MATH 170C SPRING 2007 PRACTICE PROBLEMS WITH HINTS TO SOLUTIONS

JUNE 8

1. Show that the boundary value problem for the differential equation

$$u'' = f(x, u, u'), \qquad x \in [a, b],$$

with inhomogeneous boundary conditions $u(a) = \alpha$ and $u(b) = \beta$ can be equivalently transformed into a boundary value problem with homogeneous boundary condition.

Solution. Let $g \in C^2[a, b]$ be a function such that $g(a) = \alpha$ and $g(b) = \beta$. For example, we can choose the linear function $g(x) = \alpha + \frac{\beta-\alpha}{b-a}(x - a)$. If we put $u = g + v$, or in other words, if we define a function v by $v = u - g$, then it is obvious that $v(a) = u(a) - g(a) = 0$ and $v(b) = u(b) - g(b) = 0$, i.e., v satisfies the homogeneous boundary conditions. Now by substituting $u = g + v$ into the original equation, we have

$$u'' = g'' + v'' = f(x, g + v, g' + v').$$

So v has to satisfy

$$v'' = f(x, g + v, g' + v') - g''$$

or

$$v'' = F(x, v, v')$$

with the right hand side

$$F(x, v, v'') = f(x, g + v, g' + v') - g''.$$

If we choose g to be the linear function as indicated above, we have $g''(x) = 0$ and $g'(x) = \frac{\beta-\alpha}{b-a}$ for $x \in [a, b]$.

2. Find the solution of the boundary value problem

$$y''(t) + y(t) = e^t, \qquad y(0) = y(1) = 0. \tag{1}$$

Solution. We use the shooting method to solve the problem analytically. Suppose that $u(t)$ is the solution of the equation

$$u''(t) + u(t) = e^t, \tag{2}$$

which satisfies the initial conditions

$$u(0) = 0, \qquad u'(0) = 0, \tag{3}$$

and $v(t)$ be the solution of the homogeneous equation

$$v''(t) + v(t) = 0, \tag{4}$$

which satisfies the initial condition

$$v(0) = 0, \qquad v'(0) = 1. \tag{5}$$

Then the function

$$y(t) = u(t) + s \cdot v(t)$$

satisfies $y''(t) + y(t) = e^t$ and the condition $y(0) = 0$ for *any* constant value of s. There cannot exist additional functions having this property. Therefore, a solution of (1) is found if s can be determined such that

$$u(1) + s \cdot v(1) = 0.$$

Now, let us carry out this program. The general solution of the homogeneous differential equation (4) is

$$v(t) = A\cos t + B\sin t,$$

where A and B are constants. These constants should be such that the function v satisfies the initial conditions (5). We find that $A = 0$ and $B = 1$. So the solution of the initial value problem (4) with the initial conditions (5) is

$$v(t) = \sin t.$$

The general solution of the inhomogeneous differential equation (2) is

$$u(t) = C\cos t + D\sin t + u^*(t),$$

where C and D are constants, and $u^*(t)$ is a particular solution of the differential equation (2). We look for a particular solution in the form $u^*(t) = Ke^t$ and find that the constant should be $K = \frac{1}{2}$. The constants C and D are determined from the initial conditions (3) as $C = D = -\frac{e}{2}$. So the solution of the initial value problem (2) with the initial conditions (3) is

$$u(t) = \frac{1}{2}(-e\cos t - e\sin t + e^t).$$

The function

$$y(t) = u(t) + s \cdot v(t) = \frac{1}{2}(-e\cos t - e\sin t + e^t) + s \cdot \sin t$$

satisfies (1) and the boundary condition $y(0) = 0$. We need to determine s so that y satisfies the other boundary condition $y(1) = 0$. This condition gives

$$y(1) = \frac{1}{2}(-e\cos 1 - e\sin 1 + e) + s \cdot \sin 1 = 0,$$

or

$$s = \frac{e(1 - \cos 1 - \sin 1)}{2\sin 1}.$$

3. Consider the boundary value problem

$$y''(x) = -100y(x), \qquad y(0) = y(2\pi + \epsilon) = 1, \tag{6}$$

for $\epsilon > 0$, and solve it (analytically) by the shooting method, i.e., find an initial slope s such that the solution of the initial value problem

$$Y''(x) = -100Y(x), \qquad Y(0) = 1,\, Y'(0) = s, \tag{7}$$

is equal to the solution of (6). For small ϵ, show that $s = 50\epsilon + O(\epsilon^3)$. Explain why the computational scheme corresponding to the shooting method would be difficult to apply for small ϵ.

