_n(x) + f_n(x) - f(x), \tag{139}$$

for any n, and hence

$$|f(x_k) - f(x)| \leq |f(x_k) - f_n(x_k)| + |f_n(x_k) - f_n(x)| + |f_n(x) - f(x)|, \tag{140}$$

by the triangle inequality.

Let $\varepsilon > 0$ be given. Then by uniform convergence, there is N such that

$$|f(y) - f_n(y)| \leq \varepsilon, \qquad \text{for all} \quad y \in K, \quad \text{and for all} \quad n \geq N. \tag{141}$$

In particular,

$$|f(x_k) - f(x)| \leq 2\varepsilon + |f_n(x_k) - f_n(x)|, \qquad \text{for all} \quad k, \quad \text{and for all} \quad n \geq N. \tag{142}$$

Now we fix one such n, for example, put $n = N$, and use the continuity of f_n to imply the existence of M with the property that

$$|f_n(x_k) - f_n(x)| \leq \varepsilon, \qquad \text{for all} \quad k \geq M. \tag{143}$$

Finally, this means that

$$|f(x_k) - f(x)| \leq 3\varepsilon, \qquad \text{for all} \quad k \geq M, \tag{144}$$

and since $\varepsilon > 0$ was arbitrary, we infer that f is continuous at x. □

Exercise 7.11. Find a mistake in the following purported proof.

Claim: If $\{f_n\} \subset \mathscr{C}(K)$ converges pointwise in K to a function $f : K \to \mathbb{R}$, then f is continuous in K.

Proof: Let $x \in K$, and let $\{x_k\} \subset K$ be a sequence converging to x. Then as in the preceding proof, we have

$$|f(x_k) - f(x)| \leq |f(x_k) - f_n(x_k)| + |f_n(x_k) - f_n(x)| + |f_n(x) - f(x)|. \tag{145}$$

Since f_n converges pointwise to f, both $|f(x_k) - f_n(x_k)|$ and $|f_n(x) - f(x)|$ tend to 0 as $n \to \infty$. Furthermore, $|f_n(x_k) - f_n(x)| \to 0$ as $k \to \infty$, because the function f_n is continuos. Hence by choosing k and n large enough, we can make the right hand side of (145) arbitrarily small, which means that f is continuous at x.

Corollary 7.12. *For $\alpha \in \mathbb{R}$, the function $f(x) = x^\alpha$ is continuous at each $x > 0$.*

Proof. Let $x > 0$, and pick $a, b \in \mathbb{R}$ satisfying $0 < a < x < b$. Then in the context of Example 7.8, we have a sequence $\{f_n\} \subset \mathscr{C}([a, b])$ converging uniformly in $[a, b]$ to f. The Weierstrass convergence theorem (Theorem 7.10) implies that $f \in \mathscr{C}([a, b])$. In particular, f is continuous at x. □

Exercise 7.13. For $\alpha \geq 0$, show that the function $f(x) = x^\alpha$ is continuous at 0.

Turning to the differentiability issue, for $f_n(x) = x^{\alpha_n}$, we have $f_n'(x) = \alpha_n x^{\alpha_n - 1}$. So if $\alpha_n \to \alpha$ then f_n' converges uniformly in $[a, b]$ to $g(x) = \alpha x^{\alpha-1}$, for any $[a, b] \subset (0, \infty)$. Since f_n converges to $f(x) = x^\alpha$, we expect that $f' = g$. This is confirmed in the following theorem.

Theorem 7.14. *Let $\{f_n\}$ be a sequence of differentiable functions $f_n : (a, b) \to \mathbb{R}$, such that $\{f_n\}$ converges pointwise in (a, b) to a function $f : (a, b) \to \mathbb{R}$, and that $\{f_n'\}$ converges uniformly in (a, b) to a function $g : (a, b) \to \mathbb{R}$. Then f is differentiable in (a, b), and $f'(x) = g(x)$ for $x \in (a, b)$.*

Proof. Let $x \in (a, b)$, and let $\{x_k\} \subset (a, b) \setminus \{x\}$ be a sequence converging to x. Having in mind the sequential criterion of differentiability, we want to show that $\frac{f(x_k)-f(x)}{x_k - x} \to g(x)$ as $k \to \infty$. Let $\varepsilon > 0$, and let n be such that

$$|f_m'(\xi) - g(\xi)| < \varepsilon \qquad \text{for any } \xi \in (a, b), \text{ and for all } m \geq n. \tag{146}$$

Such n exists by the uniform convergence $f_n \to g$. Moreover, since f_n is differentiable in (a, b), there exists an index K such that

$$\Big|\frac{f_n(x_k) - f_n(x)}{x_k - x} - f_n'(x)\Big| \leq \varepsilon \qquad \text{for all } k \geq K. \tag{147}$$

We have

$$\begin{aligned}\frac{f(x_k) - f(x)}{x_k - x} - g(x) = \Big(&\frac{f(x_k) - f(x)}{x_k - x} - \frac{f_n(x_k) - f_n(x)}{x_k - x}\Big) \\ &+ \Big(\frac{f_n(x_k) - f_n(x)}{x_k - x} - f_n'(x)\Big) + \big(f_n'(x) - g(x)\big),\end{aligned} \tag{148}$$

which implies that

$$\Big|\frac{f(x_k) - f(x)}{x_k - x} - g(x)\Big| \leq \Big|\frac{f(x_k) - f(x)}{x_k - x} - \frac{f_n(x_k) - f_n(x)}{x_k - x}\Big| + 2\varepsilon \qquad \text{for all } k \geq K. \tag{149}$$

Now we claim that

$$\Big|\frac{f(x_k) - f(x)}{x_k - x} - \frac{f_n(x_k) - f_n(x)}{x_k - x}\Big| < 2\varepsilon \qquad \text{for all } k \geq K, \tag{150}$$

which would complete the proof.

To establish the claim, we replace $f(x)$ and $f(x_k)$ by $f_m(x)$ and $f_m(x_k)$, respectively, where $m \in \mathbb{N}$ is understood to be large, and invoke the mean value theorem, yielding

$$\begin{aligned}\frac{f_m(x_k) - f_m(x)}{x_k - x} - \frac{f_n(x_k) - f_n(x)}{x_k - x} &= \frac{(f_m - f_n)(x_k) - (f_m - f_n)(x)}{x_k - x} \\ &= (f_m - f_n)'(\xi) = f_m'(\xi) - f_n'(\xi),\end{aligned} \tag{151}$$

for some $\xi \in (a,b)$, which may depend on m and k. However, regardless of where ξ is, our set up (146) guarantees that

$$|f_m'(\xi) - f_n'(\xi)| \leq |f_m'(\xi) - g(\xi)| + |g(\xi) - f_n'(\xi)| < 2\varepsilon \qquad \text{for all } m \geq n. \tag{152}$$

Therefore we have

$$\begin{aligned} \Big|\frac{f(x_k)-f(x)}{x_k-x} - \frac{f_n(x_k)-f_n(x)}{x_k-x}\Big| &\leq \Big|\frac{f(x_k)-f(x)}{x_k-x} - \frac{f_m(x_k)-f_m(x)}{x_k-x}\Big| \\ &\quad + \Big|\frac{f_m(x_k)-f_m(x)}{x_k-x} - \frac{f_n(x_k)-f_n(x)}{x_k-x}\Big| \\ &< \frac{|f(x_k)-f_m(x_k)| + |f(x)-f_m(x)|}{|x_k-x|} + 2\varepsilon, \end{aligned} \tag{153}$$

for all $m \geq n$. Since $f_m(y) \to f(y)$ at each $y \in (a,b)$, for each fixed k, by choosing m sufficiently large, we can make $|f(x_k)-f_m(x_k)|+|f(x)-f_m(x)|$ arbitrarily small, thus establishing the claim (150). □

Corollary 7.15. *For $\alpha \in \mathbb{R}$, $f(x) = x^\alpha$ is differentiable at each $x > 0$, with $f'(x) = \alpha x^{\alpha-1}$.*

An immediate consequence is of course that x^α is infinitely often differentiable, with $(x^\alpha)^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)x^{\alpha-n}$.

8. Power series

In Example 6.13, we established that

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \qquad \text{for } |x| < 1, \tag{154}$$

where $\alpha \in \mathbb{Q}$ and $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$. If α is a nonnegative integer, then $\binom{\alpha}{n} = 0$ for all $n > \alpha$, and hence the sum in (154) yields a polynomial of degree α. In all other cases, the sum has infinitely many nonzero terms, and the equality must be understood in the sense that

$$(1+x)^\alpha = \lim_{m\to\infty} T_m(x), \qquad \text{where} \quad T_m(x) = \sum_{n=0}^{m} \binom{\alpha}{n} x^n. \tag{155}$$

Infinite sums such as (154) are called *series*. To establish the convergence, we only used the formula $(x^\alpha)^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)x^{\alpha-n}$ in combination with the Lagrange form of Taylor's theorem (Theorem 6.11), and the fact that $\frac{|\alpha|+1}{n} \to 0$ as $n \to \infty$, cf. (127). Since these facts also hold for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we conclude that the binomial theorem (154) is true for all real exponents $\alpha \in \mathbb{R}$.

Exercise 8.1. Write out a proof that the binomial theorem (154) is true for all $\alpha \in \mathbb{R}$.

Exercise 8.2. Explicitly compute the coefficients of (154) for $\alpha = \frac{1}{2}$, $\alpha = -1$, and $\alpha = -2$.

As it stands, the binomial theorem (154) is only valid for $|x| < 1$, and thus can be used to express the power function y^α only in the range $0 < y < 2$. However, we can rescale it to derive series that are valid in larger regions. Let $c > 0$, and let $0 < y < 2c$. Then by substituting $x = \frac{y-c}{c} \in (-1,1)$ into (154), we get

$$\frac{y^\alpha}{c^\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{(y-c)^n}{c^n} \qquad \text{or} \qquad y^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} c^{\alpha-n}(y-c)^n. \tag{156}$$

This is an example of a *power series* for the function $f(y) = y^\alpha$, centred at c. It is valid in the region $y \in (0, 2c)$.

Exercise 8.3. Derive the series (156) by computing the derivatives of y^α at $y = c$, and then invoking Taylor's theorem (Theorem 6.11).

It turns out that most of the important functions used in mathematics and sciences can be expressed as power series similar to (156). In this section, we shall study the main properties of power series. Before doing so, however, we need to introduce the concept of series and establish a couple of general results.

Given a sequence $\{a_0, a_1, \ldots\}$ of real numbers, the *series* with the terms $\{a_n\}$ is

$$\sum_{n=0}^{\infty} a_n, \tag{157}$$

which is understood as an overloaded notation for all of the following.

- The sequence $\{a_n\}$.
- The sequence $\{s_m\}$, where $s_m = a_0 + \ldots + a_m$ is called the *m-th partial sum*.
- The limit $\lim\limits_{m\to\infty} s_m = \lim\limits_{m\to\infty} (a_0 + \ldots + a_m)$, if it exists.

Thus when one says that the series $\sum_n a_n$ converges, one is referring to the sequence $\{s_m\}$. On the other hand, the equality $\sum_n a_n = b$, or the statement that the sum (or the value) of the series $\sum_n a_n$ is b, would be referring to the limit $\lim\limits_{m\to\infty} s_m$.

