MATH 222 WINTER 2015 MIDTERM EXAM

FEEDBACK TO THE STUDENT

The average grade of the class at the midterm exam was 12 out of 20 points. One can see that if we do not count a few students who wrote random things on the exam paper, the grades are evenly distributed roughly between 8 and 20. If your grade is less than satisfactory, do not worry; There will be ways to lessen its effects on your final grade, and I hope that you will use them as a motivation to study.

In the submitted exam papers, there were some errors that occur frequently, and some errors that are not so common but quite serious. I would like to point those out here and give suggestions on how they can be rectified.

Explanations. The first point to be mentioned is that many students simply write formulas without any explanation whatsoever. Make sure you do not leave any doubt in the grader's mind as to whether you know the particular topic or technique under discussion. Please use *full sentences* to explain what you are doing. Ideally, there should not be any formula that is not part of a sentence. For an example of how you can do this, have a look at the midterm solutions posted on the course webpage.

Another issue is that some students write wherever there is a white space, regardless of whether it is in the middle of what they have already written. Since there is no way of knowing the order in which different blobs were written, the whole thing becomes incomprehensible to the reader. Some people draw a system of arrows to "guide" the reader, but this maze makes it even more incomprehensible. Note that *a mathematical text is still a text*, it is not graphics. Please write in a linear fashion, as the reader's time runs linearly.

Careless errors. A few people made a mistake when copying down the problem, or when copying an expression from one line to the next. Several students "expanded" $e^{\frac{1}{n}}$ as

$$e^{\frac{1}{n}} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{1}{n}\right)^n,$$

by confusing n in the exponent of $e^{\frac{1}{n}}$ with the summation index n in the Maclaurin series. A correct way to expand it would be

$$e^{\frac{1}{n}} = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \left(\frac{1}{n}\right)^k.$$

One student wrote

$$e^x = \sum_{n=0}^{\infty} (1 + x + \frac{x^2}{2!} + \ldots),$$

and then confused themselves after a few steps. Note that one should write

$$\text{either} \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \text{or} \qquad e^x = 1 + x + \frac{x^2}{2!} + \ldots.$$

These are errors that can easily be avoided with a bit of focus.

Date: March 15, 2015.

Elementary math. The following *false* identities were used by some students:

$$\log(na) = (\log a)^n,$$

$$\log \frac{a}{b} = \frac{\log a}{\log b},$$

$$1 - \cos x = \sin x,$$

One student wrote

$$\lim_{x\to 0} e^x = 0,$$

which is incorrect. If you made one of these errors, please review the basic properties of logarithms, trigonometric functions, and the exponential.

Integration and differentiation. A few students made a sign error when differentiating $\sin x$ or $\cos x$. You can easily figure out these signs if you remember how the graphs of $\sin x$ and $\cos x$ look like, or if you know how these functions behave near $x = 0$. For example, $\cos x$ is decreasing for $x > 0$ small, so $(\cos x)'$ must be negative there, and since $\sin x$ is positive for $x > 0$ small, we deduce that $(\cos x)' = -\sin x$.

One student wrote that the derivative of $\frac{1}{x}$ is $\log x$. Probably it was a careless error. If it wasn't, note that $(\log x)' = \frac{1}{x}$ and $(x^a)' = ax^{a-1}$ for all real numbers a. In particular, putting $a = -1$ into the latter formula gives the derivative of $\frac{1}{x}$.

Many students do not seem to know the product and chain rules for differentiation. For example, "derivations" such as

$$(t^2 \sin t)' = 2t \sin t, \qquad \text{and} \qquad (\cos^3 t)' = -\sin^3 t,$$

were not so uncommon. I consider such mistakes very serious since product and chain rules are at the heart of calculus. Please review the basic rules of differentiation.

The following "computation" has also occurred:

$$\int \frac{\log(1+t)}{t}\,\mathrm{d}t = \int \frac{1}{t} \cdot \log(1+t)\,\mathrm{d}t = \int \frac{1}{t}\,\mathrm{d}t \cdot \int \log(1+t)\,\mathrm{d}t.$$

The n-th term test is not a convergence test! It is a divergence test! This point cannot be emphasized enough. A lot of people simply concluded that the series $\sum a_n$ converges, because $\lim a_n = 0$. This shows that they do not understand the basics of series and hence it is a very serious mistake. Remember the harmonic series example: We have $\lim \frac{1}{n} = 0$, yet the series $\sum \frac{1}{n}$ diverges.

Convergence radius cannot depend on n. Given a power series, such as $\sum r^n x^n$, its convergence radius is a number, possibly depending on the parameter r. However, the convergence radius *cannot depend on the index n*. If you happen to have, say $R = n^2$, then you must have made a mistake somewhere.

