Problems with solutions

Math 248 Fall 2016

1. Find the coordinates of the point (x, y, z) on the plane $z = x + y + 4$ which is closest to the origin.

   Solution: The square of the distance from a point (x, y, z) on the plane to the origin $(0, 0, 0)$ is

   $$f(x, y) = x^2 + y^2 + (x + y + 4)^2. \tag{1}$$

   To find the critical points, we set up the equations

   $$\begin{aligned} \frac{\partial}{\partial x} f(x, y) &= 2x + 2(x + y + 4) = 4x + 2y + 8 = 0, \\ \frac{\partial}{\partial y} f(x, y) &= 2y + 2(x + y + 4) = 2x + 4y + 8 = 0, \end{aligned} \tag{2}$$

   whose only solution is

   $$x = y = -\tfrac{4}{3}. \tag{3}$$

   We infer that the only critical point of $f(x, y)$ is $(x^*, y^*) = (-\frac{4}{3}, -\frac{4}{3})$, and that

   $$f(x^*, y^*) = \tfrac{16}{3}. \tag{4}$$

   The question is: Is this the minimum value of f? To answer this question, we first try to show that $f(x, y)$ is large when the point (x, y) is far away from the origin. If (x, y) is *outside* the open disk $D_R = \{(x, y) : x^2 + y^2 < R^2\}$ of radius $R > 0$, that is, if $x^2 + y^2 \geq R^2$, then we have

   $$f(x, y) = x^2 + y^2 + (x + y + 4)^2 \geq x^2 + y^2 \geq R^2. \tag{5}$$

   In particular, fixing $R = 4$, we get

   $$f(x, y) \geq 16 > \tfrac{16}{3} = f(x^*, y^*) \qquad \text{for} \quad x^2 + y^2 \geq R^2, \tag{6}$$

   and since (x^*, y^*) is in the disk D_R, we conclude that any possible minimizer must be contained in the disk D_R. Now we apply the Weierstrass existence theorem in the *closed* disk $\bar{D}_R = \{(x, y) : x^2 + y^2 \leq R^2\}$, to infer that there exists a minimizer of f over the closed disk $\bar{D}_R$. By (6), a minimizer cannot be on the boundary of $\bar{D}_R$, so it must be in the open disk D_R. This means that any minimizer must be a critical point of f in D_R, but we know that there is only one critical point, implying that there is only one minimizer of f over $\bar{D}_R$, and the minimizer is the point (x^*, y^*). Note that at this point all we know is that (x^*, y^*) is the minimizer of f over $\bar{D}_R$. However, invoking (6) once again, we conclude that (x^*, y^*) is indeed the minimizer of f over $\mathbb{R}^2$. Finally, since we were asked to find the coordinates of the point (x, y, z), we note that the minimizer (x^*, y^*) corresponds to the point

   $$(x, y, z) = (-\tfrac{4}{3}, -\tfrac{4}{3}, \tfrac{4}{3}), \tag{7}$$

   on the plane $z = x + y + 4$.

Lagrange multiplier: An alternative way to find the critical points is to use a Lagrange multiplier. Thus we consider the function

$$f(x, y, z) = x^2 + y^2 + z^2, \tag{8}$$

on the plane $P = \{(x, y, z) \in \mathbb{R}^3 : \phi(x, y, z) = 0\}$, where $\phi(x, y, z) = x + y + 4 - z$. By the Lagrange multipliers theorem, $(x, y, z) \in P$ is a critical point of f on P if and only if there exists $\lambda \in \mathbb{R}$ such that $Df(x, y, z) = \lambda D\phi(x, y, z)$, that is,

$$(2x, 2y, 2z) = \lambda(1, 1, -1). \tag{9}$$

This yields $(x, y, z) = \frac{\lambda}{2}(1, 1, -1)$, and by invoking $(x, y, z) \in P$, we find

$$\tfrac{\lambda}{2} + \tfrac{\lambda}{2} + 4 - (-\tfrac{\lambda}{2}) = 0. \tag{10}$$

Hence we have $\lambda = -\frac{8}{3}$, and so $(x, y, z) = (-\frac{4}{3}, -\frac{4}{3}, \frac{4}{3})$ is the only critical point. From here on, we can proceed as in the preceding solution.

