Solutions to selected problems from the midterm exam

Math 222 Winter 2015

1. Derive the Maclaurin series for the following functions. (cf. Practice Problem 4)

(a) $L(x) = \int_0^x \frac{\log(1+t)}{t}\,\mathrm{d}t.$

Solution: We have the Maclaurin series

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \ldots, \tag{1}$$

and so

$$\frac{\log(1+t)}{t} = 1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \ldots. \tag{2}$$

Then a termwise integration yields

$$L(x) = \int_0^x \frac{\log(1+t)}{t}\,\mathrm{d}t = x - \frac{x^2}{2\cdot 2} + \frac{x^3}{3\cdot 3} - \frac{x^4}{4\cdot 4} + \ldots. \tag{3}$$

(b) $C(x) = \int_0^x \frac{1-\cos t}{t^2}\,\mathrm{d}t.$

Solution: We have the Maclaurin series

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots, \tag{4}$$

and so

$$\frac{1-\cos t}{t^2} = \frac{1}{2!} - \frac{t^2}{4!} + \frac{t^4}{6!} - \frac{t^6}{8!} + \ldots. \tag{5}$$

Then a termwise integration yields

$$C(x) = \int_0^x \frac{1-\cos t}{t^2}\,\mathrm{d}t = \frac{x}{2!} - \frac{x^3}{3\cdot 4!} + \frac{t^5}{5\cdot 6!} - \frac{t^7}{7\cdot 8!} + \ldots. \tag{6}$$

(c) $S(x) = \int_0^x \frac{e^t - e^{-t}}{t}\,\mathrm{d}t.$

Solution: We have the two Maclaurin series

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \ldots, \tag{7}$$

and

$$e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \ldots, \tag{8}$$

which imply that

$$\frac{e^t - e^{-t}}{t} = 2 + \frac{2t^2}{3!} + \frac{2t^4}{5!} + \frac{2t^6}{7!} + \ldots. \tag{9}$$

Then a termwise integration yields

$$S(x) = \int_0^x \frac{e^t - e^{-t}}{t}\,\mathrm{d}t = 2x + \frac{2x^3}{3\cdot 3!} + \frac{2x^5}{5\cdot 5!} + \frac{2x^7}{7\cdot 7!} + \ldots. \tag{10}$$

2. Decide if the following series converge. (To be compared with Practice Problem 5)

   (a) $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \log \frac{e^{\frac{1}{n}} + e^{-\frac{1}{n}}}{2}$.

   Solution: For n large, we have

$$e^{\frac{1}{n}} \approx 1 + \frac{1}{n} + \frac{1}{2n^2}, \qquad \text{and} \qquad e^{-\frac{1}{n}} \approx 1 - \frac{1}{n} + \frac{1}{2n^2}, \tag{11}$$

   which imply that

$$\log \frac{e^{\frac{1}{n}} + e^{-\frac{1}{n}}}{2} \approx \log(1 + \frac{1}{2n^2}) \approx \frac{1}{2n^2}, \tag{12}$$

   where in the last step we have used the first term of the Maclaurin series $\log(1+x) = x - \frac{x^2}{2} + \ldots$. This suggests that the original series $\sum a_n$ converges, and that it should be comparable to the series $\sum n^{-2}$. Then the limit comparison test gives us the limit

$$\lim_{n\to\infty} \frac{a_n}{n^{-2}} = \lim_{n\to\infty} n^2 \log \frac{e^{\frac{1}{n}} + e^{-\frac{1}{n}}}{2}. \tag{13}$$

   In order to compute this limit, we replace it by

$$\lim_{x\to 0} \frac{\log \frac{e^x + e^{-x}}{2}}{x^2} = \lim_{x\to 0} \frac{\log(e^x + e^{-x}) - \log 2}{x^2}, \tag{14}$$

   and manipulate it as

$$\begin{aligned} \lim_{x\to 0} \frac{\log(e^x + e^{-x}) - \log 2}{x^2} &= \lim_{x\to 0} \frac{\frac{e^x - e^{-x}}{e^x + e^{-x}}}{2x} \\ &= \left(\lim_{x\to 0} \frac{1}{e^x + e^{-x}} \right) \cdot \lim_{x\to 0} \frac{e^x - e^{-x}}{2x} \\ &= \frac{1}{2} \cdot \lim_{x\to 0} \frac{e^x - e^{-x}}{2x} = \frac{1}{2} \cdot \lim_{x\to 0} \frac{e^x + e^{-x}}{2} = \frac{1}{2}, \end{aligned} \tag{15}$$

   where we have applied L'Hôpital's rule twice, once in the first step, and once in the penultimate step. The conclusion is that

$$\lim_{n\to\infty} \frac{a_n}{n^{-2}} = \frac{1}{2}, \tag{16}$$

   and hence the series $\sum a_n$ converges.

