MATH 170C SPRING 2007 MIDTERM EXAM

SOLUTIONS

PROBLEM 1: SOLVING AN INITIAL VALUE PROBLEM

Use any numerical step-by-step method for the calculation of approximate values of the solution of the following problem for $t = 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$.

$$y'(t) = t + \sin y(t), \qquad y(1) = \pi/2.$$

Discuss the accuracy of the method you used.

SOLUTION. A direct calculation using Euler's method with step-size $\frac{1}{4}$ gives

$$\begin{aligned}
y(1) &= \tfrac{\pi}{2} \approx 1.5708, \\
y(\tfrac{5}{4}) &\approx 1.5708 + \tfrac{1}{4}[1 + \sin(1.5708)] \approx 2.0708, \\
y(\tfrac{3}{2}) &\approx 2.0708 + \tfrac{1}{4}[\tfrac{5}{4} + \sin(2.0708)] \approx 2.6027, \\
y(\tfrac{7}{4}) &\approx 2.6027 + \tfrac{1}{4}[\tfrac{3}{2} + \sin(2.6027)] \approx 3.1060, \\
y(2) &\approx 3.1060 + \tfrac{1}{4}[\tfrac{7}{4} + \sin(3.1060)] \approx 3.5524,
\end{aligned}$$

where we performed the computations up to 4 decimal digits.

Euler's method is of order 1, which means the global truncation error is bounded by a constant multiple of the step-size.

PROBLEM 2: DEGREE OF PRECISION

a). Find the degree of precision of the following quadrature rule:

$$\int_{-1}^{1} f(x)dx \approx \frac{1}{12}[11f(-\frac{3}{5}) + f(-\frac{1}{5}) + f(\frac{1}{5}) + 11f(\frac{3}{5})].$$

b). Use the above result to find the degree of precision of the following quadrature rule:

$$\int_a^b g(x)dx \approx \frac{5d}{24}[11g(a+d) + g(a+2d) + g(a+3d) + 11g(a+4d)],$$

where $d = \frac{b-a}{5}$. *Hint*: Change of variables.

SOLUTION. (a) Let us check the quadrature rule for the monomials $f(x) = x^n$, $n = 0, 1, \ldots$. When n is odd, the quadrature rule gives 0, because $f(-x) = -f(x)$ for any $x \in \mathbb{R}$. In this case, the exact value of the integral is also 0, since the integration interval is symmetric with respect to 0, and the function under integration is an odd function (A function f is said to be odd if $f(-x) = -f(x)$). So the quadrature rule is exact for all monomials of odd degree. We will check the even-degree monomials case by case. For $n = 0$,

$$\int_{-1}^{1} 1dx = 2, \quad \text{and} \quad \frac{1}{12}(11 \cdot 1 + 1 + 1 + 11 \cdot 1) = 2.$$

For $n = 2$,

$$\int_{-1}^{1} x^2 dx = \frac{2}{3}, \quad \text{and} \quad \frac{1}{12}(11 \cdot \frac{9}{25} + \frac{1}{25} + \frac{1}{25} + 11 \cdot \frac{9}{25}) = \frac{2}{3}.$$

For $n = 4$,

$$\int_{-1}^{1} x^4 dx = \frac{2}{5}, \quad \text{and} \quad \frac{1}{12}(11 \cdot \frac{81}{5^4} + \frac{1}{5^4} + \frac{1}{5^4} + 11 \cdot \frac{81}{5^4}) = \frac{446}{3 \cdot 5^4} \approx 0.24.$$

We find that the quadrature rule is not exact for x^4, so its degree of precision is less than 4. We found also that the quadrature is exact for the monomials $\{1, x, x^2, x^3\}$. Since the quadrature and the integration operations are both linear, and the monomials $\{1, x, x^2, x^3\}$ form a basis for polynomials of degree less than or equal to 3, we conclude that the degree of precision of the considered quadrature rule is 3.

For $n = 2k$ (i.e., even n), we could use the following formulas for the above calculations:

$$\int_{-1}^{1} x^{2k} dx = \left. \frac{x^{2k+1}}{2k+1} \right|_{-1}^{1} = \frac{2}{2k+1} = \frac{2}{n+1}.$$

and

$$\frac{1}{12}\left(11(-\frac{3}{5})^{2k} + (-\frac{1}{5})^{2k} + (\frac{1}{5})^{2k} + 11(\frac{3}{5})^{2k}\right) = \frac{11 \cdot 3^{2k} + 1}{6 \cdot 5^{2k}} = \frac{11 \cdot 3^n + 1}{6 \cdot 5^n}.$$

Solution. (b) Let us make a change of variables $x = ky + e$ and demand that $a = k \cdot (-1) + e$ and $b = k \cdot 1 + e$. Solving these two equations yields $k = \frac{b-a}{2}$ and $e = \frac{a+b}{2}$. Under this change of variables, the integral becomes

$$\int_a^b g(x)dx = \int_{-1}^{1} g(ky + e)kdy.$$

If we use the quadrature rule from (a) to this integral, we get

$$\begin{aligned}
\int_a^b g(x)dx &= \int_{-1}^{1} g(ky + e)kdy \\
&\approx \frac{1}{12}[11kg(-\frac{3}{5}k + e) + kg(-\frac{1}{5}k + e) + kg(\frac{1}{5}k + e) + 11kg(\frac{3}{5}k + e)] \\
&= \frac{b-a}{24}[11g(\frac{4}{5}a + \frac{1}{5}b) + g(\frac{3}{5}a + \frac{2}{5}b) + g(\frac{2}{5}a + \frac{3}{5}b) + 11g(\frac{1}{5}a + \frac{4}{5}b)] \\
&= \frac{5d}{24}[11g(a + d) + g(a + 2d) + g(a + 3d) + 11g(a + 4d)].
\end{aligned}$$