Solution by geometric reasoning: The vector $V = (1, 1, -1)$ is normal to the plane $P = \{z = x+y+4\}$, and therefore the point of intersection between the line $\gamma(t) = Vt$ and the plane P is in fact the point of P closest to the origin. To find the intersection, we solve the equation

$$-1t = 1t + 1t + 4, \tag{11}$$

which yields $t = -\frac{4}{3}$, and hence $(x, y, z) = Vt = (-\frac{4}{3}, -\frac{4}{3}, \frac{4}{3})$.

2. Find the maximum and minimum values of $f(x, y, z) = 2x-y+4z$ on the sphere $x^2+y^2+z^2 = 1$. Here you can use any type of reasonings, including geometric ones.

Solution: Let $\phi(x, y, z) = x^2+y^2+z^2-1$. By the Weierstrass theorem, there exist a maximizer and a minimizer of f on the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : \phi(x, y, z) = 0\}$, and by the first derivative test, these points must be critical points of f on S^2. Then by the Lagrange multiplier theorem, $(x, y, z) \in S^2$ is a critical point if and only if there exists $\lambda \in \mathbb{R}$ such that

$$Df(x, y, z) = \lambda D\phi(x, y, z). \tag{12}$$

We compute $Df(x, y, z) = (2, -1, 4)$ and $D\phi(x, y, z) = (2x, 2y, 2z)$, and hence (12) becomes $(2, -1, 4) = \lambda(2x, 2y, 2z)$. From this, it is clear that $\lambda \neq 0$, and so $(x, y, z) = \lambda^{-1}(1, -\frac{1}{2}, 2)$. Invoking $(x, y, z) \in S^2$, we infer

$$1 = x^2 + y^2 + z^2 = \frac{1 + (\frac{1}{2})^2 + 2^2}{\lambda^2} = \frac{21}{4\lambda}, \tag{13}$$

yielding $\lambda = \pm\frac{\sqrt{21}}{2}$. Now we can find

$$(x, y, z) = \lambda^{-1}(1, -\tfrac{1}{2}, 2) = (\pm\tfrac{2}{\sqrt{21}}, \mp\tfrac{1}{\sqrt{21}}, \pm\tfrac{4}{\sqrt{21}}). \tag{14}$$

The maximum and the minimum of f on S^2 must be among these two points. We evaluate f at these two points, as

$$f(\pm\tfrac{2}{\sqrt{21}}, \mp\tfrac{1}{\sqrt{21}}, \pm\tfrac{4}{\sqrt{21}}) = \pm\sqrt{21}, \tag{15}$$

and compare the values, to conclude that $(\frac{2}{\sqrt{21}}, -\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}})$ is the maximizer of f on S^2, and $(-\frac{2}{\sqrt{21}}, \frac{1}{\sqrt{21}}, -\frac{4}{\sqrt{21}})$ is the minimizer of f on S^2. The maximum and minimum *values* of f on S^2 are of course $\pm\sqrt{21}$.

Solution by geometric reasoning: Let us use the notations $X = (x, y, z)$ and $V = (2, -1, 4)$. Then the problem is equivalent to minimizing and maximizing the function

$$f(X) = V^T X, \tag{16}$$

over the sphere $|X|_2 = 1$. We have

$$f(X) = V^T X = |V|_2 |X|_2 \cos\theta = |V|_2 \cos\theta = \sqrt{2^2 + 1^2 + 4^2} \cos\theta = \sqrt{21}\cos\theta, \tag{17}$$

where θ is the angle between the vectors X and V. It is now clear that the maximum is obtained at $\theta = 0$ and the minimum is obtained at $\theta = \pi$. The corresponding values are

$$f_{\max} = \sqrt{21}, \qquad \text{and} \qquad f_{\min} = -\sqrt{21}. \tag{18}$$

This answers the question completely, but if we wanted to find the points at which the maximum and the minimum values are attained, the maximizer of $f(X)$ over $|X|_2 = 1$ is the vector

$$X^* = \frac{1}{|V|_2} V = \frac{1}{\sqrt{21}}(2, -1, 4) = \Big(\frac{2}{\sqrt{21}}, -\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}\Big), \tag{19}$$

and the minimizer is

$$X_* = -\frac{1}{|V|_2} V = \Big(-\frac{2}{\sqrt{21}}, \frac{1}{\sqrt{21}}, -\frac{4}{\sqrt{21}}\Big). \tag{20}$$

3. Let $f(x, y) = 5x - 7y + 4xy - 7x^2 + 4y^2$ be a function defined in the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Find the maximum and minimum values of f and where they occur.