Example 8.4. (a) Let $x \in (-1, 1)$, and let us evaluate the value of the series $\sum_{n=0}^{\infty} x^n$. The n-th term of this series is $a_n = x^n$, and the n-th partial sum is $s_n = 1 + x + x^2 + \ldots + x^n$. Then the standard argument

$$s_n - xs_n = 1 + x + x^2 + \ldots + x^n - (x + x^2 + \ldots + x^n + x^{n+1}) = 1 - x^{n+1}, \tag{158}$$

implies that

$$s_n = \frac{1 - x^{n+1}}{1 - x}. \tag{159}$$

We make a guess that s_n converges to $s = \frac{1}{1-x}$, and compute

$$s - s_n = \frac{1}{1-x} - \frac{1 - x^{n+1}}{1 - x} = \frac{x^{n+1}}{1-x}, \tag{160}$$

which leads to

$$|s - s_n| = \frac{|x^{n+1}|}{|1-x|} \leq \frac{|x|^{n+1}}{|1-x|} \to 0 \qquad \text{as } n \to \infty. \tag{161}$$

Hence we conclude that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for } x \in (-1, 1). \tag{162}$$

(b) Consider the harmonic series $\sum_n \frac{1}{n}$, with the partial sums $A_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$. Note that the indexing starts at $n = 1$. We have

$$\begin{aligned} A_{2^k} &= 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{4}\cdot 2 = \frac{1}{2}} + \underbrace{\frac{1}{5} + \ldots + \frac{1}{8}}_{>\frac{1}{8}\cdot 4 = \frac{1}{2}} + \ldots + \underbrace{\frac{1}{2^{k-1}+1} + \ldots + \frac{1}{2^k}}_{>\frac{1}{2^k}\cdot 2^{k-1} = \frac{1}{2}} \\ &\geq 1 + \frac{1}{2}\cdot k = \frac{k+2}{2}, \end{aligned} \tag{163}$$

for $k = 1, 2, \ldots$, implying that the sequence $\{A_n\}$ diverges to ∞, that is,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty. \tag{164}$$

Remark 8.5 (Leibniz criterion)**.** Suppose that $s = \sum_n a_n$, i.e., that the sequence of partial sums $s_n = a_0 + \ldots + a_n$ converges to s. We have $a_n = s_n - s_{n-1}$, and hence

$$|a_n| = |s_n - s_{n-1}| \leq |s_n - s| + |s - s_{n-1}| \to 0 \qquad \text{as } n \to \infty. \tag{165}$$

This means that the terms of a convergent series must tend to 0. For example, we can immediately tell that the series $\sum_n (-1)^n$ is divergent, because $(-1)^n \not\to 0$. Note that the converse statement is not true: The harmonic series $\sum_n \frac{1}{n}$ diverges, even though the terms $\frac{1}{n}$ tend to 0.

Now we extend the concept of series to series whose terms are functions. Given a sequence $\{g_n\}$ of functions $g_n : K \to \mathbb{R}$ with some $K \subset \mathbb{R}$, the *function series* with the terms $\{g_n\}$ is

$$\sum_{n=0}^{\infty} g_n. \tag{166}$$

As with the series of numbers, this notation may stand for either of the sequences $\{g_n\}$ and $\{f_m\}$, where $f_m = g_0 + \ldots + g_m$ is the m-th partial sum. Naturally, it may also stand for the limit of the sequence $\{f_m\}$ comprising the partial sums, provided such a limit exist. However, since $\{f_m\}$ is a sequence of functions, one must in addition specify the mode of convergence, whether it is pointwise or uniform.

Example 8.6. Let $g_n(x) = x^n$. Then the partial sums of the series $\sum_n g_n$ are

$$f_n(x) = 1 + x + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}, \tag{167}$$

as we know from Example 8.4. In Example 8.4, $x \in (-1, 1)$ was a fixed number, and the series $\sum_n x^n$ was considered to be a series of numbers. Here, we consider $\sum_n x^n$ as a series of functions $g_n(x) = x^n$, with x varying in some interval. Recall that

$$|f(x) - f_n(x)| = \frac{|x^{n+1}|}{|1 - x|} \leq \frac{|x|^{n+1}}{|1 - x|}, \qquad \text{where } f(x) = \frac{1}{1 - x}. \tag{168}$$

Now if $|x| \leq r$ for some $r < 1$, then $|1 - x| \geq 1 - |x| \geq 1 - r$, and hence

$$|f(x) - f_n(x)| \leq \frac{r^{n+1}}{1 - r}. \tag{169}$$

This shows that f_n converges to f uniformly in $[-r, r]$, for any fixed $r < 1$. In other words, the series $\sum_{n=0}^{\infty} x^n$ converges uniformly in $[-r, r]$ to $\frac{1}{1-x}$, whenever $r < 1$.

Theorem 8.7 (Weierstrass M-test, majorant test, or comparison test)**.** *Let the functions $g_n : K \to \mathbb{R}$ satisfy $|g_n(x)| \leq a_n$ for all $x \in K$ and for each n, where $\sum_n a_n$ is a convergent series of real numbers. Then there exists a function $f : K \to \mathbb{R}$ such that the function series $\sum_n g_n$ converges uniformly in K to f.*

Proof. With $f_n = g_1 + \ldots + g_n$, for $x \in K$ and $n > m$, we have

$$|f_n(x) - f_m(x)| = |g_{m+1}(x)| + \ldots + |g_n(x)| \leq a_{m+1} + \ldots + a_n \leq \sum_{k=m+1}^{\infty} a_k, \tag{170}$$

which tends to 0 when $m \to \infty$. This means that $\{f_n(x)\}$ is a Cauchy sequence of real numbers, and hence there exists $\alpha \in \mathbb{R}$ such that $f_n(x) \to \alpha$ as $n \to \infty$. We set $f(x) := \alpha$. Since $x \in K$ is arbitrary, this procedure defines a function $f : K \to \mathbb{R}$, and by construction, $f_n \to f$ pointwise in K.

It remains to show that this convergence is uniform. Let $\varepsilon > 0$. Since the quantity in the right hand side of (170) does not depend on x, there exists N such that $|f_n(x) - f_m(x)| \leq \varepsilon$

for all $x \in K$ and all $n, m \geq N$. Now for any given $x \in K$, by pointwise convergence, there exists $m \geq N$ so large that $|f_m(x) - f(x)| \leq \varepsilon$. Therefore we have

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \leq 2\varepsilon, \tag{171}$$

whenever $n \geq N$ and $x \in K$, implying that $f_n \to f$ uniformly in K. □

Definition 8.8. A *power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n, \tag{172}$$

with the *coefficients* $a_n \in \mathbb{R}$ for $n = 0, 1, \ldots$, and the *centre* $c \in \mathbb{R}$.

Arguably the most important example of a power series is the geometric series $\sum_n x^n$ studied in Example 8.6. We know that this series converges for all x satisfying $|x| < 1$. On the other hand, if $|x| > 1$, then $|x^n| = |x|^n \not\to 0$ as $n \to \infty$, meaning that the series diverges. It turns out that this is basically happens in the general case; Given a power series, there is a special interval centred at c that separates convergence and divergence behaviours.

We start with the following important observation of Niels Henrik Abel (1802-1829).

Remark 8.9 (Abel 1826)**.** (a) Suppose that (172) converges at some $x_0 \neq c$. Then it is necessary that $|a_n(x_0 - c)^n| = |a_n||x_0 - c|^n \to 0$ as $n \to \infty$. In particular, the sequence $\{|a_n||x_0 - c|^n\}$ is bounded, i.e., there is some constant M such that

$$|a_n| r^n \leq M \qquad \text{for all} \quad n, \tag{173}$$

where $r = |x_0 - c|$.

(b) Suppose that the coefficients of the series (172) satisfy the estimate (173) for some constants $r > 0$ and M. Let $0 < \rho < r$ and let $z \in [c - \rho, c + \rho]$. Then

$$|a_n(x-c)^n| \leq |a_n| \rho^n \leq M \big(\frac{\rho}{r}\big)^n, \tag{174}$$

and since $\sum_n (\frac{\rho}{r})^n < \infty$, the *Weierstrass M-test is applicable* to (172) in the interval $[c-\rho, c+\rho]$. Therefore the series (172) converges uniformly in $[c - \rho, c + \rho]$.

(c) Combining (a) and (b), we infer the following. If the power series $\sum a_n(x-c)^n$ converges at $x = x_0$, then it converges at all points in the open interval $(c - r, c + r)$ with $r = |x_0 - c|$. Moreover, the convergence is uniform in $[c - \rho, c + \rho]$ for each $0 < \rho < r$.

Definition 8.10. From (b) of the previous remark we see that it is important to find the largest value of r for which the estimate (173) holds. To this end, we let

$$A = \{r \geq 0 : \text{the sequence } \{|a_n| r^n\} \text{ is bounded}\}, \tag{175}$$

and define

$$R = \sup A, \tag{176}$$

which is called *convergence radius* of the power series $\sum a_n(x-c)^n$.

Example 8.11. (a) If $a_n = n^n$, the sequence $\{a_n r^n\}$ diverges to ∞ whatever the value of $r > 0$. Therefore we have $A = \{0\}$ and hence $R = 0$ in this case.

(b) If $a_n = n^{-n}$, the sequence $\{a_n r^n\}$ converges to 0 whatever the value of $r \geq 0$. Therefore we have $A = [0, \infty)$ and hence $R = \infty$.

(c) If $a_n = 2^n$, the sequence $\{a_n r^n\}$ is bounded for $r \leq \frac{1}{2}$ and unbounded for $r > \frac{1}{2}$. Therefore we have $A = [0, \frac{1}{2}]$ and hence $R = \frac{1}{2}$.

(d) If $a_n = n2^n$, the sequence $\{a_n r^n\}$ is bounded for $r < \frac{1}{2}$ and unbounded for $r \geq \frac{1}{2}$. Therefore we have $A = [0, \frac{1}{2})$ and hence $R = \frac{1}{2}$.

By definition, the convergence radius R has the following characteristic properties.

- Given any $r < R$, there is M such that $|a_n| \leq Mr^{-n}$ for all n.
- For any $r > R$, the sequence $\{|a_n|r^n\}$ is unbounded.

This leads to the following.

- Suppose that x satisfy $|x - c| \leq \rho < R$, and pick some r such that $\rho < r < R$. Then there is M such that $|a_n| \leq Mr^{-n}$ for all n. This implies that
$$|a_n(x - c)^n| = |a_n||x - c|^n \leq M\big(\frac{\rho}{r}\big)^n, \tag{177}$$
hence the Weierstrass M-test is applicable in the interval $[c - \rho, c + \rho]$.
- If $|x - c| > R$, then $|a_n(x - c)^n| = |a_n||x - c|^n \not\to 0$ as $n \to \infty$, and so the power series $\sum a_n(x - c)^n$ diverges.

Therefore, the convergence radius of the power series $\sum a_n(x - c)^n$ can also defined as the (extended) real number $R \in [0, \infty]$ with the property that $\sum a_n(x - c)^n$ converges whenever $|x-c| < R$ and diverges whenever $|x-c| > R$. Note also that whenever $\rho < R$, the Weierstrass M-test is applicable in the interval $[c - \rho, c + \rho]$, hence it converges uniformly in $[c - \rho, c + \rho]$.

The following result was discovered by Cauchy in 1821.

Theorem 8.12 (Ratio test). *The convergence radius R of the power series (172) is by*
$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}, \tag{178}$$
provided that the limit exists. In particular, it requires that $a_n = 0$ for only finitely many n.

Proof. Without loss of generality, we will assume that $a_n \neq 0$ for all n (Equivalently, we only consider indices $n \geq M$ for some large M).