   (b) $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \log \big(n \arctan \frac{1}{n}\big)$.

   Solution: For n large, we have

$$\arctan \frac{1}{n} \approx \frac{1}{n} - \frac{1}{3n^3}, \qquad \text{and so} \qquad \log \big(n \arctan \frac{1}{n}\big) \approx \log \big(1 - \frac{1}{3n^2}\big) \approx -\frac{1}{3n^2}. \tag{17}$$

   This suggests that the original series $\sum a_n$ converges, and that it should be comparable to the series $\sum n^{-2}$. Then the limit comparison test gives us the limit

$$\lim_{n\to\infty} \frac{a_n}{n^{-2}} = \lim_{n\to\infty} n^2 \log\big(n \arctan \frac{1}{n}\big). \tag{18}$$

In order to compute this limit, we replace it by

$$\lim_{x\to0} \frac{\log \frac{\arctan x}{x}}{x^2} = \lim_{x\to0} \frac{\log f(x)}{x^2}, \qquad \text{with} \quad f(x) = \frac{\arctan x}{x}, \tag{19}$$

and apply L'Hôpital's rule twice, as

$$\lim_{x\to0} \frac{\log f(x)}{x^2} = \lim_{x\to0} \frac{\frac{f'(x)}{f(x)}}{2x} = \lim_{x\to0} \frac{\frac{f''(x)f(x)-[f'(x)]^2}{[f(x)]^2}}{2} = \lim_{x\to0} \frac{f''(x)f(x)-[f'(x)]^2}{2[f(x)]^2}. \tag{20}$$

Now, from the Maclaurin series

$$f(x) = \frac{\arctan x}{x} = \frac{x - \frac{x^3}{3} + \dots}{x} = 1 - \frac{x^2}{3} + \dots, \tag{21}$$

we have $f(0) = 1$, $f'(0) = 0$, and $f''(0) = -\frac{2}{3}$, which yields

$$\begin{aligned} \lim_{x\to0} \frac{\log f(x)}{x^2} &= \lim_{x\to0} \frac{f''(x)f(x)-[f'(x)]^2}{2[f(x)]^2} = \frac{f''(0)f(0)-[f'(0)]^2}{2[f(0)]^2} \\ &= \frac{-\frac{2}{3}\cdot 1 - 0}{2\cdot 1} = -\frac{1}{3}. \end{aligned} \tag{22}$$

In other words, we have

$$\lim_{n\to\infty} \frac{a_n}{n^{-2}} = \lim_{n\to\infty} n^2 \log\big(n \arctan \frac{1}{n}\big) = -\frac{1}{3}, \qquad \text{or} \qquad \lim_{n\to\infty} \frac{|a_n|}{n^{-2}} = \frac{1}{3}, \tag{23}$$

and so the series $\sum a_n$ converges.

3. Determine the convergence radius of the power series $\displaystyle\sum_{n=0}^{\infty} (\sqrt{n} + a^n)x^n$, where $a \geq 0$.

Solution: This should be compared with Practice Problem 7. With the n-th term of the given series denoted by $b_n = (\sqrt{n} + a^n)x^n$, an application of the ratio test leads to

$$L = \lim_{n\to\infty} \frac{|b_{n+1|}}{|b_n|} = \lim_{n\to\infty} \frac{(\sqrt{n+1} + a^{n+1})|x|^{n+1}}{(\sqrt{n} + a^n)|x|^n} = |x| \cdot \lim_{n\to\infty} \frac{\sqrt{n+1} + a^{n+1}}{\sqrt{n} + a^n}. \tag{24}$$

In order to compute the limit, we need to have some idea on how fast the individual terms in $\sqrt{n} + a^n$ grows with as $n \to \infty$. In particular, note that a^n grows faster than $\sqrt{n}$ if $a > 1$, and that a^n does not grow at all if $a \leq 1$. Therefore we split the problem into two cases. First, assume that $a > 1$. Then we have

$$\lim_{n\to\infty} \frac{\sqrt{n+1} + a^{n+1}}{\sqrt{n} + a^n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{a^n} + \frac{a^{n+1}}{a^n}}{\frac{\sqrt{n}}{a^n} + \frac{a^n}{a^n}} = \frac{0 + a}{0 + 1} = a. \tag{25}$$

