PRACTICE PROBLEMS FOR THE MIDTERM

MATH 248 FALL 2016

1. Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and let $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}^n$ be smooth functions satisfying
$$M\gamma''(t) = -(D\phi(\gamma(t))^T, \qquad t \in \mathbb{R}.$$
(a) Show that the quantity
$$E(t) = \tfrac{1}{2}(\gamma'(t))^T M\gamma'(t) + \phi(\gamma(t)),$$
is independent of $t \in \mathbb{R}$, i.e., E is a *conserved* along $\gamma(t)$.
(b) Let $V \in \mathbb{R}^n$, and suppose that $D_V\phi(x) = 0$ for all $x \in \mathbb{R}^n$. Then show that the quantity
$$p(t) = V^T M\gamma'(t),$$
is independent of $t \in \mathbb{R}$. Consider the case $n = 3$, $M = I$, and $\phi(x) = x_3$, and construct such a conserved quantity $p(t)$ by appropriately choosing a vector V.
(c) Suppose that $\phi(x) = g(x^T Mx)$ with some smooth function $g : \mathbb{R} \to \mathbb{R}$, and assume that M is invertible. Then show that
$$\gamma''(t) = \psi(\gamma(t))\gamma(t), \qquad t \in \mathbb{R},$$
where $\psi : \mathbb{R}^n \to \mathbb{R}$ is given by
$$\psi(x) = -2g'(x^T Mx), \quad x \in \mathbb{R}^n.$$
Write down the equation for γ'' when $n = 3$, $M = I$, and $\phi(x) = 1/|x|$.
(d) In the setting of (c), let $A \in \mathbb{R}^{n\times n}$ be an antisymmetric matrix, and show that the quantity
$$L(t) = \gamma(t)^T A\gamma'(t),$$
is independent of $t \in \mathbb{R}$. Consider the case $n = 3$ and $M = I$, and construct such a conserved quantity $L(t)$ by appropriately choosing an antisymmetric matrix A.

2. Let $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ be a smooth, matrix valued function of a single variable.
(a) Compute $(A^2)'$ and $(A^3)'$.
(b) Recall that the *trace* of a matrix $X = (x_{ik}) \in \mathbb{R}^{n\times n}$ is
$$\mathrm{tr}X = \sum_{k=1}^{n} x_{kk} = x_{11} + \ldots + x_{nn}.$$
Show that $(\mathrm{tr}A)' = \mathrm{tr}A'$.

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(c) Assuming that $A(t)$ is an invertible diagonal matrix for all $t \in \mathbb{R}$, show that
$$(\det A)' = \det(A)\operatorname{tr}(A^{-1}A').$$
(d) Assuming that $A(t)$ is invertible for all $t \in \mathbb{R}$, show that
$$(A^{-1})' = -A^{-1}A'A^{-1}.$$

3. Show that the differentiation rules $(f \pm g)' = f' \pm g'$ and $(fg)' = f'g + fg'$ for single variable functions are special cases of the multivariable chain rule.
4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function satisfying $f(tx) = tf(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$. Show that if f is differentiable at 0, then f is linear, i.e., $f(x) = b^T x$ for some $b \in \mathbb{R}^n$.
5. (a) Show that
$$\partial_i |x| = \frac{x_i}{|x|} \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}, \qquad \text{where} \quad |x| = \sqrt{x_1^2 + \ldots + x_n^2}.$$
(b) Let $\phi : (0, \infty) \to \mathbb{R}$ be a smooth function, and let $u(x) = \phi(|x|)$ for $x \in \mathbb{R}^n \setminus \{0\}$. Show that
$$\Delta u(x) = \partial_1^2 u(x) + \ldots + \partial_n^2 u(x) = \phi''(|x|) + \frac{n-1}{|x|}\phi'(|x|),$$
for $x \in \mathbb{R}^n \setminus \{0\}$.
(c) Find all smooth functions $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ of the above form, satisfying the equation $\Delta u = 0$ in $\mathbb{R}^n \setminus \{0\}$.
6. Show that the equation $z^3 + ze^{xy} - xy = 0$ defines z as a function of $(x, y) \in \mathbb{R}^2$. Is $z = z(x, y)$ differentiable?
7. Show that the equation $x + y + z + \cos(xyz) = 0$ can be solved for $z = z(x, y)$ in an open set containing the origin. Find the plane tangent to $z = z(x, y)$ at the origin.
8. The point $p = (1, -1, 1)$ lies on the surfaces
$$x^3(y^3 + z^3) = 0, \qquad (x - y)^3 - z^2 = 7.$$
Show that, in an open set containing p, the curve of intersection of the surfaces can be parameterized by x, that is, the curve can be described by a system of equations of the form $\{y = f(x),\, z = g(x)\}$.