Let α be the limit in (178) and suppose that $|x| < \rho < \alpha$. Then there exists N such that $\frac{|a_n|}{|a_{n+1}|} \geq \rho$ for all $n \geq N$, which leads to the estimate
$$|a_{n+1}| \leq \rho^{-1}|a_n| \leq \ldots \leq \rho^{N-1-n}|a_N| \qquad \text{for} \quad n \geq N, \tag{179}$$
or $|a_n| \leq |a_N|\rho^N\rho^{-n}$ for $n > N$. Thus we have
$$|a_nx^n| \leq |a_N|\rho^N\big(\frac{|x|}{\rho}\big)^n \qquad \text{for} \quad n > N, \tag{180}$$
and so $\sum a_nx^n$ converges. This means that $\alpha \leq R$.

Now suppose that $|x| > \alpha$. Then there exists N such that $\frac{|a_n|}{|a_{n+1}|} \leq |x|$ for all $n \geq N$, which leads to the estimate $|a_nx^n| \geq |a_N||x|^N$ for $n > N$. Since $a_N \neq 0$, the series $\sum a_nx^n$ diverges, and hence $R \leq \alpha$. □

Example 8.13. (a) For $a_n = n!$, we have $\frac{|a_n|}{|a_{n+1}|} = \frac{1}{n+1} \to 0$ as $n \to \infty$. Therefore the convergence radius of the series $\sum n!x^n$ is 0.

(b) For $a_n = \frac{1}{n!}$, we have $\frac{|a_n|}{|a_{n+1}|} = n + 1 \to \infty$ as $n \to \infty$. Therefore the convergence radius of the series $\sum \frac{x^n}{n!}$ is ∞.

(c) For $a_n = (-1)^n n^3 2^n$, we have $\frac{|a_n|}{|a_{n+1}|} = \frac{n^3}{2(n+1)^3} \to \frac{1}{2}$ as $n \to \infty$. Therefore the convergence radius of the series $\sum(-1)^n n^3 2^n x^n$ is $\frac{1}{2}$.

(d) Let the sequence $\{a_n\}$ be given by $\{1, 1, 3, 3, 3^2, 3^2, 3^3, 3^3, \ldots\}$. Then the value of $\frac{|a_n|}{|a_{n+1}|}$ alternates between 1 and $\frac{1}{3}$, and hence the ratio test as stated is not applicable.

Exercise 8.14. Find the convergence radius of the series described in (d) of the preceding example. *Hint*: Use Definition 8.10 directly.

Now we turn to the question of termwise differentiating power series.

Theorem 8.15. *Let $0 < R \leq \infty$ be the convergence radius of the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n. \tag{181}$$

Then the power series

$$g(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}, \tag{182}$$

has convergence radius equal to R, and it holds that

$$f' = g \qquad \text{in } (c-R, c+R), \tag{183}$$

where in case $R = \infty$ it is understood that $(c-R, c+R) = \mathbb{R}$.

Proof. Without loss of generality, we will assume that $c = 0$. It is obvious that the convergence radius R' of the power series representing g is at most R, that is, $R' \leq R$. To prove the other direction, let $|x| < r < R$. Then there is some M such that $|a_n| r^n \leq M$ for all n, which implies that

$$n|a_n||x|^n \leq Mn\big(\frac{|x|}{r}\big)^n. \tag{184}$$

Since $|x| < r$, we have $\sum n(\frac{|x|}{r})^n < \infty$, and so $R \leq R'$.

Now we will show that $f' = g$ in $(c-R, c+R)$. To this end, let

$$f_m(x) = \sum_{n=0}^{m} a_n (x-c)^n, \qquad \text{and} \qquad g_m(x) = \sum_{n=1}^{m} n a_n (x-c)^{n-1}. \tag{185}$$

Then it is clear that $f_m' = g_m$ in $(c-R, c+R)$ for all m. Moreover, $f_m \to f$ and $g_m \to g$ both uniformly in $(c-r, c+r)$ for any $r < R$. By Theorem 7.14 this shows that f is differentiable in $(c-r, c+r)$ with $f' = g$. Since $r < R$ is arbitrary, we have $f' = g$ in $(c-R, c+R)$. $\square$

Example 8.16. So far, the only examples of power series we have considered are the binomial series. Since $(y^\alpha)' = \alpha y^{\alpha-1}$, differentiating a binomial series results in another binomial series, and hence would not serve as an interesting example of differentiation of power series. However, if we look at the right hand side of the equation $(y^\alpha)' = \alpha y^{\alpha-1}$, we notice that all powers of y can be obtained as the result of differentiation, except the power y^{-1}. Thus we can ask the question what would the function whose derivative is equal to y^{-1}, and try to construct such as function by using power series. We have

$$(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1} \qquad \text{for } |x| < 1, \tag{186}$$

Then in view of the ratio test (Theorem 8.12) and of Theorem 8.15, the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \ldots \tag{187}$$

satisfies $f'(x) = (1+x)^{-1}$ for $|x| < 1$. This function will in fact turn out to be the logarithm function, which is explored in the next section.

9. The exponential and logarithm functions

The repository of functions we have so far can be generated by starting with the power functions $f(x) = x^a$, $a \in \mathbb{R}$, and by combining them by using finitely many addition, subtraction, multiplication, quotient, and composition operations. In view of Theorem 4.6, the derivative of such a function will be a function from the same repository. The methods that can potentially generate new types of functions are inverse functions and function sequences. The latter is the most general method, and we shall here focus on a special yet powerful method of *differential equations*.

Roughly speaking, a differential equation is an equation involving a function and its derivatives. For example, $f'(x) + f(x) - x = 0$ is a differential equation. Given a differential equation, one is interested in finding a function f satisfying the equation. Such a function is called a *solution* of the differential equation. The simplest differential equation is

$$f' = 0. \tag{188}$$

Lemma 9.1. *Let $f : (a, b) \to \mathbb{R}$ be differentiable in (a, b), and suppose that $f'(x) = 0$ for all $x \in (a, b)$. Then f is constant, i.e., there is $c \in \mathbb{R}$ such that $f(x) = c$ for $x \in (a, b)$.*

Proof. Suppose that f is not constant, i.e., that $f(x) \neq f(y)$ for some $x, y \in (a, b)$. Without loss of generality, assume that $x < y$. Then by the mean value theorem (Theorem 5.4), there exists $\xi \in (x, y)$ such that $f'(\xi) = \frac{f(y)-f(x)}{y-x} \neq 0$, leading to contradiction. □

For any $c \in \mathbb{R}$, the function $f : (a, b) \to \mathbb{R}$ defined by $f(x) = c$ is a solution of the differential equation $f' = 0$ in (a, b), and hence this differential equation has many solutions. In order to pinpoint single solution, we could specify the value of f at a point $\alpha \in (a, b)$, as $f(\alpha) = \beta$. For example, the only solution of the problem

$$f' = 0, \qquad f(0) = 1, \tag{189}$$

considered in $\mathbb{R}$, is $f(x) = 1$ for all $x \in \mathbb{R}$.

The next natural examples of differential equations would be $f'(x) = 1$, $f'(x) = x$, etc., and more generally, we may consider the problem of finding f satisfying $f' = g$, where g is a given function. This leads to the problem of *antidifferentiation* or *integration*, and will be discussed in Section 12.

The main subject of this section is the differential equation

$$f' = f, \tag{190}$$

and we look for its solution in the form of a power series $f(x) = \sum a_n x^n$. Formally differentiating the power series, we find

$$f'(x) = (a_0 + a_1 x + a_2 x^2 + \ldots)' = 0 + a_1 + 2a_2 x + \ldots = \sum_{n=0}^{\infty} (n + 1)a_{n+1}x^n, \tag{191}$$

which should be equal to $f(x) = \sum a_n x^n$. This yields $a_1 = a_0$, $2a_2 = a_1$, $3a_3 = a_2$, $\ldots$, $na_n = a_{n-1}$, or

$$a_n = \frac{a_{n-1}}{n} = \frac{a_{n-2}}{n(n-1)} = \ldots = \frac{a_0}{n!}. \tag{192}$$

Note that under the convention $0! = 1$, the equality $a_n = a_0/n!$ is true even for $n = 0$. Hence all the coefficients $\{a_n\}$ can be written in terms of the single coefficient a_0. By choosing $a_0 = 1$, we are led to the exponential function.

Definition 9.2. The *exponential function* is given by the power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots, \tag{193}$$

whose convergence radius (e.g. by the ratio test) is ∞, and so in particular $\exp \in \mathscr{C}^\infty(\mathbb{R})$.

Very often we will omit the parentheses and write $\exp x$ instead of $\exp(x)$.

Definition 9.3 (Euler 1748)**.** We define the *Euler number* by $e = \exp 1$.

Remark 9.4. Putting $x = 0$ into (193), we get $\exp 0 = 1$. Moreover, termwise differentiating the power series (193), we have

$$\exp' x = 0 + 1 + \frac{2x}{2} + \frac{3x^2}{3!} + \ldots + \frac{nx^{n-1}}{n!} = \exp x, \tag{194}$$

confirming that the exponential function is indeed a solution of (190).

Theorem 9.5 (Law of addition)**.** *We have*

$$\exp(x + y) = \exp(x)\exp(y) \qquad \textit{for} \quad x, y \in \mathbb{R}. \tag{195}$$

Proof. For $a \in \mathbb{R}$, let $g(x) = \exp(x)\exp(a - x)$. Then we have

$$g'(x) = \exp(x)\exp(a - x) - \exp(x)\exp(a - x) = 0, \tag{196}$$

for all $x \in \mathbb{R}$, which, by $g(0) = \exp(a)$ and by Lemma 9.1, implies that

$$g(x) = \exp(x)\exp(a - x) = \exp(a) \qquad \text{for} \quad a, x \in \mathbb{R}. \tag{197}$$

Putting $a = x + y$, we get (195). $\square$

Corollary 9.6. *a)* $\exp(-x)\exp(x) = 1$*, and so* $\exp(x) > 0$ *for all* $x \in \mathbb{R}$*.*
b) The map $\exp : \mathbb{R} \to (0, \infty)$ *is strictly increasing and surjective.*
c) The only function satisfying $f' = f$ *in* $\mathbb{R}$ *with* $f(0) = 1$ *is the exponential function.*
d) $\exp x = e^x$ *for all* $x \in \mathbb{R}$*, where* $e = \exp 1$*, cf. Definition 9.3.*

Proof. Putting $y = -x$ into the law of addition (195), we infer

$$\exp(-x)\exp(x) = 1 \qquad \text{for } x \in \mathbb{R}. \tag{198}$$

We have

$$\exp x = 1 + x + \frac{x^2}{2!} + \ldots \geq 1 + x \qquad \text{for} \quad x \geq 0, \tag{199}$$

which implies that $\exp n \to \infty$ as $n \to \infty$. Moreover, since $\exp(-x) = \frac{1}{\exp x}$, we have

$$0 < \exp(-x) \leq \frac{1}{1 + x} \qquad \text{for} \quad x \geq 0, \tag{200}$$

and so in particular, $\exp(-n) \to 0$ as $n \to \infty$. We conclude that $\exp : \mathbb{R} \to (0, \infty)$ is surjective. Furthermore, we have $\exp : \mathbb{R} \to (0, \infty)$ is strictly increasing, as $\exp' x = \exp x > 0$ for all $x \in \mathbb{R}$. Hence $\exp : \mathbb{R} \to (0, \infty)$ is a bijection. This establishes a) and b).

Now we shall establish c). Suppose that f is a solution, and let $g(x) = f(x)\exp(-x)$. Then $g(0) = f(0)\exp(0) = 1$, and

$$g'(x) = f'(x)\exp(-x) + f(x)(\exp(-x))' = f(x)\exp(-x) - f(x)\exp(-x) = 0, \tag{201}$$

where we have used $f'(x) = f(x)$ and $(\exp(-x))' = -\exp(-x)$. Invoking Lemma 9.1, we infer $g(x) = c$ for some constant $c \in \mathbb{R}$. Since $g(0) = 1$, we must have $c = 1$, that is, $g(x) = f(x)\exp(-x) = 1$ for all $x \in \mathbb{R}$. This means that $f(x) = \exp(x)$ for $x \in \mathbb{R}$.

