# MATH 248 PROBLEM SET 3

DUE TUESDAY NOVEMBER 8

1. In an idealized situation, each of the planets moves along an ellipse

$$x = a\cos\theta, \qquad y = b\sin\theta,$$

in the $xy$-plane, where $a \geq b > 0$ are constants, and $\theta$ is a real parameter representing an angular coordinate with respect to the centre of the ellipse. The Sun is located at $(c, 0)$ where $c = \sqrt{a^2 - b^2}$, and the motion of the planet is determined by *Kepler's equation*

$$kt = \theta - \varepsilon \sin\theta,$$

where $t$ is time measured from an instant the planet passes through $(a, 0)$, $k > 0$ is a constant depending on the unit of time, and $\varepsilon = \frac{c}{a}$ is called the *eccentricity* of the ellipse.
(a) Show that Kepler's equation can be solved for $\theta = \theta(t)$ for all $t \in \mathbb{R}$.
(b) Show that $\theta(t)$ is differentiable, and compute $\theta'(t)$ in terms of $\theta(t)$.
(c) Find the points at which $\theta'$ takes its minimal and maximal values.

2. Sketch the level curves $f(x, y) = C$ of the function

$$f(x, y) = (x^2 + y^2)^2 - 2(x^2 - y^2),$$

for several different values of $C \in \mathbb{R}$. In particular, the case $C = 0$ yields the *lemniscate* of Jacob Bernoulli. Find the set of points at which the implicit function theorem does not apply to the equation $f(x, y) = C$ to yield $y = y(x)$. Then find the set of points at which $y'(x) = 0$, and show that they lie on a circle.

3. Suppose that the real variables $p$, $v$, $t$, and $u$ satisfy the equations

$$f(p, v, t, u) = 0, \qquad g(p, v, t, u) = 0,$$

and that these two equations can be solved for any two of the four variables as functions of the other two. Then the symbol $\partial u/\partial p$, for example, is ambiguous. We denote by $(\partial u/\partial p)_t$ the partial derivative of $u$ with respect to $p$, with $u$ and $v$ considered as functions of $p$ and $t$, and by $(\partial u/\partial p)_v$ the partial derivative of $u$ with respect to $p$, with $u$ and $t$ considered as functions of $p$ and $v$. With this notation, show that

$$\left(\frac{\partial u}{\partial p}\right)_v = \left(\frac{\partial u}{\partial t}\right)_v \left(\frac{\partial t}{\partial p}\right)_v = \left(\frac{\partial u}{\partial t}\right)_p \left(\frac{\partial t}{\partial p}\right)_v + \left(\frac{\partial u}{\partial p}\right)_t.$$

All functions that appear in this problem may be assumed to be defined and differentiable in a suitable set to make the arguments work.

*Date*: Fall 2016.

4. Let $M \subset \mathbb{R}^3$ be the set obtained by rotating the circle $\{(0, y, z) \in \mathbb{R}^3 : (y-2)^2 + z^2 = 1\}$ about the $z$-axis. By explicitly constructing coordinate charts, show that $M$ is a manifold in $\mathbb{R}^3$. The manifold $M$ is an example of a *torus*.
5. Let $A \in \mathbb{R}^{n \times n}$ be an invertible, symmetric matrix. Then in each of the following cases, show that $M$ is a manifold, and determine the dimension of $M$.
   (a) $M = \{x \in \mathbb{R}^n : x^T A x = 1\} \subset \mathbb{R}^n$. (Generalized sphere)
   (b) $M = \{X \in \mathbb{R}^{n \times n} : X^T A X = A\} \subset \mathbb{R}^{n \times n}$. (Generalized orthogonal transformations)
6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = (y - x^2)(y - 2x^2)$. Show that the origin $(0, 0)$ is a local minimum of $f$ restricted to any line that passes through the origin, but the origin is *not* a local minimum of $f$.
7. (a) Let $f : (a, b) \to \mathbb{R}$ be three times differentiable in $(a, b)$, and let $x, c \in (a, b)$. Show that there exists $\xi \in (x, c) \cup (c, x)$, such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(\xi)}{6}(x-c)^3.$$

   *Hint*: Proof of Theorem 7.5 from the differentiation notes.
   (b) Introduce the notion of the third derivative $D^3 f$ for functions $f : \mathbb{R}^n \to \mathbb{R}$, and give a quantitative information on the error of the quadratic approximation

$$f(y + V) \approx f(y) + Df(y)V + \frac{1}{2}V^T D^2 f(y)V,$$

   in the spirit of Remark 7.8 from the differentiation notes.