Now assume that $0 \le a \le 1$. In this case, we have

$$\lim_{n\to\infty} \frac{\sqrt{n+1}+a^{n+1}}{\sqrt{n}+a^n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{\sqrt{n}} + \frac{a^{n+1}}{\sqrt{n}}}{\frac{\sqrt{n}}{\sqrt{n}} + \frac{a^n}{\sqrt{n}}} = \frac{1+0}{1+0} = 1. \tag{26}$$

Based on these computations, we conclude that

$$L = \lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = \begin{cases} a|x| & \text{if } a > 1, \\ |x| & \text{if } 0 \le a \le 1. \end{cases} \tag{27}$$

In the first case, we have convergence for $a|x| < 1$ and divergence for $a|x| > 1$, hence the convergence radius is $R = \frac{1}{a}$. In the second case, we have convergence for $|x| < 1$ and divergence for $|x| > 1$, hence the convergence radius is $R = 1$. We can also write it in a single formula, as

$$R = \min\{1, \frac{1}{a}\}. \tag{28}$$

4. Compute the unit tangent $T(t)$, the principal unit normal $N(t)$, and the curvature $\kappa(t)$, for each of the following plane curves.
   (a) $X(t) = (t - \sin t, 1 - \cos t), \qquad 0 < t < 2\pi.$
   Solution: By a direct computation, we have

$$X'(t) = (1 - \cos t, \sin t) \tag{29}$$

and

$$|X'(t)|^2 = (1-\cos t)^2 + \sin^2 t = 1 + \cos^2 t - 2\cos t + \sin^2 t = 2 - 2\cos t, \tag{30}$$

which yield

$$T(t) = \frac{X'(t)}{|X'(t)|} = \Big(\frac{1-\cos t}{\sqrt{2-2\cos t}}, \frac{\sin t}{\sqrt{2-2\cos t}}\Big) = \Big(\frac{\sqrt{1-\cos t}}{\sqrt{2}}, \frac{\sin t}{\sqrt{2-2\cos t}}\Big). \tag{31}$$

Going further, we compute

$$X''(t) = (\sin t, \cos t), \tag{32}$$

and

$$X'(t) \times X''(t) = \begin{vmatrix} 1-\cos t & \sin t \\ \sin t & \cos t \end{vmatrix} = (1-\cos t)\cos t - \sin^2 t = \cos t - 1, \tag{33}$$

leading to

$$\kappa(t) = \frac{|X'(t) \times X''(t)|}{|X'(t)|^3} = \frac{1-\cos t}{(2-2\cos t)^{\frac{3}{2}}} = \frac{1}{2\sqrt{2-2\cos t}}, \tag{34}$$

where we have taken into account the fact that $\cos t < 1$ for $0 < t < 2\pi$, and thus $|\cos t - 1| = 1 - \cos t$. The latter fact also implies that $X'(t) \times X''(t) = \cos t - 1 < 0$, so that the curve is "bending to the right", as the minimal turn from X' to X'' is clockwise. Hence the principal normal $N(t)$ is equal to $T(t)$, rotated 90 degree clockwise:

$$N(t) = \Big(\frac{\sin t}{\sqrt{2-2\cos t}}, -\frac{\sqrt{1-\cos t}}{\sqrt{2}}\Big) \tag{35}$$

Note: By using the double-angle formula $\cos t = \cos^2(\frac{t}{2}) - \sin^2(\frac{t}{2})$, we get

$$1 - \cos t = 1 - \cos^2(\tfrac{t}{2}) + \sin^2(\tfrac{t}{2}) = \sin^2(\tfrac{t}{2}) + \cos^2(\tfrac{t}{2}) - \cos^2(\tfrac{t}{2}) + \sin^2(\tfrac{t}{2}) = 2\sin^2(\tfrac{t}{2}). \quad (36)$$