Finally, we turn to d). For any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\exp(nx) = \exp(x + \ldots + x) = \exp(x) \cdots \exp(x) = (\exp x)^n, \tag{202}$$

and

$$\exp(-nx) = \frac{1}{\exp(nx)} = \frac{1}{(\exp x)^n} = (\exp x)^{-n}, \tag{203}$$

showing that $\exp(nx) = (\exp x)^n$ for all $n \in \mathbb{Z}$. Putting $x = \frac{1}{n}$, we get $\exp 1 = (\exp \frac{1}{n})^n$, or $\exp(\frac{1}{n}) = e^{\frac{1}{n}}$. This implies that

$$\exp \tfrac{n}{m} = (\exp \tfrac{1}{m})^n = (e^{\frac{1}{m}})^n = e^{\frac{n}{m}} \qquad \text{for} \quad n \in \mathbb{Z}, \quad m \in \mathbb{N}. \tag{204}$$

Let $x \in \mathbb{R}$, and let $\{x_n\} \subset \mathbb{Q}$ be a sequence converging to x. Then by definition, we have $e^{x_n} \to e^x$ as $n \to \infty$. On the other hand, by continuity, we have $\exp x_n \to \exp x$. Since $\exp x_n = e^{x_n}$, we infer

$$|\exp x - e^x| \leq |\exp x - \exp x_n| + |e^{x_n} - e^x|, \tag{205}$$

meaning that for any $\varepsilon > 0$, by choosing n large enough, we conclude that $|\exp x - e^x| < \varepsilon$. Therefore, we have $\exp x = e^x$ for all $x \in \mathbb{R}$. □

Exercise 9.7. Find all solutions of the differential equation $f' = af$, where $a \in \mathbb{R}$ is given.

Since the exponential $\exp : \mathbb{R} \to (0, \infty)$ is bijective, its inverse function is well defined.

Definition 9.8. The inverse function $\log : (0, \infty) \to \mathbb{R}$ is called the *logarithm function*.

Theorem 9.9. *a)* $\log' x = \dfrac{1}{x}$ *for* $x > 0$.

b) $\log(1 + x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ *for* $|x| < 1$.

Proof. a) Let $x > 0$ and let $y = \log x$. Then by Theorem 4.6d), we get

$$\log' x = \frac{1}{\exp' y} = \frac{1}{\exp y} = \frac{1}{\exp(\log x)} = \frac{1}{x}. \tag{206}$$

b) By the ratio test, the convergence radius of the given series is 1, so the function

$$\lambda(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x - 1)^n, \tag{207}$$

is well defined for $x \in (0, 2)$. We will show that $\lambda(x) = \log x$ for $x \in (0, 2)$. A termwise differentiation of the series for $\lambda(x)$ gives

$$\lambda'(x) = \sum_{n=1}^{\infty} (-1)^{n-1} (x - 1)^{n-1} = \sum_{n=1}^{\infty} (1 - x)^{n-1} = \frac{1}{1 - (1 - x)} = \frac{1}{x}, \tag{208}$$

provided that $|1 - x| < 1$, i.e., that $x \in (0, 2)$. Now let $g(x) = x \exp(-\lambda(x))$. Then for $x \in (0, 2)$ we have

$$g'(x) = \exp(-\lambda(x)) - x \exp(-\lambda(x)) \lambda'(x) = 0, \tag{209}$$

meaning that $g(x) = g(1) = \exp(-\lambda(1)) = \exp(0) = 1$ for $x \in (0, 2)$. □

Exercise 9.10. Prove the following.

(a) $\log(ab) = \log a + \log b$, for $a > 0$ and $b > 0$.
(b) $\log(a^x) = x \log a$, for $a > 0$ and $x \in \mathbb{R}$.
(c) Given any $a > 0$, the power series

$$\log x = \log(a) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n a^n} (x - a)^n, \tag{210}$$

is valid for $0 < x < 2a$.

Example 9.11. By Theorem 9.9, $\log(1+x)$ behaves like x for $x \approx 0$. Hence we expect for instance that $n\log(1+\frac{1}{n}) \to 1$ as $n \to \infty$. To establish it rigorously, we use L'Hôpital's rule (Corollary 5.9), as

$$\lim_{n\to\infty} \frac{\log(1+x_n)}{x_n} = \lim_{n\to\infty} \frac{1/(1+x_n)}{1} = 1, \tag{211}$$

where $\{x_n\} \subset (0,\infty)$ is any sequence converging to 0. By continuity of the exponential function, this leads to the following remarkable limit

$$\lim_{n\to\infty} \big(1+\tfrac{1}{n}\big)^n = \lim_{n\to\infty} \exp\big(n\log(1+\tfrac{1}{n})\big) = 1. \tag{212}$$

Lemma 9.12. *We have $e \notin \mathbb{Q}$. That is, the Euler number is irrational.*

Proof. Suppose that $e = \frac{n}{m}$ for some positive integers n and m. Then we have

$$\frac{m}{n} = e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} + \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!}, \tag{213}$$

and multiplying this by $n!$ yields

$$\frac{m}{n}\cdot n! = \underbrace{\sum_{k=0}^{n}(-1)^k\frac{n!}{k!}}_{A} + \underbrace{\sum_{k=n+1}^{\infty}(-1)^k\frac{n!}{k!}}_{B}. \tag{214}$$

It is clear that $\frac{m}{n}\cdot n! = m\cdot(n-1)!$ and that the sum

$$A = \sum_{k=0}^{n}(-1)^k\frac{n!}{k!} = \sum_{k=0}^{n}(-1)^k(k+1)(k+2)\cdots(n-1)n, \tag{215}$$

is an integer. Then (214) implies that B must be an integer. We will now show that B is *not* an integer, which would be a contradiction. Note that

$$B = \sum_{k=n+1}^{\infty}(-1)^k\frac{n!}{k!} = (-1)^{n+1}\left(\frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \ldots\right). \tag{216}$$

Multiplying both sides by $(-1)^{n+1}$, we infer that

$$(-1)^{n+1}B = \frac{1}{n+1} - \left(\frac{n!}{(n+2)!} - \frac{n!}{(n+3)!}\right) - \left(\frac{n!}{(n+4)!} - \frac{n!}{(n+5)!}\right) - \ldots \leq \frac{1}{n+1}, \tag{217}$$

and

$$\begin{aligned}(-1)^{n+1}B &= \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \left(\frac{n!}{(n+3)!} - \frac{n!}{(n+4)!}\right) + \ldots \\ &\geq \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} = \frac{1}{n+2},\end{aligned} \tag{218}$$

leading to

$$0 < \frac{1}{n+2} \leq |B| \leq \frac{1}{n+1} \leq \frac{1}{2}. \tag{219}$$

Thus B cannot be an integer, and the proof is established. □

10. The sine and cosine functions

In the preceding section, we considered the differential equation $f' = f$, which led us to the exponential and logarithm functions. The differential equation $f' = f$ is one of the simplest examples of first order equations, as it involves f' but not f'' or any higher order derivatives. In this section, we will look at the second order equation

$$f'' = -f, \tag{220}$$

and as before, we look for its solution in the form of a power series $f(x) = \sum a_n x^n$. Formally differentiating the power series, we find

$$\begin{aligned} f''(x) &= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots)'' = (0 + a_1 + 2a_2 x + 3a_3 x^2 \ldots)' \\ &= 0 + 0 + 2a_2 + 6a_3 x \ldots = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n \end{aligned} \tag{221}$$

which should be equal to $-f(x) = \sum(-a_n)x^n$. This yields $2a_2 = -a_0$, $2 \cdot 3a_3 = -a_1$, $3 \cdot 4a_4 = -a_2$, $4 \cdot 5a_5 = -a_3$, $\ldots$, $(n+1)(n+2)a_{n+2} = -a_n$, or

$$a_n = -\frac{a_{n-2}}{n(n-1)} = \frac{a_{n-4}}{n(n-1)(n-2)(n-3)} = \ldots = (-1)^{\frac{n}{2}} \cdot \frac{a_0}{n!} \qquad \text{for } n \text{ even}, \tag{222}$$

and similarly,

$$a_n = (-1)^{\frac{n-1}{2}} \cdot \frac{a_1}{n!} \qquad \text{for } n \text{ odd}. \tag{223}$$

Hence every a_n can be computed in terms of either a_0 or a_1. We can writte

$$a_{2k} = (-1)^k \frac{a_0}{(2k)!}, \qquad \text{and} \qquad a_{2k+1} = (-1)^k \frac{a_1}{(2k+1)!}, \qquad k = 0, 1, \ldots, \tag{224}$$

and therefore

$$f(x) = a_0 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \tag{225}$$

Choosing either a_0 or a_1 to be 1 and the other to be 0, we are led to the following definition.

Definition 10.1. The *sine function* is given by the power series

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, \tag{226}$$

and the *cosine function* is given by

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots. \tag{227}$$

As with exp and log, we often write $\sin x$ and $\cos x$ instead of $\sin(x)$ and $\cos(x)$.

Remark 10.2. The convergence radii of both series (e.g. by the ratio test) are ∞. By termwise differentiation, we infer that

$$\sin' x = \cos x \qquad \text{and} \qquad \cos' x = -\sin x \qquad \text{for } x \in \mathbb{R}. \tag{228}$$

This implies that both $\sin x$ and $\cos x$ satisfy the differential equation (220). It is immediate from (227) that

$$\cos(-x) = \cos x \qquad \text{for } x \in \mathbb{R}, \tag{229}$$

and by differentiating this, we get

$$\sin(-x) = -\sin x \qquad \text{for } x \in \mathbb{R}. \tag{230}$$

Furthermore, we have $\sin 0 = 0$ and $\cos 0 = 1$, and so for $A, B \in \mathbb{R}$, the function

$$f(x) = A\cos(x) + B\sin(x), \tag{231}$$

satisfies $f'' + f = 0$ in $\mathbb{R}$, $f(0) = A$, and $f'(0) = B$.

Theorem 10.3 (Law of addition)**.** *For $x, y \in \mathbb{R}$, we have*

$$\begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y), \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y). \end{aligned} \tag{232}$$

Proof. For $a \in \mathbb{R}$, let

$$g(x) = \cos(a-x)\sin(x) + \sin(a-x)\cos(x). \tag{233}$$

We expect that $g(x) = \text{const}$. The heuristic motivation for this choice is as follows.

- $\sin x$ is the solution at x of the problem $f'' + f = 0$, $f(0) = 0$, $f'(0) = 1$.
- For this solution, we have $f(x) = \sin x$ and $f'(x) = \cos x$ at x.
- $g(x)$ is the solution at $a - x$ of the problem $f'' + f = 0$, $f(0) = \sin x$, $f'(0) = \cos x$. Since $g(x)$ is the solution $\sin x$ further evolved for a "time" $a - x$, we expect $g(x)$ to be $\sin x$ evaluated at $x + (a - x) = a$.