Solution: By the Weierstrass existence theorem, there exist a maximizer and a minimizer in the (closed) square $\bar{Q} = \{(x, y) : 0 \leq x \leq 1,\, 0 \leq y \leq 1\}$. If there is a maximizer (or a minimizer) in $Q = \{(x, y) : 0 < x < 1,\, 0 < y < 1\}$, then it must be a critical point. From the equations

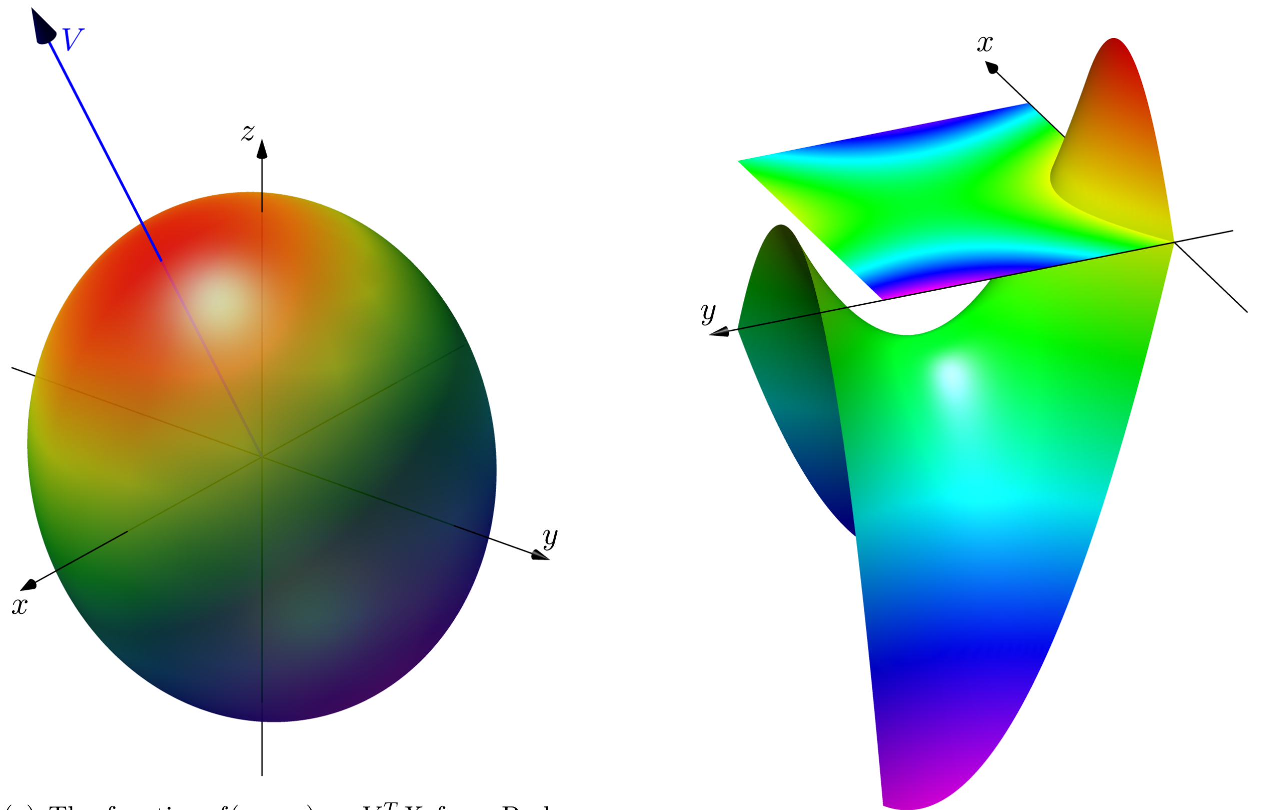
$$\begin{aligned} \frac{\partial}{\partial x} f(x, y) &= 5 + 4y - 14x = 0, \\ \frac{\partial}{\partial y} f(x, y) &= -7 + 4x + 8y = 0, \end{aligned} \tag{21}$$

it follows that $(x^*, y^*) = (\frac{17}{32}, \frac{39}{64})$ is the only critical point of f in Q. For later reference, let us compute

$$f(x^*, y^*) = f(\tfrac{17}{32}, \tfrac{39}{64}) = -\tfrac{103}{128}. \tag{22}$$