This gives simplifications to many of the formulas above. For example, we have

$$T(t) = \frac{X'(t)}{|X'(t)|} = \big(\sin(\tfrac{t}{2}), \cos(\tfrac{t}{2})\big), \quad (37)$$

where we have also used $\sin t = 2\sin(\frac{t}{2})\cos(\frac{t}{2})$, and

$$\kappa(t) = \frac{1}{4\sin(\frac{t}{2})}. \quad (38)$$

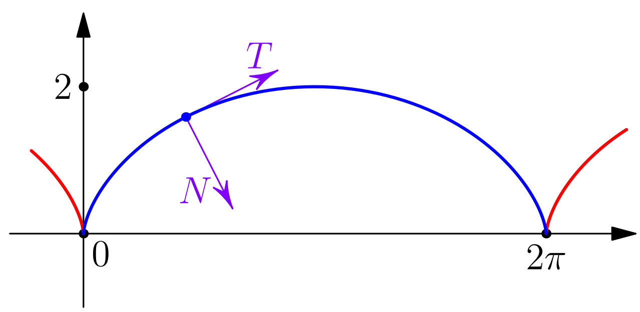


Figure 1: Cycloid, given by $X(t) = (t - \sin t, 1 - \cos t)$. The blue part corresponds to the parameter values $0 < t < 2\pi$.

(b) $X(t) = (\cos^3 t, \sin^3 t), \qquad 0 < t < \frac{\pi}{2}$.
Solution: A direct computation yields

$$X'(t) = (-3\cos^2 t \sin t, 3\sin^2 t \cos t), \quad (39)$$

and

$$|X'(t)|^2 = 9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t = 9\sin^2 t \cos^2 t(\cos^2 t + \sin^2 t) = 9\sin^2 t \cos^2 t. \quad (40)$$

Since both $\sin t$ and $\cos t$ are positive for $0 < t < \frac{\pi}{2}$, we have

$$|X'(t)| = 3|\sin t \cos t| = 3\sin t \cos t, \quad (41)$$

and therefore

$$T(t) = \frac{X'(t)}{|X'(t)|} = \Big(-\frac{3\cos^2 t \sin t}{3\sin t \cos t}, \frac{3\sin^2 t \cos t}{3\sin t \cos t}\Big) = (-\cos t, \sin t). \quad (42)$$

Furthermore, we have

$$X''(t) = (6\cos t \sin^2 t - 3\cos^3 t, 6\sin t \cos^2 t - 3\sin^3 t), \quad (43)$$

and

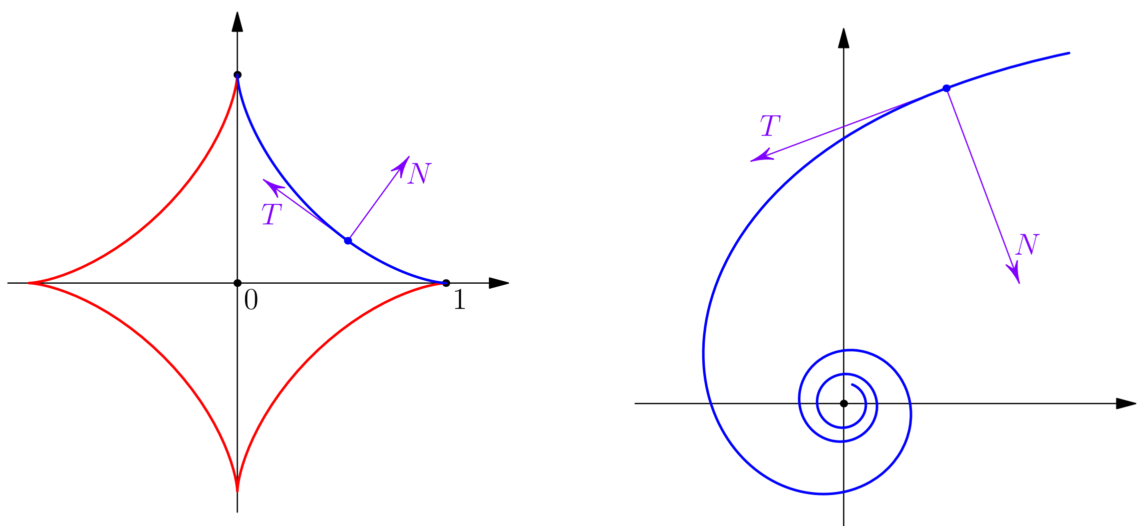
$$
\begin{aligned}
X'(t) \times X''(t) &= \begin{vmatrix} -3\cos^2 t \sin t & 3\sin^2 t \cos t \\ 6\cos t \sin^2 t - 3\cos^3 t & 6 \sin t \cos^2 t - 3\sin^3 t \end{vmatrix} \\
&= -18\cos^4 t \sin^2 t + 9\cos^2 t \sin^4 t - 18 \sin^4 t \cos^2 t + 9 \sin^2 t \cos^4 t \\
&= -9\sin^4 t \cos^2 t - 9 \sin^2 t \cos^4 t = -9 \sin^2 t \cos^2 t.
\end{aligned} \tag{44}
$$