Now we compute

$$\begin{aligned} g'(x) = \sin(a-x)\sin(x) &+ \cos(a-x)\cos(x) \\ &- \cos(a-x)\cos(x) - \sin(a-x)\sin(x) = 0, \end{aligned} \tag{234}$$

for all $x \in \mathbb{R}$, which, by $g(0) = \sin(a)$ and by Lemma 9.1, implies that

$$g(x) = \cos(a-x)\sin(x) + \sin(a-x)\cos(x) = \sin(a) \qquad \text{for} \quad a, x \in \mathbb{R}. \tag{235}$$

Putting $a = x + y$, we get the first equation in (232). For the second equation, we may use

$$g(x) = \cos(a-x)\cos(x) - \sin(a-x)\sin(x), \tag{236}$$

and proceed similarly. $\square$

A notational convention that is prevalent in this context is that the powers $(\sin x)^n$ and $(\cos x)^n$ are written as $\sin^n x$ and $\cos^n x$.

Corollary 10.4. *a)* $\sin^2 x + \cos^2 x = 1$ *for all $x \in \mathbb{R}$.*
b) The only function satisfying $f'' + f = 0$ in $\mathbb{R}$ with $f(0) = 0$ and $f'(0) = 1$ is sin.
c) The only function satisfying $f'' + f = 0$ in $\mathbb{R}$ with $f(0) = 1$ and $f'(0) = 0$ is cos.

Proof. a) Putting $y = -x$ into the second equality of (232), and taking into account (229), (230), and the fact that $\cos 0 = 1$, we get $\sin^2 x + \cos^2 x = 1$.

b) Let f be a twice differentiable function in $\mathbb{R}$, satisfying $f'' + f = 0$ in $\mathbb{R}$ with $f(0) = 0$ and $f'(0) = 1$. Let

$$F(x) = \cos(x)f(x) - \sin(x)f'(x) \qquad \text{and} \qquad G(x) = \sin(x)f(x) + \cos(x)f'(x). \tag{237}$$

Then we have

$$\begin{aligned} F'(x) &= -\sin(x)f(x) + \cos(x)f'(x) - \cos(x)f'(x) - \sin(x)f''(x) = 0, \\ G'(x) &= \cos(x)f(x) + \sin(x)f'(x) - \sin(x)f'(x) + \cos(x)f''(x) = 0, \end{aligned} \tag{238}$$

implying that both functions F and G are constant functions. Since $F(0) = f(0) = 0$ and $G(0) = f'(0) = 1$, we get

$$\cos(x)f(x) - \sin(x)f'(x) = 0 \qquad \sin(x)f(x) + \cos(x)f'(x) = 1 \qquad \text{for all } x \in \mathbb{R}. \tag{239}$$

Multiplying the first equation by $\cos x$, the second by $\sin x$, and then summing them, yield $(\cos^2 x + \sin^2 x)f(x) = \sin x$, or $f(x) = \sin x$. $\square$

Exercise 10.5. (a) Prove c) of the preceding theorem.
(b) $\sin 2x = 2\sin x \cos x$

Our next task is to identify the zero sets

$$Z(\sin) = \{x \in \mathbb{R} : \sin x = 0\} \qquad \text{and} \qquad Z(\cos) = \{x \in \mathbb{R} : \cos x = 0\}. \tag{240}$$

It is obvious that $Z(\sin) \neq \varnothing$ as $\sin 0 = 0$. We now show that $Z(\cos)$ is nonempty.

Lemma 10.6. *There exists $x > 0$ such that $\cos x = 0$.*

Proof. Suppose that there is no $x > 0$ with $\cos x = 0$. Since $\cos 0 = 1$, this yields that $\cos x > 0$ for all $x \geq 0$, and hence by (228), sin is a strictly increasing function in $[0, \infty)$. As $\sin 0 = 0$, we have $\sin x > 0$ for $x > 0$, and so (228) implies that cos is a strictly decreasing function in $(0, \infty)$. Now consider the sequence $\{\cos n : n \in \mathbb{N}\} \subset (0, 1]$, which is a bounded and decreasing sequence. By the monotone convergence theorem (Theorem 2.15), there exists $\alpha \in \mathbb{R}$ such that $\cos n \to \alpha$ as $n \to \infty$. In particular, noting that $\sin 1 > 0$, there exists $n > 1$ such that $|\cos(n) - \cos(n+1)| < \sin 1$. Furthermore, taking into account that $\cos' x = -\sin x$, by the mean value theorem (Theorem 5.4) there exists $\xi \in (n, n+1)$ such that $|\sin \xi| < \sin 1$. This contradicts the assertion that sin is strictly increasing in $[0, \infty)$, and therefore there is some $x > 0$ with $\cos x = 0$. $\square$

The following theorem states that the zero set of $\sin : \mathbb{R} \to \mathbb{R}$ is given by $\pi\mathbb{Z} := \{\pi n : n \in \mathbb{Z}\}$ with some $\pi > 0$. As $a\mathbb{Z} = b\mathbb{Z}$ if and only if $a = b$, this *defines* the number π uniquely. In other words, we are defining the number π as the smallest positive solution of $\sin x = 0$.

Theorem 10.7. *We have $Z(\sin) = \pi\mathbb{Z} = \{\pi n : n \in \mathbb{Z}\}$ for some constant $\pi > 0$.*

Proof. Let $K = Z(\sin)$, and let $\tau > 0$ satisfy $\cos \tau = 0$, cf. Lemma 10.6. Then we have $\sin(2\tau) = 2\sin(\tau)\cos(\tau) = 0$, and hence $2\tau \in K$. As $\{t \in K : t > 0\}$ is nonempty, the number

$$\pi = \inf\{t \in K : t > 0\}, \tag{241}$$

is well defined. In order to show that $\pi \in K$, let $\{t_k\} \subset K$ be a sequence satisfying $t_k \to \pi$ as $k \to \infty$. Then by continuity, $\sin(t_k) \to \sin(\pi)$ as $k \to \infty$. On the other hand, $\sin(t_k) = 0$ for all k, which implies that $\sin(\pi) = 0$, that is, $\pi \in K$. Furthermore, since

$$\sin(x \pm \pi) = \sin(x)\cos(\pi) \pm \cos(x)\sin(\pi) = \sin(x)\cos(\pi), \tag{242}$$

we conclude that $\pi n \in K$ for all $n \in \mathbb{Z}$.

Now we wish to show that $\pi > 0$. To this end, we write

$$\sin x = x - \frac{x^3}{3!} + \ldots = x\big(1 - \frac{x^2}{3!} + \ldots\big) = xg(x), \tag{243}$$

where we have introduced the function

$$g(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}. \tag{244}$$

The convergence radius of the latter series is ∞, and so in particular we have $g \in \mathscr{C}^\infty(\mathbb{R})$. Since $g(0) = 1$, by continuity, g has no zeroes in $(-\varepsilon, \varepsilon)$ for some small $\varepsilon > 0$. Hence in the interval $(-\varepsilon, \varepsilon)$, the only solution to $\sin x = 0$ is $x = 0$, which means that $\pi > 0$.

Finally, suppose that $r \in K$. Then there is $n \in \mathbb{Z}$ such that $n\pi \leq r < (n+1)\pi$, or $0 \leq r - n\pi < \pi$. But $\sin(r - n\pi) = \sin(r)\cos(-nt) + \cos(r)\sin(-n\pi) = 0$, hence $r - n\pi = 0$ by the minimal property of π. This proves that $K \subset \pi\mathbb{Z}$. $\square$

Corollary 10.8. *a) We have $\sin x = \cos(x - \frac{\pi}{2})$ for all $x \in \mathbb{R}$.*
b) $\sin x = \sin(x + \tau)$ for all $x \in \mathbb{R}$ if and only if $\tau = 2\pi n$ for some $n \in \mathbb{Z}$. In other words, the periods of the sine function are precisely the numbers $2\pi n$, $n \in \mathbb{Z}$.

c) $\sin x > 0$ *for* $0 < x < \pi$. *In fact, we have* $\sin x \geq \frac{2}{\pi}x$ *for* $0 \leq x \leq \frac{\pi}{2}$.
d) $\sin(\frac{\pi}{2} - x) = \sin(\frac{\pi}{2} + x)$ *and* $\sin(\pi - x) = \sin x$ *for all* $x \in \mathbb{R}$.

Proof. c) Since $\cos 0 = 1$, there is $\varepsilon > 0$ such that $\cos x > 0$ for $x \in (-\varepsilon, \varepsilon)$. Then sin is strictly increasing in $(-\varepsilon, \varepsilon)$, and as $\sin 0 = 0$, we have $\sin x > 0$ for $x \in (0, \varepsilon)$. Now taking into account that $\sin x \neq 0$ for $x \in (0, \pi)$, we infer $\sin x > 0$ for $x \in (0, \pi)$. This shows that $\sin x$ is concave for $x \in [0, \pi]$, as $\sin'' x = -\sin x$.

Since $\sin 0 = \sin \pi = 0$, by Rolle's theorem (Theorem 5.3), there is $\xi \in (0, \pi)$ such that $\sin' \xi = 0$, i.e., that $\cos \xi = 0$. This implies that $\sin(2\xi) = 2\sin(\xi)\cos(\xi) = 0$, and as the zeroes of sin are the numbers $0, \pm\pi, \pm 2\pi, \ldots$, and $0 < \xi < \pi$, we conclude that $2\xi = \pi$ or $\xi = \frac{\pi}{2}$. Therefore, we have $\cos \frac{\pi}{2} = 0$, and $|\sin \frac{\pi}{2}| = 1$ by $\sin^2(\frac{\pi}{2}) + \cos^2(\frac{\pi}{2}) = 1$. In fact we have $\sin \frac{\pi}{2} = 1$, as in the preceding paragraph we have just proven that $\sin x > 0$ for $x \in (0, \pi)$. Then by concavity we get $\sin x \geq \frac{2}{\pi}x$ for $x \in [0, \frac{\pi}{2}]$.

a) Let $g(x) = \cos(x - \frac{\pi}{2})$ for $x \in \mathbb{R}$. Then $g'(x) = -\sin(x - \frac{\pi}{2})$ and $g''(x) = -\cos(x - \frac{\pi}{2}) = -g(x)$ for $x \in \mathbb{R}$. Moreover, we have $g(0) = \cos(-\frac{\pi}{2}) = \cos(\frac{\pi}{2}) = 0$ and $g'(0) = -\sin(-\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$. Thus Corollary 10.4b) implies that $g(x) = \sin x$ for $x \in \mathbb{R}$.

d) By repeatedly applying a), we infer $\sin(\frac{\pi}{2} - x) = \cos(-x) = \cos(x) = \sin(x + \frac{\pi}{2})$ and $\sin(\pi - x) = \cos(\frac{\pi}{2} - x) = \cos(x - \frac{\pi}{2}) = \sin(x)$.

b) Recall that $\sin x = 0$ if and only if $x = \pi n$ for some $n \in \mathbb{Z}$. Hence if τ is a period of the sine function then $\tau = \pi n$ for some $n \in \mathbb{Z}$. Moreover, π cannot be a period because $\sin(-\frac{\pi}{2}) = -\sin\frac{\pi}{2} = 1$. We claim that 2π is a period. Indeed, the function $g(x) = \sin(x + 2\pi)$ satisfies $g(0) = \sin(2\pi) = 0$, $g'(0) = \cos(2\pi) = \sin(\frac{\pi}{2} - 2\pi) = -\sin(\frac{3\pi}{2}) = -\sin(-\frac{\pi}{2}) = 1$, and $g'' = -g$, meaning that $g(x) = \sin x$ for all $x \in \mathbb{R}$. This makes $2\pi n$ a period for any $n \in \mathbb{Z}$. Since $\cos \pi = \cos(-\pi) = \sin(-\frac{\pi}{2}) = -1$, we have

$$\sin(\tfrac{\pi}{2} + 2n\pi + \pi) = \sin(\tfrac{\pi}{2} + 2\pi n)\cos(\pi) + \cos(\tfrac{\pi}{2} + 2\pi n)\sin(\pi) = -\sin(\tfrac{\pi}{2}), \tag{245}$$

and hence any of the numbers $(2n+1)\pi$, $n \in \mathbb{Z}$, is not a period. Therefore, the periods of the sine function are precisely the numbers $2\pi n$, $n \in \mathbb{Z}$. □

11. Other trigonometric functions

Definition 11.1. Define the *tangent function* by

$$\tan x = \frac{\sin x}{\cos x} \qquad \text{for } x \in \mathbb{R} \setminus (\tfrac{\pi}{2} + \pi\mathbb{Z}), \tag{246}$$

and the *cotangent function* by

$$\cot x = \frac{\cos x}{\sin x} \qquad \text{for } x \in \mathbb{R} \setminus \pi\mathbb{Z}, \tag{247}$$

where $\frac{\pi}{2} + \pi\mathbb{Z} = \{\frac{\pi}{2} + \pi n : n \in \mathbb{Z}\}$.