Let us also compute the values of f at the four corners of the square $\bar{Q}$:

$$f(0, 0) = 0, \qquad f(0, 1) = -3, \qquad f(1, 0) = -2, \qquad f(1, 1) = -1. \tag{23}$$



(a) The function $f(x, y, z) = V^T X$ from Problem 2 for $X = (x, y, z)$ on the unit sphere is depicted by a colour density plot on the sphere. Red represents high values, and blue represents low values.

(b) The graph and a colour density plot of the function $f(x, y)$ from Problem 3 over the unit square $\bar{Q} = \{0 \le x \le 1,\, 0 \le y \le 1\}$.

Figure 1: Illustrations for Problem 2 and Problem 3.

It remains to check the four sides of $\bar{Q}$. At the bottom edge $\ell_1 = \{0 < x < 1,\, y = 0\}$, the values of f are recorded in the single variable function

$$g_1(x) = f(x, 0) = 5x - 7x^2. \tag{24}$$

It is easy to find the critical point $x_1^* = \frac{5}{14}$, which gives

$$f(x_1^*, 0) = g_1(x_1^*) = \tfrac{25}{28}. \tag{25}$$

At the top edge $\ell_2 = \{0 < x < 1,\, y = 1\}$, we have

$$g_2(x) = f(x, 1) = 5x - 7 + 4x - 7x^2 + 4 = -7x^2 + 9x - 3, \tag{26}$$

whose critical point is $x_2^* = \frac{9}{14}$, with the corresponding value

$$f(x_2^*, 1) = g_2(x_2^*) = -\tfrac{3}{28}. \tag{27}$$

As for the left edge $\ell_3 = \{x = 0,\, 0 < y < 1\}$, we have

$$g_3(y) = f(0, y) = -7y + 4y^2. \tag{28}$$

The critical point is $y_3^* = \frac{7}{8}$, and the function value is

$$f(0, y_3^*) = g_3(y_3^*) = -\tfrac{49}{16}. \tag{29}$$

Finally, at the right edge $\ell_4 = \{x = 1,\ 0 < y < 1\}$, the values of f are

$$g_4(y) = f(1, y) = 5 - 7y + 4y - 7 + 4y^2 = 4y^2 - 3y - 2. \tag{30}$$

The only critical point of g_4 is $y_4^* = \frac{3}{8}$, with the value

$$f(1, y_4^*) = g_4(y_4^*) = -\tfrac{41}{16}. \tag{31}$$

Now, by comparing the values (22), (23), (25), (27), (29), and (31), we conclude that the maximum value of f over $\bar{Q}$ is $\frac{25}{28}$, which occurs at $(x_1^*, 0) = (\frac{5}{14}, 0)$, and the minimum value of f over $\bar{Q}$ is $-\frac{49}{16}$, which occurs at $(0, y_3^*) = (0, \frac{7}{8})$.

4. Find the maximum and minimum values of the function $f(x, y) = 5x^2 - 22xy + 5y^2 + 8$ in the disk $x^2 + y^2 \leq 25$.

   Solution: By the Weierstrass existence theorem, there exist a maximizer and a minimizer in the (closed) disk $\bar{D} = \{(x, y) : x^2 + y^2 \leq 25\}$. If there is a maximizer (or a minimizer) in $D = \{(x, y) : x^2 + y^2 < 25\}$, then it must be a critical point. From the equations

$$\frac{\partial}{\partial x} f(x, y) = 10x - 22y = 0, \qquad \frac{\partial}{\partial y} f(x, y) = -22x + 10y = 0, \tag{32}$$

   it follows that $(x^*, y^*) = (0, 0)$ is the only critical point of f in D. For later reference, let us compute