Because $X'(t) \times X''(t) < 0$, we see that the curve is "bending to the right," which means that the principal normal $N(t)$ is equal to $T(t)$, rotated 90 degree clockwise:

$$N(t) = (\sin t, \cos t). \tag{45}$$

Finally, let us compute the curvature, as

$$\kappa(t) = \frac{|X'(t) \times X''(t)|}{|X'(t)|^3} = \frac{9\sin^2 t \cos^2 t}{27 \sin^3 t \cos^3 t} = \frac{1}{3 \sin t \cos t}. \tag{46}$$



(a) Astroid, given by $X(t) = (\cos^3 t, \sin^3 t)$. The blue part corresponds to the parameter values $0 < t < \frac{\pi}{2}$.

(b) Hyperbolic spiral, given by $X(t) = \big(\frac{\cos t}{t}, \frac{\sin t}{t}\big)$, $t > 0$.

Figure 2: Curves from Question 4(b) and Question 4(c).

(c) $X(t) = \big(\dfrac{\cos t}{t}, \dfrac{\sin t}{t}\big), \qquad t > 0.$

Solution: Let us start with

$$X'(t) = \Big(\frac{-t\sin t - \cos t}{t^2}, \frac{t \cos t - \sin t}{t^2}\Big) \tag{47}$$

and

$$
\begin{aligned}
&(-t\sin t - \cos t)^2 + (t\cos t - \sin t)^2 \\
&\qquad = t^2 \sin^2 t + \cos^2 t + 2t \sin t \cos t + t^2 \cos^2 t + \sin^2 t - 2t \sin t \cos t \\
&\qquad = t^2 + 1,
\end{aligned} \tag{48}
$$

which imply that

$$|X'(t)| = \frac{\sqrt{t^2+1}}{t^2}, \tag{49}$$

and

$$T(t) = \frac{X'(t)}{|X'(t)|} = \frac{1}{\sqrt{t^2+1}}(-t\sin t - \cos t, t\cos t - \sin t). \tag{50}$$

Furthermore, we have

$$\begin{aligned} X''(t) &= \Big(\frac{t^2(-t\cos t) - 2t(-t\sin t - \cos t)}{t^4}, \frac{t^2(-t\sin t) - 2t(t\cos t - \sin t)}{t^4}\Big) \\ &= \Big(\frac{-t^3\cos t + 2t^2\sin t + 2t\cos t}{t^4}, \frac{-t^3\sin t - 2t^2\cos t + 2t\sin t}{t^4}\Big). \end{aligned} \tag{51}$$

From the preliminary computation

$$\begin{aligned} &\begin{vmatrix} -t\sin t - \cos t & t\cos t - \sin t \\ -t^3\cos t + 2t^2\sin t + 2t\cos t & -t^3\sin t - 2t^2\cos t + 2t\sin t \end{vmatrix} \\ &= (-t\sin t - \cos t)(-t^3\sin t - 2t^2\cos t + 2t\sin t) - (t\cos t - \sin t)(-t^3\cos t + 2t^2\sin t + 2t\cos t) \\ &= t^4 - 2t^2 + 2t^2 = t^4, \end{aligned} \tag{52}$$

we infer that

$$X'(t) \times X''(t) = \frac{t^4}{t^2 \cdot t^4} = \frac{1}{t^2}, \tag{53}$$

which in its turn gives

$$\kappa(t) = \frac{|X'(t) \times X''(t)|}{|X'(t)|^3} = \frac{1}{t^2} \cdot \frac{t^6}{(t^2+1)^{\frac{3}{2}}} = \frac{t^4}{(t^2+1)^{\frac{3}{2}}}. \tag{54}$$

Finally, since $X'(t) \times X''(t) > 0$, the curve is "bending to the left", as the minimal turn from X' to X'' is counter-clockwise, and hence the principal normal $N(t)$ is equal to $T(t)$, rotated 90 degree counter-clockwise:

$$N(t) = \frac{1}{\sqrt{t^2+1}}(-t\cos t + \sin t, -t\sin t - \cos t). \tag{55}$$