PRACTICE PROBLEMS FOR THE FINAL EXAM

MATH 248 FALL 2016

1. Show that $M = \{(x, y) \in \mathbb{R}^2 : (x + y)^5 - xy = 1\}$ is a manifold in $\mathbb{R}^2$. Sketch this curve.
2. Let $(x_*, y_*, z_*) \in \mathbb{R}^3$, and let a, b, c be positive numbers. Then with
$$\phi(x, y, z) = \frac{(x - x_*)^2}{a^2} + \frac{(y - y_*)^2}{b^2} + \frac{(z - z_*)^2}{c^2},$$
show that $M = \{(x, y, z) \in \mathbb{R}^3 : \phi(x, y, z) = 1\}$ is a manifold in $\mathbb{R}^3$. Do you recognize this surface?
3. Find and classify the critical points of $f(x, y) = (x^2 + y^2)e^{x^2-y^2}$.
4. Find, with full justifications, the minimum value of
$$f(x, y) = (x - y)y + \frac{1}{x - y} + \frac{1}{y},$$
over the set $\Omega = \{(x, y) : x > y > 0\}$.
5. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ and $\eta : \mathbb{R} \to \mathbb{R}^n$ be two smooth curves. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and define $d(p, q) = (p - q)^T A(p - q)$ for $p, q \in \mathbb{R}^n$. Suppose that $p = \gamma(s_*)$ and $q = \eta(t_*)$ are two points such that $d(p, q) \le d(\gamma(s), \eta(t))$ for any other pair of points $\gamma(s)$ and $\eta(t)$ on the two curves. Then show that $(p - q)^T A\gamma'(s_*) = (p - q)^T A\eta'(t_*) = 0$.
6. Compute the area of the region enclosed by the loops of Bernoulli's lemniscate
$$(x^2 + y^2)^2 - 2(x^2 - y^2) = 0.$$
Possible approaches include direct computation, polar coordinates, and other suitable change of coordinates.
7. Compute the integral
$$\int_A xyz(1 - x - y - z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$$
where A is the tetrahedron defined by
$$A = \{(x, y, z) : x \ge 0, y \ge 0, z \ge 0, x + y + z \le 1\}.$$
8. Let $T \subset \mathbb{R}^3$ be the solid torus in $\mathbb{R}^3$ obtained by rotating the disk
$$\{(0, y, z) \in \mathbb{R}^3 : (y - a)^2 + z^2 \le b^2\}$$
about the z-axis, where $|b| \le |a|$. Compute the volume of T.
9. Let $S \subset \mathbb{R}^3$ be the intersection of the cylinders $x^2 + y^2 \le 1$ and $y^2 + z^2 \le 1$. Compute its volume.

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10. Let $u(x, y) = \log(x^2 + y^2)$ be defined for $(x, y) \neq (0, 0)$, and let γ be the unit circle oriented counter-clockwise. For any given 1-form $\alpha = f\mathrm{d}x + g\mathrm{d}y$, define the so-called *Hodge star* operation
$$\star\alpha = g\mathrm{d}x - f\mathrm{d}y.$$
Intuitively, $\star\alpha$ is obtained by rotating α by $90°$. Compute the line integral $\int_\gamma \star\mathrm{d}u$.
11. Let $(x_k, y_k, q_k) \in \mathbb{R}^3$, $k = 1, \ldots, n$, and let
$$u(x, y) = \sum_{k=1}^{n} q_k \log\left((x - x_k)^2 + (y - y_k)^2\right).$$
Compute the line integral $\int_\gamma \star\mathrm{d}u$, where γ is a counterclockwise-oriented smooth closed curve enclosing the points $(x_1, y_1), \ldots, (x_n, y_n)$, and the Hodge star is defined as in the previous exercise.
12. Let C be the curve of intersection of the cylinder $x^2+y^2 = 1$ and the surface $z = xy+1$, oriented counterclockwise around the cylinder. Compute the line integral
$$\int_C z(x-1)\,\mathrm{d}y + y(x+1)\,\mathrm{d}z.$$
13. Let $F(x) = |x|x$, where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Compute $\operatorname{div} F$, and the volume integral
$$\int_{B_r} |x|\,\mathrm{d}^3x,$$
where $B_r = \{x \in \mathbb{R}^3 : |x| < r\}$.
14. Let $E \subset \mathbb{R}^3$ be the the solid ellipsoid defined by
$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} \leq 1.$$
For $x \in \partial E$, let $n(x) \in \mathbb{R}^3$ be the outward unit normal to ∂E at x, and let $V(x) \in \mathbb{R}^3$ be a vector field defined by $V(x) = (0, 0, x_3)$. Compute the surface integral
$$\int_{\partial E} V(x) \cdot n(x)\,\mathrm{d}^2x,$$
where $V \cdot n = V_1n_1 + V_2n_2 + V_3n_3$. Here ∂E denotes the boundary of E, and d^2x is the surface area element of ∂E.