$$f(x^*, y^*) = f(0, 0) = 8. \tag{33}$$

   Now we parameterize the boundary of $\bar{D}$ as

$$(x(t), y(t)) = (5 \cos t, 5 \sin t), \qquad t \in \mathbb{R}. \tag{34}$$

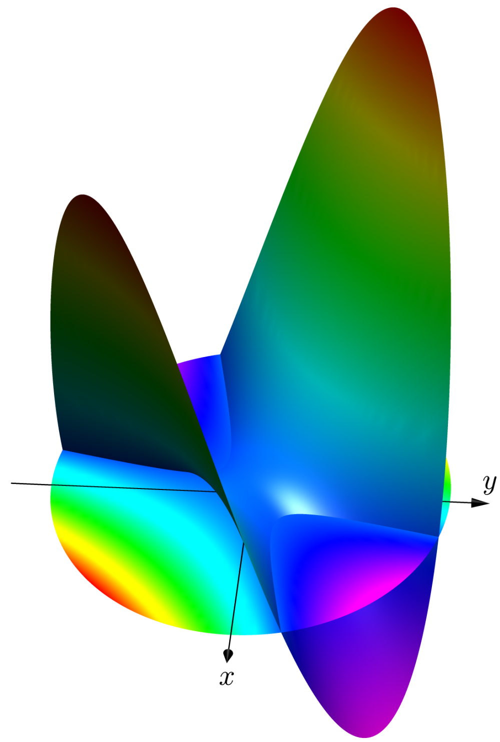
   Note that for any t_1 and t_2 satisfying $t_2 = t_1 + 2\pi n$ with some integer n, we have $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$, meaning that the parameter values t_1 and t_2 correspond to the same point on the boundary (circle) of $\bar{D}$. The values of f along the boundary gives rise to the function

$$\begin{aligned} g(t) = f(x(t), y(t)) &= 5 \cdot 25 \cos^2 t - 22 \cdot 25 \sin t \cos t + 5 \cdot 25 \sin^2 t + 8 \\ &= 125 - 22 \cdot 25 \sin t \cos t + 8 = 133 - 275 \sin 2t, \end{aligned} \tag{35}$$

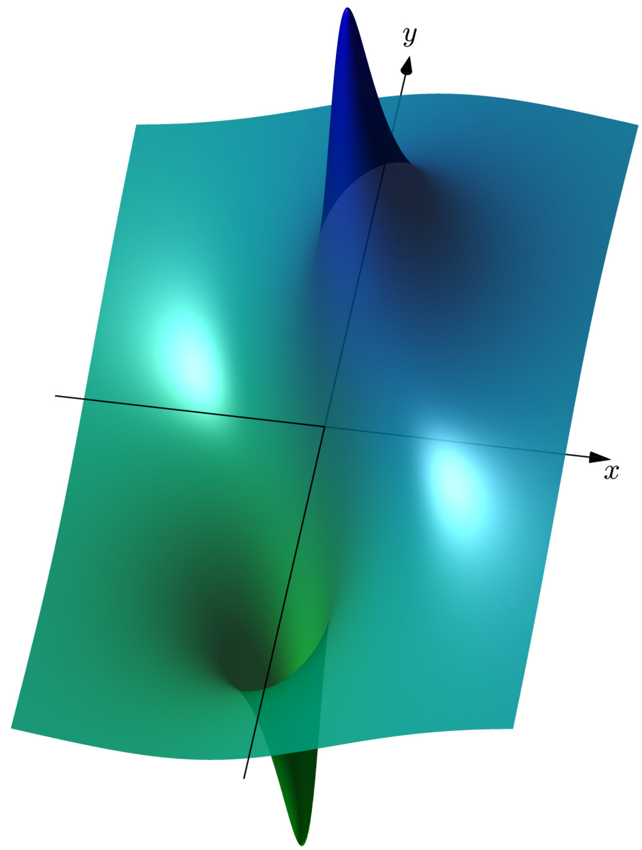
   where we have used the identity $\sin 2t = 2 \sin t \cos t$. From the properties of sine, we infer that the minimum of g is obtained at $2t = \frac{\pi}{2} + 2\pi n$ for integer n, and the maximum of g is obtained at $2t = \frac{3\pi}{2} + 2\pi n$ for integer n. In terms of t, the minimum is at $t = \frac{\pi}{4} + \pi n$, which means that there are two minimizers corresponding to $t_1 = \frac{\pi}{4}$ and to $t_2 = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$. Similarly, the maximum is obtained at $t = \frac{3\pi}{4} + \pi n$, giving two maximizers corresponding to $t_3 = \frac{3\pi}{4}$ and to $t_4 = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$. The values of f at these points are

