MATH 170C SPRING 2007 PRACTICE PROBLEMS

APRIL 29

1. A *Riemann sum* associated with an integral $\int_a^b f(x)dx$ is an approximation of the form
$$S_n = \sum_{k=0}^{n} f(t_k)(s_{k+1} - s_k),$$
where
$$a = s_0 \le t_0 \le s_1 \le t_1 \le s_2 \le \ldots \le s_n \le t_n \le s_{n+1} = b.$$
Any sequence of such sums in which the subdivision of $[a, b]$ is refined in such a way that $\max_k(s_{k+1} - s_k) \to 0$ tends to the Riemann integral I if it exists.
Show that the approximations afforded by the composite midpoint rule, the trapezoidal rule, and Simpson's rule are Riemann sums. Display the values of $s_1, s_2, \ldots, s_n$ in each case.
2. Given the following rounded values of the function
$$f(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2}$$
calculate approximate values of the integral
$$P(1) = \sqrt{\frac{2}{\pi}} \int_0^1 e^{-t^2/2} dt \approx 0.682689$$
by use of the composite trapezoid rule with $h = 1$, $\frac{1}{2}$, and $\frac{1}{4}$, and compare the results with the rounded true value. In each case, obtain an upper bound on the error analytically and verify that it is conservative.

Table 1. Rounded values of $f(x)$

x	$f(x)$
0.00	0.797885
0.25	0.773336
0.50	0.704131
0.75	0.602275
1.00	0.483941

3. At what interval in x and to how many decimal places must $f(x) = \cos x$ be tabulated in order to evaluate $\int_0^5 f(x)dx$ correctly to six decimal places by using
   (a) composite trapezoid rule,
   (b) composite Simpson's rule?

Additional note: Let Q be a real number, and let $\hat{Q}$ be its approximation. We say that $\hat{Q}$ is correct to n decimal places when $\hat{Q}$ has exactly n digits after the decimal point *and* its last digit has uncertainty ± 1 or less, i.e., when $|\hat{Q} - Q| \leq 10^{-n}$. For example, 3.15 is an approximation of π, correct to 2 decimal places.

4. Find the degree of precision of the following quadrature rule

$$\int_{-1}^{1} f(x)dx \approx \frac{2}{3}[2f(-\frac{1}{2}) - f(0) + 2f(\frac{1}{2})].$$

5. Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)) \qquad t \in [a, b], \\ y(a) = y_a, \end{cases} \tag{1}$$

and assume that $f(t, y)$ satisfies the Lipschitz condition in the variable y on $[a, b] \times (-\infty, \infty)$, so that the above initial value problem has a unique solution on $[a, b]$.

Let w_i, $i = 0, 1, \ldots, n$, be the result of Euler's method applied to (1) with stepsize $h = (b - a)/n$, and let u_i, $i = 0, 1, \ldots, 2n$, be the result of the same method applied to (1) with stepsize $h/2$. Devise a formula to extrapolate w_i and u_{2i} to get a better approximation to $y(a + hi)$, and estimate the convergence of the resulting extrapolation. *Hint*: This is analogous to both the (Richardson) extrapolation for numerical differentiation formulas and the Romberg integration algorithm.