MATH 248 PROBLEM SET 2

DUE THURSDAY OCTOBER 20

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} xy & \text{for } -|x| < y < |x|, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

   (a) Compute the partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ at the origin, if they exist. Check if $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ at the origin.
   (b) How do you reconcile this with the theorem on symmetricity of the Hessian?
2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{2}x + x^2 \sin \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases} \tag{2}$$

   (a) Show that f is differentiable everywhere, and that $f'(0) \neq 0$.
   (b) Show that f is not invertible in $(-r, r)$ for any $r > 0$.
   (c) How do you reconcile this with the inverse function theorem?
3. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be differentiable in $\mathbb{R}^n$. Prove the following.
   (a) $f + g$ is differentiable in $\mathbb{R}^n$, with $D(f+g) = Df + Dg$.
   (b) fg is differentiable in $\mathbb{R}^n$, with $D(fg) = fDg + gDf$.
   (c) For $m \in \mathbb{N}$, f^m is differentiable in $\mathbb{R}^n$, with $D(f^m) = mf^{m-1}Df$.
   (d) $D_{V+W}f = D_V f + D_W f$ for $V, W \in \mathbb{R}^n$.
4. (a) Show that $u(x,t) = \frac{1}{\sqrt{t}} \exp(-\frac{x^2}{4t})$ satisfies $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for all $x \in \mathbb{R}$ and $t > 0$.
   (b) If $g(u,v) = f(au + bv, cu + dv)$, where a, b, c, d are constants, and $f = f(x,y)$ is a twice differentiable function in $\mathbb{R}^2$, then show that

$$\frac{\partial^2 g}{\partial u \partial v} = ab\frac{\partial^2 f}{\partial x^2} + (ad + bc)\frac{\partial^2 f}{\partial x \partial y} + cd\frac{\partial^2 f}{\partial y^2}. \tag{3}$$

5. Let $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$\Psi(r,\phi,\theta) = \begin{pmatrix} x(r,\phi,\theta) \\ y(r,\phi,\theta) \\ z(r,\phi,\theta) \end{pmatrix} = \begin{pmatrix} r\cos\theta\cos\phi \\ r\cos\theta\sin\phi \\ r\sin\theta \end{pmatrix}, \tag{4}$$

   where $(r,\phi,\theta) \in \mathbb{R}^3$ are the coordinates of the domain, and (x, y, z) are the components of the function Ψ.

Date: Fall 2016.

(a) Compute the Jacobian of Ψ, and show that Ψ is differentiable everywhere in $\mathbb{R}^3$.
(b) Determine the points at which the Jacobian of Ψ is invertible. What would the inverse function theorem give if you apply this theorem near every such point?
(c) Let $u : \mathbb{R}^3 \to \mathbb{R}$ be differentiable in $\mathbb{R}^3$, and let $v = u \circ \Psi$. Does Dv exist everywhere in $\mathbb{R}^3$? Find $Dv = (\partial_r v, \partial_\phi v, \partial_\theta v)$ in terms of $Du = (\partial_x u, \partial_y u, \partial_z u)$. Then compute Du in terms of Dv.
(d) Let $u : \mathbb{R}^3 \to \mathbb{R}$ be twice differentiable in $\mathbb{R}^3$, and let $v = u \circ \Psi$. Compute

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \tag{5}$$

in terms of the (first and second) derivatives of v.