$$g(t_1) = g(t_2) = 133 - 275 = -142, \qquad \text{and} \qquad g(t_3) = g(t_4) = 133 + 275 = 408. \tag{36}$$

   A comparison of the preceding values with (33) makes it clear that the maximum value of f in $\bar{D}$ is 408, and the minimum value is -142.



(a) The graph and a colour density plot of the function $f(x,y)$ from Problem 4 over the disk $\bar{D} = \{x^2 + y^2 \le 25\}$.



(b) The graph of the function $f(x,y)$ from Problem 5, over a square centred at the origin.

Figure 2: Illustrations for Problem 4 and Problem 5.

5. Find the maximum and minimum values of

$$f(x,y) = \frac{x+y}{2+x^2+y^2}.$$

Solution: The critical points of f must satisfy

$$\begin{aligned} \frac{\partial}{\partial x} f(x,y) &= \frac{2+x^2+y^2-(x+y)\cdot 2x}{(2+x^2+y^2)^2} = \frac{2-x^2+y^2-2xy}{(2+x^2+y^2)^2} = 0, \\ \frac{\partial}{\partial y} f(x,y) &= \frac{2+x^2+y^2-(x+y)\cdot 2y}{(2+x^2+y^2)^2} = \frac{2+x^2-y^2-2xy}{(2+x^2+y^2)^2} = 0, \end{aligned} \tag{37}$$

which imply the equations

$$\begin{aligned} 2-x^2+y^2-2xy &= 0, \\ 2+x^2-y^2-2xy &= 0. \end{aligned} \tag{38}$$

By adding and subtracting one of the equations from the other, we arrive at

$$xy = 1, \qquad x^2 - y^2 = 0, \tag{39}$$

whose only solutions are

$$(x_1, y_1) = (1, 1), \qquad \text{and} \qquad (x_2, y_2) = (-1, -1). \tag{40}$$

For the values of f at the critical points, we have

$$f(x_1, y_1) = \frac{1}{2}, \qquad \text{and} \qquad f(x_2, y_2) = -\frac{1}{2}. \tag{41}$$

The question is: Are they the maximum and minimum values of f? To answer this question, we first try to show that $f(x, y)$ is close to 0 when the point (x, y) is far away from the origin. Given (x, y), let $r = \sqrt{x^2 + y^2}$ be the distance from (x, y) to the origin $(0, 0)$. Then we have $|x| \leq r$ and $|y| \leq r$, and so

$$|f(x, y)| = \frac{|x + y|}{2 + x^2 + y^2} \leq \frac{|x| + |y|}{2 + x^2 + y^2} \leq \frac{2r}{2 + r^2}. \tag{42}$$

Moreover, if (x, y) is *outside* the open disk $D_R = \{(x, y) : x^2 + y^2 < R^2\}$ of radius $R > 0$, that is, if $r \geq R$, then we have

$$|f(x, y)| \leq \frac{2r}{2 + r^2} \leq \frac{2r}{r^2} = \frac{2}{r} \leq \frac{2}{R}. \tag{43}$$

In particular, fixing $R = 8$, we get

$$|f(x, y)| \leq \frac{1}{4} \qquad \text{for} \quad x^2 + y^2 \geq R^2. \tag{44}$$