Remark 11.2. We have

$$\tan' x = \frac{\sin' x \cos x - \cos' x \sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}, \tag{248}$$

and hence the tangent function is strictly increasing in the interval $(\frac{\pi}{2} + \pi n, \frac{\pi}{2} + \pi n + \pi)$ for each $n \in \mathbb{Z}$. Let $\{x_k\} \subset (-\frac{\pi}{2}, \frac{\pi}{2})$ be a sequence converging to $\frac{\pi}{2}$. Then $\sin x_k \to 1$ and $\cos x_k \to 0$ with $\cos x_k > 0$, yielding $\tan x_k \to \infty$. Similarly, we get $\tan x_k \to -\infty$ for any sequence $\{x_k\} \subset (-\frac{\pi}{2}, \frac{\pi}{2})$ converging to $-\frac{\pi}{2}$. Moreover, since

$$\tan(x + \pi) = \frac{-\sin x}{\sin(x + \pi + \frac{3\pi}{2})} = \frac{-\sin x}{\sin(x - \frac{\pi}{2})} = \frac{-\sin x}{-\sin(\frac{\pi}{2} - x)} = \frac{-\sin x}{-\cos(-x)} = \tan(x), \tag{249}$$

the periods of the tangent function are precisely the numbers πn, $n \in \mathbb{Z}$.

Exercise 11.3. Prove the following.

(a) The cotangent function is strictly decreasing in the interval $(\pi n, \pi n + \pi)$ for each $n \in \mathbb{Z}$.
(b) We have $\cot x_k \to \infty$ for any sequence $\{x_k\} \subset (0, \pi)$ converging to 0, and $\cot x_k \to -\infty$ for any sequence $\{x_k\} \subset (0, \pi)$ converging to π.
(c) The periods of the cotangent function are precisely the numbers πn, $n \in \mathbb{Z}$.

Definition 11.4. Since $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is continuous, strictly increasing, and surjective, its inverse exists. The inverse function $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is called the *arctangent function*.

Lemma 11.5. *We have*

$$\arctan' x = \frac{1}{1+x^2} \qquad \text{for } x \in \mathbb{R}, \tag{250}$$

and

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \qquad \text{for } -1 < x < 1. \tag{251}$$

Proof. Let $x \in \mathbb{R}$ and $y = \arctan x$. Then we have

$$\arctan' x = \frac{1}{\tan' y} = \cos^2 y = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}. \tag{252}$$

We can expand the latter expression in power series as

$$\arctan' x = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad \text{for } -1 < x < 1. \tag{253}$$

Then taking into account that $(x^{2n+1})' = (2n+1)x^{2n}$, we infer

$$\Big(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}\Big)' = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad \text{for } -1 < x < 1. \tag{254}$$

In other words, the function

$$f(x) = \arctan x - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad (-1 < x < 1), \tag{255}$$

satisfies $f'(x) = 0$ for $x \in (-1, 1)$. We also have $f(0) = 0$ because $\tan 0 = 0$ implies that $\arctan 0 = 0$, and thus $f \equiv 0$ in $(-1, 1)$. Therefore, we conclude that (251) holds. ☐

Exercise 11.6. Show that $\cot : (0, \pi) \to \mathbb{R}$ is bijective, and compute the derivative of its inverse function. Then expand the inverse function in power series for $-1 < x < 1$.

Next, we turn to possible inverse functions of sin and cos. Since $\sin' x = \cos x > 0$ for $x \in (\frac{\pi}{2}, \frac{\pi}{2})$, the sine function is strictly increasing in $[\frac{\pi}{2}, \frac{\pi}{2}]$. Hence the sine function restricted to the interval $[\frac{\pi}{2}, \frac{\pi}{2}]$ is injective, and $\sin([\frac{\pi}{2}, \frac{\pi}{2}]) = [-1, 1]$.

Definition 11.7. The inverse function $\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ of $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ is called the *arcsine function*.

Lemma 11.8. *We have*

$$\arcsin' x = \frac{1}{\sqrt{1-x^2}} \qquad \text{for } -1 < x < 1, \tag{256}$$

and

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{x^{2n+1}}{2n+1} = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \ldots \tag{257}$$

for $-1 < x < 1$.

Proof. Let $x \in (-1,1)$ and $y = \arcsin x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then we have

$$\arcsin' x = \frac{1}{\sin' y} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}. \tag{258}$$

We can expand the latter expression by using the binomial series as

$$\arcsin' x = \big(1 + (-x^2)\big)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n x^{2n} \qquad \text{for } -1 < x < 1, \tag{259}$$

where

$$\binom{-1/2}{n} = \frac{1}{n!} \cdot \big(-\frac{1}{2}\big)\big(-\frac{1}{2}-1\big)\cdots\big(-\frac{1}{2}-n+1\big) = \frac{(-1)^n(2n-1)!!}{2^n n!} = (-1)^n \frac{(2n-1)!!}{(2n)!!}. \tag{260}$$

Then taking into account that $(x^{2n+1})' = (2n+1)x^{2n}$, we infer

$$\Big(\sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{x^{2n+1}}{2n+1}\Big)' = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n x^{2n} \qquad \text{for } -1 < x < 1. \tag{261}$$

In other words, the function

$$f(x) = \arcsin x - \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{x^{2n+1}}{2n+1} \qquad (-1 < x < 1), \tag{262}$$

satisfies $f'(x) = 0$ for $x \in (-1,1)$. We also have $f(0) = 0$ because $\sin 0 = 0$ implies that $\arcsin 0 = 0$, and thus $f \equiv 0$ in $(-1,1)$. Therefore, we conclude that (257) holds. $\square$

Exercise 11.9. Show that $\cos : [0,\pi] \to [-1,1]$ is bijective, and compute the derivative of its inverse function $\arccos : [-1,1] \to [0,\pi]$, the *arccosine function*. Then expand $\arccos x$ in power series for $-1 < x < 1$.

Exercise 11.10. Starting with the differential equation $f'' = f$, develop a theory for the hyperbolic functions $\sinh x$, $\cosh x$, $\tanh x$, etc.

12. Antidifferentiation

In the preceding two sections, we have considered the differential equations $f' = f$ and $f'' + f = 0$, which lead us to the functions $\exp x$, $\log x$, $\sin x$, $\cos x$, $\arctan x$, etc. Indeed, if f satisfies $f' = f$, then $f(x) = A \exp x$ for some constant $A \in \mathbb{R}$, and if f satisfies $f'' + f = 0$, then $f(x) = A \sin x + B \cos x$ for some constants $A, B \in \mathbb{R}$. In this section, we consider the problem of finding f satisfying

$$f' = g, \tag{263}$$

where g is a given function. Given g, finding f is called *antidifferentiation*, and f is called an *antiderivative of g*.

Remark 12.1. Suppose that g is a function defined on (a,b), and let $F' = G' = g$ on (a,b), that is, let F and G be antiderivatives of g. Then $(F-G)' = F' - G' = 0$ on (a,b), and from Lemma 9.1 we infer that

$$F(x) = G(x) + C, \qquad x \in (a,b), \tag{264}$$

for some constant $C \in \mathbb{R}$. On the other hand, if $G' = g$ on (a,b), and if $C \in \mathbb{R}$, then a new function F defined by (264) is also an antiderivetive of g, because

$$F'(x) = (G(x) + C)' = G'(x) + 0 = g(x), \qquad x \in (a,b). \tag{265}$$

What this means is that the antiderivative of a given function can only be found *up to an additive constant*, and that if we know one antiderivative of a given function, all other antiderivatives are found by adding an arbitrary constant to it.

Definition 12.2. Let G be an antiderivative of g on some interval (a, b), i.e., let $G'(x) = g(x)$ for $x \in (a, b)$. Then the set of all antiderivatives of g is denoted by

$$\int g(x)\mathrm{d}x = \{G + C : C \in \mathbb{R}\}, \tag{266}$$

which is called the *indefinite integral of g*. Alternatively and more informally, it is a standard practice to think of the indefinite integral as a notation for infinitely many functions (one function for each value of $C \in \mathbb{R}$), and write

$$\int g(x)\mathrm{d}x = G(x) + C, \tag{267}$$

where $C \in \mathbb{R}$ is considered to be an "arbitrary constant."

Example 12.3. (a) We have $G' = 0$ for the zero function $G(x) = 0$, i.e., $G \equiv 0$ is an antiderivative of $g \equiv 0$. Hence we can write

$$\int 0\,\mathrm{d}x = 0 + C = C. \tag{268}$$

(b) More generally, for $\alpha \in \mathbb{R}$ we have $(x^\alpha)' = \alpha x^{\alpha-1}$ at each $x > 0$, i.e., $G(x) = \frac{1}{\alpha}x^\alpha$ is an antiderivative of $g(x) = x^{\alpha-1}$ on the interval $(0, \infty)$, for each $\alpha \in \mathbb{R} \setminus \{0\}$. Hence we have

$$\int x^{\alpha-1}\mathrm{d}x = \frac{x^\alpha}{\alpha} + C, \tag{269}$$

for $\alpha \in \mathbb{R} \setminus \{0\}$. Note that if $\alpha \in \mathbb{N}$, the relation $G'(x) = g(x)$ is true for $x \in \mathbb{R}$, and if $\alpha \in \{-1, -2, -3, \ldots\}$, it is true for $x \in \mathbb{R} \setminus \{0\}$.

(c) As for the case $\alpha = 0$, we recall $(\log x)' = \frac{1}{x}$ for $x > 0$, which leads to

$$\int \frac{\mathrm{d}x}{x} = \log x + C. \tag{270}$$

(d) Since $(e^x)' = e^x$ for $x \in \mathbb{R}$, we have

$$\int e^x\mathrm{d}x = e^x + C. \tag{271}$$

(e) From $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$ for $x \in \mathbb{R}$, we infer

$$\int \cos x\,\mathrm{d}x = \sin x + C, \qquad \text{and} \qquad \int \sin x\,\mathrm{d}x = -\cos x + C. \tag{272}$$

(f) Similarly, we have

$$\int \frac{\mathrm{d}x}{1 + x^2} = \arctan x + C, \qquad \text{and} \qquad \int \frac{\mathrm{d}x}{\sqrt{1 - x^2}} = \arcsin x + C, \tag{273}$$

where the former is valid for $x \in \mathbb{R}$, and the latter is for $x \in (-1, 1)$.

Exercise 12.4. By a direct guess, find the indefinite integrals of the following functions.

(a) $g(x) = (\cos x)^{-2}$.
(b) $g(x) = 2 \sin x \cos x$.
(c) $g(x) = e^{2x}$.
(d) $g(x) = 2^x$.

