CRITICAL POINTS

TSOGTGEREL GANTUMUR

Abstract. Critical points and minimization problems.

Contents

1. Introduction

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Then we know from the level surface theorem that near any point $x \in \mathbb{R}^n$ with $Du(x) \neq 0$, the level sets of u are manifolds. Hence in a certain sense, "interesting things" happen only at or near the *critical points*, that is, the points x_* such that $Du(x_*) = 0$. In fact, in most situations, understanding a function can be identified with locating its critical points and studying its behaviour near the critical points. Critical points also arise in optimization problems, where one is tasked to investigate maximum and minimum values of a function.

Definition 1.1. Let $K \subset \mathbb{R}^n$, and let $u : K \to \mathbb{R}$. We say that u has a *local maximum* at $y \in K$ if there exists $\delta > 0$ such that $u(x) \leq u(y)$ for all $x \in Q_\delta(y) \cap K$. A *local strict maximum* at y is defined by the condition $u(x) < u(y)$ for all $x \in Q_\delta(y) \cap K \setminus \{y\}$. The notions of *local minimum* and *local strict minimum* may be defined similarly.

If u is defined in some larger set $U \supset K$, and we are only interested in the restriction of u to K, then we indicate it by expressions such as local maximum *with respect to* K, and local maximum *over* K. The following result is called the *first derivative test*, or *Fermat's theorem*.

Lemma 1.2. *Let $U \subset \mathbb{R}^n$ be open, and let $M \subset U$ be a manifold. Suppose that $u : U \to \mathbb{R}$ is differentiable. If u has a local maximum over M at $y \in M$, then we have*

$$D_V u(y) = 0 \qquad \text{for all} \quad V \in T_y M. \tag{1}$$

Proof. Let $V \in T_y M$, and let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a differentiable curve satisfying $\gamma(0) = y$ and $\gamma'(0) = V$. Put $g(t) = u(\gamma(t))$, and note that $g'(0) = D_V u(y)$. Anticipating a contradiction, suppose that $g'(0) > 0$. By definition, we have $g(t) = g(0) + h(t)t$ with $h(t) \to g'(0)$ as $t \to 0$. Hence by continuity, there exists $t > 0$ arbitrarily small, such that $h(t) > \frac{1}{2}g'(0) > 0$. This gives $g(t) > g(0) + \frac{1}{2}g'(0)t$, meaning that $g(0)$ cannot be a local maximum. The case $g'(0) < 0$ can be treated similarly, by considering values $g(t)$ with small $t < 0$. □

Definition 1.3. Let $U \subset \mathbb{R}^n$ be open, and let $M \subset U$ be a manifold. Suppose that $u : U \to \mathbb{R}$ is differentiable. If $D_V u(x) = 0$ for all $V \in T_x M$, then $x \in M$ is called a *critical point* of u over M, and the value $u(x) \in \mathbb{R}$ is called a *critical value* of u over M.

Date: 2016/12/15 18:15 (EST). Incomplete draft.

Remark 1.4. Let $\Psi : \Omega \to M$ be a local parameterization of M, and let $x = \Psi(\xi)$ for some $\xi \in \Omega$. Since $D_V u(x) = Du(x)V$ and the columns of $D\Psi(\xi)$ form a basis of $T_x M$, we infer that x is a critical point of u over M if and only if $Du(x)D\Psi(\xi) = 0$. Furthermore, noting that $Dv(\xi) = Du(x)D\Psi(\xi)$ for $v = u \circ \Psi$, we conclude that x is a critical point of u over M if and only if ξ is a critical point of v in Ω.

2. Conormals and Lagrange multipliers

Definition 2.1. Given a manifold $M \subset \mathbb{R}^n$ and its point $p \in M$, the *conormal space* of M at p is defined as

$$N_p^* M = \{\alpha \in \mathbb{R}^{n*} : \alpha V = 0 \text{ for all } V \in T_p M\}, \tag{2}$$

where we recall that $\mathbb{R}^{n*} = \mathbb{R}^{1\times n}$ is the space of n-dimensional row vectors.

Let $\Psi : \Omega \to M$ be a local parameterization of M, and let $p = \Psi(q)$, $q \in \Omega$. Then $\alpha \in N_p^* M$ if and only if $\alpha D\Psi(q) = 0$, that is, $\alpha \in \operatorname{coker} D\Psi(q)$. So we have $N_p^* M = \operatorname{coker} D\Psi(q)$.

Now let $U \subset \mathbb{R}^n$ be open, and suppose that $M \cap U$ is described by the equation $\phi(x) = 0$, where $\phi : U \to \mathbb{R}^k$ is a continuously differentiable function with $D\phi(x)$ surjective for all $x \in M \cap U$. We know that $T_p M = \ker D\phi(p)$. For any $\beta \in \mathbb{R}^{k*}$, we have $\beta D\phi(p) D\Psi(q) = 0$, and hence $\beta D\phi(p) \in N_p^* M$. In other words, $\operatorname{coran} D\phi(p) \subset N_p^* M$.

By assumption, the column rank of $D\phi(p)$ is k, and so the row rank must also be k, that is, $\dim(\operatorname{coran} D\phi(p)) = k$. On the other hand, the column rank of $D\Psi(q)$ is $m = n - k$, and so $\dim(\operatorname{coran} D\Psi(q)) = m$. Invoking the rank-nullity theorem, the dimension of $\operatorname{coker} D\Psi(q)$ is $n - m = k$. Since we have the inclusion $\operatorname{coran} D\phi(p) \subset \operatorname{coker} D\Psi(q)$ with dimensions agreeing, we conclude that

$$N_p^* M = \operatorname{coran} D\phi(p) = \operatorname{coker} D\Psi(q). \tag{3}$$

The following result is now straightforward.

Theorem 2.2 (Lagrange multipliers)**.** *Let $U \subset \mathbb{R}^n$ be open, and let $M \subset U$ be a manifold. Suppose that $u : U \to \mathbb{R}$ is differentiable. Then $p \in M$ is a critical point of u over M if and only if $Du(p) \in N_p^* M$. In addition, suppose that M is described by the equation $\phi(x) = 0$, where $\phi : U \to \mathbb{R}^k$ is a continuously differentiable function with $D\phi(x)$ surjective for all $x \in M$. Then $Du(p) \in N_p^* M$ if and only if there exists $\lambda \in \mathbb{R}^{k*}$ such that*

$$Du(p) = \lambda D\phi(p). \tag{4}$$

Proof. By definition, p is a critical point if and only if $Du(p)V = 0$ for all $V \in T_p M$. The latter condition is equivalent to $Du(p) \in N_p^* M$. By (3), we have $Du(p) \in N_p^* M$ if and only if $Du(p)$ is a linear combination of the rows of $D\phi(p)$. $\square$