Since both points (x_1, y_2) are (x_2, y_2) are in the disk D_R, we conclude that any possible maximizers and minimizers must be contained in the disk D_R. Now we apply the Weierstrass existence theorem in the *closed* disk $\bar{D}_R = \{(x, y) : x^2 + y^2 \leq R^2\}$, to infer that there exist a maximizer and a minimizer of f over the closed disk $\bar{D}_R$. By (44), neither a maximizer nor a minimizer can be on the boundary of $\bar{D}_R$, so they must be in the open disk D_R. This means that any maximizer must be a critical point of f in D_R, and comparing the values (41), we infer that there is only one maximizer of f over $\bar{D}_R$, and the maximizer is the point (x_1, y_1). Similarly, there is only one minimizer of f over $\bar{D}_R$, and the minimizer is the point (x_2, y_2). Note that at this point all we know is that (x_1, y_1) is the maximizer of f over $\bar{D}_R$ and that (x_2, y_2) is the minimizer of f over $\bar{D}_R$. However, invoking (44) once again, we conclude that (x_1, y_1) is indeed the maximizer of f over $\mathbb{R}^2$ and that (x_2, y_2) is indeed the minimizer of f over $\mathbb{R}^2$. The final answer is that the maximum value of f in $\mathbb{R}^2$ is $\frac{1}{2}$, and the minimum value is $-\frac{1}{2}$.

6. Find the most economical dimensions of a closed rectangular box of volume 3 cubic units if the cost of the material per square unit for (i) the top and bottom is 2, (ii) the front and back is 2 and (iii) the other two sides is 8.

Solution: Let us denote the width of the box by x, the height by z, and the depth by y. Then the combined area of the top and bottom faces is $2xy$, the area of the front and back faces is $2xz$, and the area of the other two sides is $2yz$. Thus the problem is to find the minimizer of

$$F(x, y, z) = 4xy + 4xz + 16yz, \qquad \text{subject to} \quad xyz = 3. \tag{45}$$

We look for the solution satisfying $x > 0$, $y > 0$, and $z > 0$, because if one of x, y, and z is 0, the volume of the box cannot be equal to 3. Then by using the volume constraint $xyz = 3$, we can express z in terms of x and y, resulting in the reformulation of the problem as minimizing

$$f(x, y) = 4xy + \frac{12}{y} + \frac{48}{x}, \tag{46}$$

over the quadrant $H = \{(x, y) : x > 0, y > 0\}$. First, let us find the critical points of f. The relevant equations are

$$\begin{aligned} \frac{\partial}{\partial x} f(x, y) = 4y - \frac{48}{x^2} = 0, \\ \frac{\partial}{\partial y} f(x, y) = 4x - \frac{12}{y^2} = 0, \end{aligned} \tag{47}$$

which lead us to $x^2 y = 12$ and $xy^2 = 3$. If we divide one equation by the other, we get $x = 4y$, and this in turn yields that

$$(x^*, y^*) = (\sqrt[3]{48}, \sqrt[3]{\tfrac{3}{4}}), \tag{48}$$

is the only critical point of f over H. Note that the corresponding z-value is $z^* = y^* = \sqrt[3]{48}$. (The variables y and z play indistinguishable roles in the original problem, so for quick calculations, we could have set $y = z$ from the beginning and could have transformed the whole problem into a single variable minimization problem.)

The question is now if (x^*, y^*) is indeed a minimizer of f over H. Intuitively, from (46) it is clear that $f(x, y)$ tends to ∞ if $x > 0$ and $y > 0$ are small, or if they are large. To make it precise, given $R > 0$, let

$$Q_R = \{(x, y) : \frac{1}{R} < x < R^2, \frac{1}{R} < y < R^2\}. \tag{49}$$

We want to show that if the point (x, y) is outside the square Q_R with $R > 0$ large, then $f(x, y)$ is large. Suppose that $x \leq \frac{1}{R}$ (Figure 3(b), blue region). Then we have

$$f(x, y) = 4xy + \frac{12}{y} + \frac{48}{x} \geq \frac{48}{x} \geq 48R. \tag{50}$$

Similarly, for $y \leq \frac{1}{R}$ (Figure 3(b), green region plus part of the blue region), we have

$$f(x, y) = 4xy + \frac{12}{y} + \frac{48}{x} \geq \frac{12}{y} \geq 12R. \tag{51}$$

Now suppose that $x > \frac{1}{R}$ and $y \geq R^2$ (Figure 3(b), red region). Then we have

$$f(x, y) = 4xy + \frac{12}{y} + \frac{48}{x} \geq 4xy \geq 4 \cdot \frac{1}{R} \cdot R^2 = 4R. \tag{52}$$