The ratio test involves a limit. In the same vein, to apply the ratio test to the series $\sum a_n$, one *must take the limit* of $\frac{|a_{n+1}|}{|a_n|}$ as $n \to \infty$. It does *not* tell you anything if $\frac{|a_{n+1}|}{|a_n|}$ is, say, less than 1 for some particular n. For example, for the series $\sum \frac{1}{n}$, we have $\frac{|a_{n+1}|}{|a_n|} = \frac{n}{n+1} < 1$ for each n, but it does not mean that the series converges.

Power series never include x^{-n} term. A power series, and in particular a Maclaurin series of a function, cannot contain a term of the form, e.g., $\frac{1}{x}$ or $\frac{1}{x^2}$. So for example,

$$\frac{\cos x}{x^2} = \frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}{x^2} = \frac{1}{x^2} - \frac{1}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots,$$

is *not* a Maclaurin series.

First expand, then integrate or differentiate. If you are asked to find a Maclaurin series of a function defined in terms of an integral, such as

$$F(x) = \int_0^x f(t)\,\mathrm{d}t,$$

it is always a good idea to expand $f(t)$ into its Maclaurin series before integrating, because integrating a power series is very easy. In most cases, $f(t)$ is a function such as $\sin t^2$, which is not integrable in terms of elementary functions, so attempting to find the integral before expanding would most certainly be futile.

When manipulating series, always check a first few terms explicitly. This rule could even be elevated to the status of a "golden rule." Let me illustrate this rule by an example. Say, we want to find the Maclaurin series of

$$S(x) = \int_0^x \frac{e^t - e^{-t}}{t}\,\mathrm{d}t.$$

Following the "first expand, then integrate" rule, we expand

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \qquad e^{-t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!},$$

and thus

$$\frac{e^t - e^{-t}}{t} = \frac{1}{t}\sum_{n=0}^{\infty} \frac{t^n}{n!} - \frac{1}{t}\sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{n-1}}{n!} - \sum_{n=0}^{\infty} (-1)^n \frac{t^{n-1}}{n!}. \qquad (*)$$

We can now integrate this expression term by term, and get some result, but we feel that we are missing something, because we are just blindly manipulating symbols. In particular, for $n = 0$, we have $t^{n-1} = t^{-1}$, which is worrisome. This is where the "golden rule" comes in: We should explicitly write down a first few terms of all the series in question, and see exactly what is going on! In fact, to avoid any "blind manipulation", we have to do it from the beginning. So we write

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \dots, \qquad \text{and} \qquad e^{-t} = 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \dots.$$

It is now clear what happens when we subtract the two series:

$$e^t - e^{-t} = 0 + 2t + 0 + 2\cdot\frac{t^3}{3!} + 0 + 2\cdot\frac{t^5}{5!} + \dots,$$

and so

$$\frac{e^t - e^{-t}}{t} = 2 + 2\cdot\frac{t^2}{3!} + 2\cdot\frac{t^4}{5!} + \dots. \qquad (**)$$

We understand now how some of the terms cancel and why there is no t^{-1} term in the end! Now we integrate it term by term and get the final answer, see the full solution.

If you still want to go back to $(*)$, and write the answer in the form of a sum, we can proceed as follows. Since now we can see through the fog of the sum symbols $\sum$, we have

$$\frac{e^t - e^{-t}}{t} = \sum_{n=0}^{\infty} \frac{t^{n-1}}{n!} - \sum_{n=0}^{\infty} (-1)^n \frac{t^{n-1}}{n!} = \sum_{n=0}^{\infty} (1 - (-1)^n) \frac{t^{n-1}}{n!} = \sum_{k=0}^{\infty} \frac{2t^{2k}}{(2k+1)!},$$

because $1 - (-1)^n = 0$ for n even, and $1 - (-1)^n = 2$ for n odd. Note that we could have easily derived this formula from $(**)$, so that a manipulation of the general index n could have been avoided until the last moment.

Radius of curvature $\frac{1}{\kappa}$ is not the same as $|X(t)|$. A number of students found the curvature κ by using the assumption that the radius of curvature $\frac{1}{\kappa}$ is the same as $|X(t)|$.

This assumption is wrong! Note that $|X(t)|$ is simply the distance from the point $X(t)$ to the origin. The radius of the osculating circle is $\frac{1}{\kappa}$, but this circle does not have to be centred at the origin!

Some more simple mistakes. A few students did not seem to know the difference between a scalar and a vector. For example, they have the *wrong* idea that the curvature is a vector, and that the unit tangent vector is a scalar (that is, a number). Moreover, some students mixed space and plane curves, and tried to compute the binormal vector for a plane curve. Note that the *plane curve* is a curve in *two* dimensions, such as $X(t) = (\sin t, t \cos t)$. A *space curve* would have *three* components, e.g., $X(t) = (t, t^2, \cos t)$.

Finally, a standard convention to write a vector in components is to use brackets, as in $(1, 2, 4)$ and $(\cos t, 1 + t^2)$. Angle brackets can also be used, as in $\langle 3, -1 \rangle$. Please *do not omit the brackets*, and avoid writing, e.g., $T(t) = \cos t, 1 + t^2$.