Example 12.5. (a) Since $(2x^3 + e^x)' = 2(x^3)' + (e^x)' = 6x^2 + e^x$, we have

$$\int (6x^2 + e^x)\mathrm{d}x = 2x^3 + e^x + C. \tag{274}$$

(b) Let $\alpha \in \mathbb{R}$ be a constant. Then we have $(\sin(\alpha x))' = \alpha \cos(\alpha x)$, and hence

$$\int \cos(\alpha x) \mathrm{d}x = \frac{\sin(\alpha x)}{\alpha} + C \qquad \text{for} \quad \alpha \neq 0. \tag{275}$$

(c) We have $(\log \log x)' = \frac{1}{\log x} \cdot \frac{1}{x}$, which means that

$$\int \frac{\mathrm{d}x}{x \log x} = \log \log x + C. \tag{276}$$

Remark 12.6. Recall the opening paragraph of Section 9, where we discussed the repository of functions that are generated by starting with the power functions x^a ($a \in \mathbb{R}$), and by combining them by using finitely many addition, subtraction, multiplication, quotient, and composition operations. This repository has now been extended, since we can start with the exponential, logarithm, trigonometric and inverse trigonometric functions, in addition to the power functions x^a ($a \in \mathbb{R}$). Let us call the resulting functions *elementary functions*. Then the derivative of an elementary function is an elementary function, because we have differentiation rules that tell us how to compute $(f+g)'$, $(fg)'$, $(f \circ g)'$, etc., based on the knowledge of f' and g', cf. Theorem 4.6. Each differentiation rule can be applied "in reverse" to compute antiderivatives of a large number of elementary functions. However, these "antidifferentiation rules" cannot give antiderivatives of all elementary functions, because as discovered by Joseph Liouville around 1840, there are elementary functions whose antiderivatives are not elementary. As a reflection, for example, there is no useful formula that gives an antiderivative of fg, based on antiderivatives of f and g. This makes antidifferentiation of elementary functions somewhat of a challenge, as opposed to differentiation, which is completely straightforward. Nevertheless, there exist algorithms, such as the Risch algorithm, that can decide whether an elementary function is the derivative of an elementary function, and if so, compute the antiderivative. More generally, by using the Riemann integral, we can construct an antiderivative of, say, any continuous function as the limit of a sequence of functions, and thus demonstrate that continuous functions admit antiderivatives. Note that for certain pathological functions g, there is no G satisfying $G' = g$. Although the Risch algorithm and the Riemann integral are beyond the scope of these notes, in what follows we will develop a few useful antidifferentiation rules, and will show that functions defined by power series admit antiderivatives.

Lemma 12.7. *a) Let $F' = f$ on (a,b), and let $\alpha \in \mathbb{R} \setminus \{0\}$ be a nonzero constant. Then we have $(\alpha F)' = \alpha f$ on (a,b), that is,*

$$\int \alpha f(x) \mathrm{d}x = \alpha \int f(x) \mathrm{d}x. \tag{277}$$

b) Let $F' = f$ on (a,b), and let $\alpha \in \mathbb{R}$. Then we have

$$(F(\alpha x))' = \alpha F'(\alpha x) = \alpha f(\alpha x) \qquad \textit{for} \quad x \in (a,b). \tag{278}$$

In other words, for $\alpha \neq 0$ we have

$$\int f(\alpha x) \mathrm{d}x = \frac{1}{\alpha} F(x) + C = \frac{1}{\alpha} \int f(x) \mathrm{d}x. \tag{279}$$

c) Let $F' = f$ and $G' = g$ on (a,b). Then we have $(F+G)' = f + g$ on (a,b), that is,

$$\int (f(x) + g(x)) \mathrm{d}x = \int f(x) \mathrm{d}x + \int g(x) \mathrm{d}x. \tag{280}$$

Exercise 12.8. Give a detailed proof of the preceding lemma. Is (277) true for $\alpha = 0$?

The chain rule of differentiation leads to the following rule for antidifferentiation.

Theorem 12.9 (Substitution)**.** *Let $F' = f$ on (a, b), and let $\phi : (c, d) \to (a, b)$ be a differentiable function. Then $F \circ \phi$ is an antiderivative of $(f \circ \phi)\phi'$ on (c, d), that is, we have*

$$\int f(\phi(x))\phi'(x)\mathrm{d}x = F(\phi(x)) + C = \Big(\int f(y)\mathrm{d}y\Big)\Big|_{y=\phi(x)}. \tag{281}$$

Proof. Taking into account that $F' = f$, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}F(\phi(x)) = F'(\phi(x))\phi'(x) = f(\phi(x))\phi'(x) \qquad \text{for} \quad x \in (c, d), \tag{282}$$

which shows that $F \circ \phi$ is an antiderivative of $(f \circ \phi)\phi'$ on (c, d). □

Recognizing if a given integral is amenable to substitution is the same as deciding if the expression under the integral can be written in the form $f(\phi(x))\phi'(x)$.

Example 12.10. We have

$$\begin{aligned}\int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x &= \frac{1}{2}\int \frac{(x^2)'}{\sqrt{1+x^2}}\,\mathrm{d}x = \frac{1}{2}\int \frac{(1+x^2)'}{\sqrt{1+x^2}}\,\mathrm{d}x = \frac{1}{2}\Big(\int \frac{1}{\sqrt{y}}\,\mathrm{d}y\Big)\Big|_{y=1+x^2} \\ &= \frac{1}{2}\Big(\int y^{-\frac{1}{2}}\mathrm{d}y\Big)\Big|_{y=1+x^2} = y^{\frac{1}{2}}\Big|_{y=1+x^2} + C = \sqrt{1+x^2} + C.\end{aligned} \tag{283}$$

Exercise 12.11. Compute the following indefinite integrals.

$$\text{(a)} \quad \int \cos^2 x \sin x\,\mathrm{d}x \qquad \text{(b)} \quad \int (8x+2)e^{2x^2+x}\,\mathrm{d}x \qquad \text{(c)} \quad \int \frac{\sin\log x}{x}\,\mathrm{d}x$$

There is no "product rule" for antidifferentiation, and the following statement is basically the best we can do, in the sense that it is the most useful antidifferentiation rule that can be derived from the product rule of differentiation. In practical terms, this rule allows us to replace fg' by $f'g$ under integration.

Theorem 12.12 (Integration by parts)**.** *Let f and g be functions differentiable on (a, b), and let $F' = f'g$ on (a, b). Then $fg - F$ is an antiderivative of fg' on (a, b), that is, we have*

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x. \tag{284}$$

Proof. By a direct computation, we infer

$$(fg - F)' = f'g + fg' - F' = f'g + fg' - f'g = fg', \tag{285}$$

which shows that $fg - F$ is an antiderivative of fg' on (a, b). □

Example 12.13. We have

$$\begin{aligned}\int \log x\,\mathrm{d}x &= \int \log x \cdot (x)'\,\mathrm{d}x = x\log x - \int (\log x)' \cdot x\,\mathrm{d}x = x\log x - \int \frac{1}{x}\cdot x\,\mathrm{d}x \\ &= x\log x - \int \mathrm{d}x = x\log x - x + C.\end{aligned} \tag{286}$$

Exercise 12.14. Compute the following indefinite integrals.

$$\text{(a)} \quad \int xe^x\mathrm{d}x \qquad \text{(b)} \quad \int x^2\cos x\,\mathrm{d}x \qquad \text{(c)} \quad \int e^x \sin x\,\mathrm{d}x$$

Finally, we show that functions defined by power series admit antiderivatives.

Theorem 12.15. *Let $0 < R \leq \infty$ be the convergence radius of the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n. \tag{287}$$

Then the power series

$$F(x)=\sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-c)^{n+1}, \tag{288}$$

has convergence radius equal to R, and it holds that

$$F'=g \qquad \text{in } (c-R,c+R), \tag{289}$$

where in case $R=\infty$ it is understood that $(c-R,c+R)=\mathbb{R}$.

Proof. Without loss of generality, we will assume that $c=0$. It is obvious that the convergence radius R' of the power series representing F is at least R, that is, $R'\geq R$. To prove the other direction, let $|x|>r>R$. Then the sequence $\{|a_n|r^n\}$ is unbounded, meaning that for any given $M\in\mathbb{R}$, there are infinitely many n for which $|a_n|r^n>M$. For those n, we have

$$\frac{|a_n||x|^n}{n+1}\geq M\frac{(|x|/r)^n}{n+1}. \tag{290}$$

Since $|x|>r$, we have $\frac{(|x|/r)^n}{n+1}\to\infty$ as $n\to\infty$, and so $\frac{|a_n||x|^n}{n+1}>M$ for infinitely many n. As M was arbitrary, this implies that $R'\leq R$.

The partial sums F_m of F converges to F uniformly in $(c-r,c+r)$ for any $r<R$, and the analogous statement is true for the partial sums f_m of f. Moreover, we have $F_m'=f_m$, and by invoking Theorem 7.14 we infer that $F'=f$ in $(c-R,c+R)$. □

Example 12.16. We have

$$\frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}, \tag{291}$$

with the convergence radius equal to ∞. Hence the function

$$\mathrm{Si}(x)=x-\frac{x^3}{3\cdot 3!}+\frac{x^5}{5\cdot 5!}-\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)\cdot(2n+1)!}, \tag{292}$$

is defined for $x\in\mathbb{R}$, and satisfies $\mathrm{Si}'(x)=\frac{\sin x}{x}$ for all $x\in\mathbb{R}$. The function $\mathrm{Si}(x)$ is known to be non-elementary, and called the *sine integral*.

Appendix A. Sets and functions

A *set* is a collection of its elements, where the elements must be distinct from each other. The simplest way to describe a set is to list its elements in between curly brackets, as in $\{1,2,3\}$ and $\{a,b,d\}$. The set without any element is called an *empty set*, and denoted by $\varnothing$ or $\{\}$. We write $a\in A$, if a is an element of A, and $a\notin A$ otherwise. Thus we have $a\notin\varnothing$ for any a. If all elements of A are also elements of another set B, i.e., if for all $a\in A$ it holds that $a\in B$, then we say that A is a *subset* of B (or B is a *superset* of A), and write $A\subset B$. For example $\{3,2\}\subset\{1,2,3\}$. Given any set A, we have $A\subset A$ and $\varnothing\subset A$, so in particular $\varnothing\subset\varnothing$. Two sets A and B are *equal*, that is, $A=B$, if and only if $A\subset B$ and $B\subset A$. Thus $\{3,2,1\}=\{1,2,3\}$ and $\{1,2,1\}=\{1,2\}$. The notation $B\supset A$ means $A\subset B$.

Given a set A, a powerful method to generate a new set is to take the set of all subsets of A. The resulting set is called the *power set* of A, and denoted by $\mathscr{P}(A)$ or 2^A. For example, the only subset of $\varnothing$ is $\varnothing$ itself, and hence $\mathscr{P}(\varnothing)=\{\varnothing\}$, and $\mathscr{P}(\{\varnothing\})=\{\varnothing,\{\varnothing\}\}$. Another example is $2^{\{a,c,e\}}=\{\varnothing,\{a\},\{c\},\{e\},\{a,c\},\{a,e\},\{c,e\},\{a,c,e\}\}$.

Two sets can be combined in various ways to construct a new set. Let A and B be sets.

- The *union* of A and B, denoted by $A\cup B$, is the set consisting of the elements that belong to at least one of A and B. In other words, $a\in A\cup B$ if and only if $a\in A$ or $a\in B$. Thus $\{1,2,3\}\cup\{2,4\}=\{1,2,3,4\}$.