Finally, for $y > \frac{1}{R}$ and $x \geq R^2$ (Figure 3(b), yellow region plus part of the red region), we have

$$f(x, y) = 4xy + \frac{12}{y} + \frac{48}{x} \geq 4xy \geq 4 \cdot R^2 \cdot \frac{1}{R} = 4R, \tag{53}$$

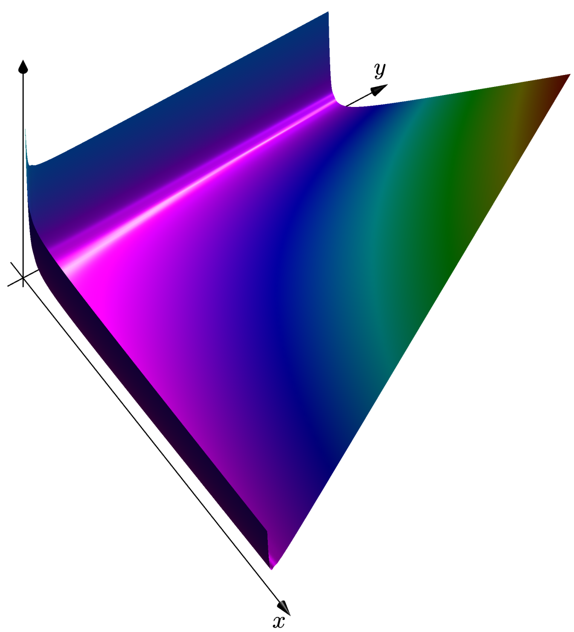
and a combination of the last four formulas gives

$$f(x,y) \geq 4R, \qquad \text{for} \quad (x,y) \notin Q_R. \tag{54}$$

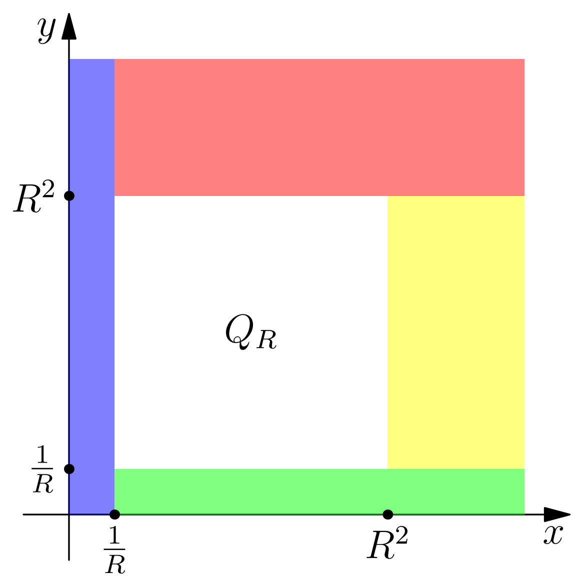
Therefore, by choosing $R > 0$ sufficiently large, we can ensure that $f(x,y) > f(x^*, y^*)$ for all (x,y) outside Q_R, meaning that any minimizer of f over H must be contained in Q_R. Let us fix such a value for R. Then as usual, the Weierstrass existence theorem guarantees the existence of a minimizer of f over the closed set $\bar{Q}_R$, where

$$\bar{Q}_R = \{(x,y) : \frac{1}{R} \leq x \leq R^2, \, \frac{1}{R} \leq y \leq R^2\}. \tag{55}$$

We have chosen $R > 0$ so large that $f(x,y) > f(x^*, y^*)$ for all (x,y) outside Q_R, which rules out the possibility that a minimizer over $\bar{Q}_R$ is on the boundary of $\bar{Q}_R$. Hence all minimizers are in Q_R, and at least one such minimizer exists. Since Q_R is open, all minimizers must be critical points, but we have only one critical point, thus we infer that (x^*, y^*) is the only minimizer of f over $\bar{Q}_R$. Finally, recalling that $f(x,y) > f(x^*, y^*)$ for all (x,y) outside Q_R, we conclude that (x^*, y^*) is the only minimizer of f over H.



(a) The function $f(x,y)$ from Problem 6.



(b) Various regions used in the solution.

Figure 3: Illustrations for Problem 6.