MATH 248 PROBLEM SET 5

DUE THURSDAY DECEMBER 1

1. Let $f, g : Q \to \mathbb{R}$ be integrable functions. Show that fg and $|f|$ are integrable.
2. Compute the volume of the 4-dimensional ball
$$B_r = \{x \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 < r^2\},$$
of radius $r > 0$.
3. Compute the mass of the solid ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ if its mass density is given by $\mu(x, y, z) = x^2 + y^2 + z^2$.
4. Let $K \subset \mathbb{R}^3$ be a Jordan region, representing a solid body with mass density given by a continuous function $\mu(x) \geq 0$, $x \in K$. Then according to Newton's *law of universal gravitation*, the gravitational force exerted by this body on a point mass m located at $y \in \mathbb{R}^3$ is equal to
$$F(y) = Gm \int_K \frac{\mu(x)}{|x - y|^2} \frac{x - y}{|x - y|} \, \mathrm{d}^3 x, \tag{1}$$
where $|\xi| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$ for $\xi \in \mathbb{R}^3$, and $G > 0$ is the gravitational constant, whose numerical value depends on the unit system one is using to measure force, mass, and distance. Note that the integral in (1) should be thought of as a collection of 3 integrals, as we have $x - y \in \mathbb{R}^3$ under the integral, and the result is the force vector $F(y) \in \mathbb{R}^3$.
   (a) Consider a 3-dimensional solid ball B of radius a centred at the origin, with a spherically symmetric mass density $\mu(x) = g(|x|) \geq 0$, for some continuous function g of a single variable. Show that the gravitational force exerted by B on a point mass m located at $\xi = (R, 0, 0)$ with $R > a$, is the same as if its mass were all concentrated at the origin, that is,
$$F = (-\frac{GMm}{R^2}, 0, 0),$$
where M is the total mass of B.
   (b) Consider now a spherical shell S, defined by $a \leq |x| \leq b$, with spherically symmetric, continuous mass density function. Show that this shell exerts *no* gravitational force on a point mass m located at the point $(c, 0, 0)$ *inside* S (that is, $|c| < a$).

Date: Fall 2016.