- The *intersection* of A and B, denoted by $A \cap B$, is the set consisting of the elements that belong to both A and B. In other words, $a \in A \cap B$ if and only if $a \in A$ and $a \in B$. Thus $\{1,2,3\} \cap \{2,4\} = \{2\}$.
- The *set difference* between A and B, denoted by $A \setminus B$, is the set consisting of the elements that belong to A but not to B. In other words, $a \in A \setminus B$ if and only if $a \in A$ and $a \notin B$. Thus $\{1,2,3\} \setminus \{2,4\} = \{1,3\}$ and $\{2,4\} \setminus \{1,2,3\} = \{4\}$.

Exercise A.1. Let A, B, and C be sets. Prove the following.
(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(c) $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$.
(d) $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$.
(e) $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

One can observe a kind of parallel between sets and nonnegative integers. Let us denote by $|A|$ the number of elements[1] of A. For example, $|\{a,b,c\}| = 3$ and $|\varnothing| = 0$. We say A is a *finite set* if $|A|$ is a finite number. Let A and B be finite sets. Then $A \subset B$ implies $|A| \leq |B|$, and $|2^A| = 2^{|A|}$. Moreover, if $A \cap B = \varnothing$ then $|A \cup B| = |A| + |B|$, and if $B \subset A$ then $|A \setminus B| = |A| - |B|$. In general, we have $|A \cup B| + |A \cap B| = |A| + |B|$ and $|A \setminus B| = |A| - |A \cap B|$.

Thus very roughly speaking, set union and set difference are constructions corresponding to addition and subtraction of positive integers. There is also a construction associated to multiplication. To explain it, we introduce an *ordered pair* (a,b) of elements $a \in A$ and $b \in B$. The difference between $\{a,b\}$ and (a,b) is that the order of the elements matters in (a,b), that is, $(a,b) \neq (b,a)$ unless $a = b$, while $\{a,b\} = \{b,a\}$. Then the *product* $A \times B$ of the sets A and B is the set of all possible ordered pairs (a,b), where $a \in A$ and $b \in B$ are arbitrary elements. This can be written as

$$A \times B = \{(a,b) : a \in A,\, b \in B\}. \tag{293}$$

For example, $A \times B = \{(2,1),(2,2),(2,4),(3,1),(3,2),(3,4)\}$ if $A = \{2,3\}$ and $B = \{1,2,4\}$. We also have $A \times \varnothing = \varnothing \times A = \varnothing$. For finite sets we have $|A \times B| = |A| \cdot |B|$.

The definition (293) is an example of *set-builder notation*. This notation allows one to define a subset of a given set X according to whether or not $x \in X$ satisfies a particular property. A general form of set-builder notation is

$$Y = \{x \in X : P(x)\}, \tag{294}$$

where $P(x)$ is a logical expression that may be true or false depending on the variable x. By construction, we have $Y \subset X$. As an example, let $X = \{1,2,3,4,5,6,8,9\}$ and let $P(x) = (x \text{ is even})$. Then $E = \{x \in X : P(x)\}$, or equivalently $E = \{x \in X : x \text{ is even}\}$ would be the set $E = \{2,4,6,8\}$. In order to make sense of (293) in this framework, one can define ordered pairs by $(a,b) = \{\{a\},\{a,b\}\}$. This implements the correct behaviour since $(b,a) = \{\{b\},\{a,b\}\} \neq \{\{a\},\{a,b\}\}$ unless $a = b$. Note that $\{a\}$ and $\{a,b\}$ are subsets of $A \cup B$, i.e., elements of $\mathscr{P}(A \cup B)$, and hence $\{\{a\},\{a,b\}\}$ is an element of $\mathscr{P}(\mathscr{P}(A \cup B))$. Now (293) can be reinterpreted as

$$A \times B = \{x \in \mathscr{P}(\mathscr{P}(A \cup B)) : \text{there exist } a \in A \text{ and } b \in B \text{ such that } x = (a,b)\}. \tag{295}$$

In logical expressions, we allow the symbols $\in$, $=$, and the logical connectives "not, and, or, implies, equivalent," as well as the quantifiers "there exist," and "for all."

Exercise A.2. Let A, B, C, and D be sets. Prove the following.
(a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
(b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

[1] The notation $\#A$ is also often used in place of of $|A|$.

(c) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.
(d) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

A construction of fundamental importance is a *map* or a *function* $f : A \to B$, which assigns an element $b = f(a) \in B$ to every $a \in A$. We say that $f(a)$ is called the *image* of a under f, or the *value* of f at a. The set A is called the *domain* of f, and B is called the *codomain*. For example, with $A = \{2, 3\}$ and $B = \{1, 2, 4\}$, we may define a function $g : A \to B$ by $g(2) = 1$ and $g(3) = 4$. The *image of a subset* $S \subset A$ under f is defined to be

$$f(S) = \{b \in B : \text{there exists } a \in A \text{ such that } b = f(a)\} \subset B, \tag{296}$$

that is, the collection of all the images $f(a)$ for $a \in S$. A convenient abbreviation for this is

$$f(S) = \{f(a) : a \in S\}. \tag{297}$$

The image $f(A)$ of the entire domain A is called the *range* or[2] the *image* of f. If $f(A) = B$, then we say that the map f is *onto*, or *surjective*.

Given $b \in f(A)$, the set $\{a \in A : f(a) = b\} \subset A$ is called the *preimage of* b *under* f, and denoted by $f^{-1}(b)$. More generally, for $C \subset B$, the set $\{a \in A : f(a) \in C\} \subset A$ is called the *preimage of* C *under* f, and denoted by $f^{-1}(C)$. Obviously, $a \in f^{-1}(f(a))$ and $f^{-1}(f(A)) = A$. If $f(a) = f(a')$ implies $a = a'$ for all $a, a' \in A$, that is, if $|f^{-1}(b)| = 1$ for all $b \in f(A)$, then we say that f is *one to one*, or *injective*. A map that is both injective and surjective is said to be *bijective*. If $f : A \to B$ is bijective, then for any $b \in B$, there exists a unique $a \in A$ such that $f(a) = b$, and we define the *inverse function* $f^{-1} : B \to A$ by $f^{-1}(b) = a$. Given two functions $f : A \to B$ and $g : B \to C$, their *composition* $g \circ f : A \to B$ is defined by $(g \circ f)(a) = g(f(a))$ for $a \in A$. If $f : A \to B$ is biijective, then $f \circ f^{-1} = \mathrm{id} : B \to B$ and $f^{-1} \circ f = \mathrm{id} : A \to A$. For $S \subset A$, we let $i : S \to A$ be the map defined by $i(s) = s$ for all $s \in S$. This map is called the *natural injection* of S into A. Then for $f : A \to B$, the composition $f \circ i : S \to B$ is called the *restriction* of f to S, and denoted by $f|_S$.

Example A.3. (a) Given any set A, the *identity map* $\mathrm{id} : A \to A$ is defined by $\mathrm{id}(a) = a$ for all $a \in A$. Obviously, the identity map is bijective, and it is its own inverse.
(b) Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers, and let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by $f(n) = 2n$ for $n \in \mathbb{N}$. Then the image of f is the set of all positive even integers $f(\mathbb{N}) = \{2n : n \in \mathbb{N}\}$. It is injective, but not surjective, as the image misses the odd integers.
(c) Let $X = \{1, 2, 3, 5\}$. Then $Z = \{2x : x \in X\}$ would be the set $Z = \{2, 4, 6, 10\}$, and

$$C = \{2x : x \in X,\ x \text{ is odd}\}, \tag{298}$$

would be the set $C = \{2, 6, 10\}$.
(d) Let the functions $p : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $d : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be defined by $p(x) = m \cdot n$ for $x = (m, n) \in \mathbb{N} \times \mathbb{N}$ and $d(n) = (n, n)$ for $n \in \mathbb{N}$. Then the composition $f \circ d : \mathbb{N} \to \mathbb{N}$ is simply $(f \circ d)(n) = n^2$, and $d \circ f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is $(d \circ f)((m, n)) = (mn, mn)$.

Exercise A.4. Let $f : X \to Y$ be a function. Let $A, B \subset X$ and $C, D \subset Y$ be sets. Prove the following.
(a) $f(A \cup B) = f(A) \cup f(B)$.
(b) $f(A \cap B) \subset f(A) \cap f(B)$.
(c) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
(d) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Exercise A.5. Let $f : A \to B$ and $g : B \to C$ be functions. Prove the following.
(a) $(f \circ g) \circ h = f \circ (g \circ h)$.

[2]Note that some references use the range as synonymous to the codomain.

(b) If f and g are bijective, then so is $f \circ g$, and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Given two sets A and B, a function $f : A \to B$ is completely described by its *graph* $\Gamma(f) = \{(a, f(a)) : a \in A\} \subset A \times B$. In fact, given a subset Γ of $A \times B$, with the property that for each $a \in A$ there is one and only one element of the form $(a, b) \in \Gamma$, we can define a function f by setting $f(a) = b$ whenever $(a, b) \in \Gamma$. Therefore, functions are nothing more than a special kind of subsets of a product set. In particular, if A and B are finite sets, then the number of possible functions $f : A \to B$ is not more than $2^{|A|\cdot|B|}$. Note that strictly speaking, a function is a triple (f, A, B), or (Γ, A, B), so the functions $f_1 : A \to B_1$ and $f_2 : A \to B_2$ are considered different when $B_1 \neq B_2$, even if $\Gamma(f_1) = \Gamma(f_2)$.

The concept of functions can be generalized to that of relations. A *relation* between two sets A and B is simply a subset $\square \subset A \times B$. In this context, we write $a \square b$ to mean $(a, b) \in \square$, and say that a and b satisfies R if $a \square b$. For example, let the relation $\leq$ between the two sets $A = \{2, 3\}$ and $B = \{1, 2, 4\}$ be defined by the subset $\{(2, 4), (2, 2), (3, 4)\} \subset A \times B$. Then $2 \leq 4$ is true, but $2 \leq 1$ is false. Note that since there are more than one $b \in B$ with $2 \leq b$, this relation is not a function.

Finally, we present a set-theoretic construction that is a reminiscent of division. A relation $R \subset A \times A$ is called *reflexive* if $(a, a) \in R$ for all A, *symmetric* if $(a, b) \in R$ implies $(b, a) \in R$, and *transitive* if $(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$. An *equivalence relation* is a relation that is reflexive, symmetric, and transitive. Let $\sim$ be an equivalence relation on A. Then an *equivalence class* of $a \in A$ under $\sim$ is the set $[a] = \{b \in A : b \sim a\}$, and the *quotient* of A by the relation $\sim$ is defined as $A/\!\sim = \{[a] : a \in A\}$. This is well defined, as the following argument shows. If $b \in [a]$ and $c \in [b]$ then $c \sim a$ by transitivity, meaning that $b \in [a]$ implies $[b] \subset [a]$. Moreover, since $a \in [a]$ by reflexivity, $b \in [a]$ implies $a \in [b]$ by symmetry, and hence $[a] \subset [b]$. We conclude that $[a]$ and $[c]$ for $a, c \in A$ are either disjoint or identical, since if $b \in [a] \cap [c]$ then $[a] = [b] = [c]$. In other words, $a \sim c$ if and only if $[a] = [c]$, and an equivalence relation can be completely described by the equivalence classes. For example, let $A = \{1, 2, 3\}$, and let $R = \{(1, 1), (1, 3), (3, 1), (2, 2), (3, 3)\} \subset A \times A$. Then it is straightforward to verify that R is an equivalence relation on A, and $A/R = \{\{1, 3\}, \{2\}\}$. Note in this case that $[1] = [3] = \{1, 3\}$ and $[2] = \{2\}$.