We see that the quadrature rule (b) applied to $g(x)$ is equal to the quadrature rule (a) applied to $k \cdot g(kx + e)$. Since the change of variables is linear and $k \neq 0$, polynomial degree will be preserved under this change of variables, i.e., if $g(x)$ is a polynomial of degree p, then $k \cdot g(kx + e)$ will be a polynomial of degree p. So, the quadrature rule

(b) applied to a polynomial $g(x)$ of degree p is equal to the quadrature rule (b) applied to the polynomial $k \cdot g(kx+e)$ of degree p. If $p \leq 3$, we know from (a) that this rule is equal to the integral of $k \cdot g(kx+e)$ over $[-1,1]$, which in turn is equal to the integral of $g(x)$ over $[a,b]$. This proves that the quadrature rule (b) has degree of precision 3.

Problem 3: Numerical integration

Compute the integral

$$\int_0^1 \frac{\cos x}{\sqrt{x}} dx, \tag{1}$$

correctly to 2 decimal places. *Hint*: Make the change of variables $x = t^2$.

Solution. Performing the change of variables, we have

$$\int_0^1 \frac{\cos x}{\sqrt{x}} dx = \int_0^1 \frac{\cos t^2}{t} d(t^2) = \int_0^1 \frac{\cos t^2}{t} 2tdt = \int_0^1 2\cos t^2 dt.$$

We have derived in a practice problem solution that if one wants to compute a real number with n correct decimal digits, it is sufficient to compute it with an accuracy $\leq \frac{1}{2} \cdot 10^{-n}$ and round the computed value to a nearest number with n decimal digits after the decimal point. We have also derived that the total error of the composite trapezoid rule applied to $\int_a^b f(x)dx$ can be bounded by

$$|\hat{E}_T(h)| \leq (b-a)\varepsilon + \frac{(b-a)h^2}{12} \max_{\xi \in [a,b]} |f''(\xi)|, \tag{2}$$

and the total error of composite Simpson's rule applied to the same integral satisfies

$$|\hat{E}_S(h)| \leq (b-a)\varepsilon + \frac{(b-a)h^4}{2880} \max_{\xi \in [a,b]} |f^{(iv)}(\xi)|, \tag{3}$$

where in both formulas, ε is the accuracy with which $f(x)$ is calculated in the quadrature formula.

Let us find the first four derivatives of $\cos t^2$:

$$\begin{aligned}
(\cos t^2)' &= (-\sin t^2) \cdot 2t = -2t \sin t^2, \\
(\cos t^2)'' &= (-2t)' \sin t^2 - 2t(\sin t^2)' = -2\sin t^2 - 4t^2 \cos t^2 \\
(\cos t^2)''' &= 8t^3 \sin t^2 - 12t \cos t^2 \\
(\cos t^2)^{(iv)} &= 16t^4 \cos t^2 + 48t^2 \sin t^2 - 12\cos t^2.
\end{aligned}$$

We can bound the second and the fourth derivatives for the interval $t \in [0,1]$ as

$$\begin{aligned}
|(\cos t^2)''| &\leq 6 \\
|(\cos t^2)^{(iv)}| &\leq |16t^4 - 12| + 48 \leq 60.
\end{aligned}$$

By using (2), we can bound the total error of the composite trapezoid rule applied to (1) by

$$\hat{E}_T(h) \leq \varepsilon + \frac{h^2}{12} \cdot 2 \cdot 6 = \varepsilon + h^2.$$

We will choose h (and ε) so that the right hand side is bounded by $\frac{1}{2}10^{-2}$ to ensure that $\hat{E}_T(h) \leq \frac{1}{2}10^{-2}$. Neglecting ε as $\varepsilon \ll 10^{-2}$, we get $h \leq \frac{1}{10\sqrt{2}}$, or in other words, we need at least $\lceil 10\sqrt{2} \rceil = 15$ panels. Choosing 15 panels gives an upper bound $\frac{1}{2}10^{-2} - (\frac{1}{15})^2 = \frac{1}{1800} \approx 0.0005$ for ε, which is clearly larger than the accuracy of a typical calculator.

By using (3), let us bound the error of composite Simpson's rule applied to (1):

$$\hat{E}_S(h) \leq \varepsilon + \frac{h^4}{2880} \cdot 2 \cdot 60 = \varepsilon + \frac{h^4}{24}.$$

We will choose h (and ε) so that the right hand side is bounded by $\frac{1}{2}10^{-2}$ to ensure that $\hat{E}_S(h) \leq \frac{1}{2}10^{-2}$. Neglecting ε as $\varepsilon \ll 10^{-2}$, we get $h \leq \sqrt[4]{0.12} \approx 0.58$, or in other words, we need at least 2 panels. Choosing 2 panels gives an upper bound $\frac{1}{2}10^{-2} - \frac{1}{24}(\frac{1}{2})^2 = \frac{23}{9600} \approx 0.002$ for ε.

The conclusion is that using composite Simpson's rule saves time as it needs only 2 panels (so 5 evaluations of the function cos) while the composite trapezoid rule needs 15 panels (16 evaluations).

The composite trapezoid rule with 15 panels gives

$$\int_0^1 2\cos t^2 dt \approx 1.8078,$$

and the composite Simpson rule with 2 panels gives

$$\int_0^1 2\cos t^2 dt \approx 1.8090.$$

Both of the above computations give 1.81 after rounding up to 2 decimal digits.