Solution. The solution to the initial value problem (7) is

$$Y(x) = \cos(10x) + \frac{s}{10}\sin(10x).$$

The value of the parameter s is found by requiring

$$Y(2\pi + \epsilon) = \cos(20\pi + 10\epsilon) + \frac{s}{10}\sin(20\pi + 10\epsilon) = 1. \tag{8}$$

By using the periodicity of the trigonometric functions, we can rewrite this condition as

$$\cos(10\epsilon) + \frac{s}{10}\sin(10\epsilon) = 1.$$

Now using the Taylor's formulae

$$\cos x = 1 - \frac{1}{2}x^2 + O(x^4)$$

and

$$\sin x = x - O(x^3),$$

for small values of x, we have

$$\begin{aligned}\cos(10\epsilon) + \frac{s}{10}\sin(10\epsilon) &= 1 - \frac{1}{2}(10\epsilon)^2 + O(\epsilon^4) + \frac{s}{10}[10\epsilon - O(\epsilon^3)]\\ &= 1 - 50\epsilon^2 + O(\epsilon^4) + s[\epsilon - O(\epsilon^3)].\end{aligned}$$

Equating this with 1, we have the equation

$$s[\epsilon - O(\epsilon^3)] = 50\epsilon^2 + O(\epsilon^4),$$

and solving this equation for s, we find

$$s = 50\epsilon + O(\epsilon^3).$$

In a computational scheme, we would solve the initial value problem (7) numerically. Similarly to the above, from (8) we calculate that for small ϵ,

$$Y(2\pi + \epsilon) = 1 - 50\epsilon^2 + s\epsilon + O(\epsilon^3),$$

which is very close to 1. So the boundary condition is almost satisfied no matter what value s has, and we need to compute $Y(2\pi + \epsilon)$ with high accuracy in order to find s with reasonable accuracy.

4. Consider the initial value problem $y' = y$, $y(0) = 1$, and show that the approximate solution from Euler's method with step-size h is given by $w_k = (1 + h)^k$.

Solution. By applying Euler's method to the problem, we have

$$w_{k+1} = w_k + hw_k = (1 + h)w_k, \qquad k = 0, 1, \ldots,$$

with $w_0 = 1$. Solving this recurrence gives

$$w_{k+1} = (1 + h)w_k = (1 + h)^2 w_{k-1} = \ldots = (1 + h)^{k+1} w_0 = (1 + h)^{k+1}.$$

5. Show that the differential equation $y'(t) = \alpha t$ with $\alpha \in \mathbb{R}$ is solved exactly by the explicit trapezoid method.

SOLUTION. The exact solution of the equation is

$$y(t) = y(0) + \frac{\alpha}{2}t^2.$$

Upon applying the explicit trapezoid method to the original problem, we have

$$w_{k+1} = w_k + \frac{h}{2}(\alpha t_k + \alpha t_{k+1}) = w_k + \frac{h}{2}(\alpha hk + \alpha h(k+1)) = w_k + \alpha h^2 k + \frac{\alpha h^2}{2},$$

for $k = 0, 1, \ldots$, with $w_0 = y(0)$.

Recalling that w_k is the provided approximation to $y(hk)$, we have to prove that

$$w_k = y(hk) = y(0) + \frac{\alpha}{2}h^2k^2, \qquad k = 0, 1, \ldots.$$

We will do it by induction. For $k = 0$, we have $w_0 = y(0)$ by construction. Now, assume that $w_k = y(hk)$ holds and prove this implies $w_{k+1} = y(hk + h)$.

$$\begin{aligned} w_{k+1} &= w_k + \alpha h^2 k + \frac{\alpha h^2}{2} = y(0) + \frac{\alpha}{2}h^2k^2 + \alpha h^2 k + \frac{\alpha h^2}{2} \\ &= y(0) + \frac{\alpha h^2}{2}(k^2 + 2k + 1) = y(0) + \frac{\alpha h^2}{2}(k+1)^2. \end{aligned}$$

6. Compute the weights for the polynomial interpolatory quadratures with equidistant quadrature points

$$x_k = a + (k+1)\frac{b-a}{n+2}, \qquad k = 0, 1, \ldots, n,$$

for $n = 0, 1, 2$ and obtain representations of the quadrature errors.

PARTIAL SOLUTION ($n = 1$). The quadrature nodes are $x_0 = a + \frac{1}{3}(b-a)$ and $x_1 = a + \frac{2}{3}(b-a)$. We will find the corresponding weights w_0 and w_1 by requiring that the quadrature rule is exact for the polynomials $p_0(x) = 1$ and $p_1(x) = x$. Let us compute the integrals

$$\int_a^b 1dx = b - a, \qquad \text{and} \qquad \int_a^b xdx = \tfrac{1}{2}(b-a)^2.$$

So the exactness conditions for the quadrature read as

$$w_0 \cdot p_0(x_0) + w_1 \cdot p_0(x_1) = w_0 \cdot 1 + w_1 \cdot 1 = b - a,$$

and

$$w_0 \cdot p_1(x_0) + w_1 \cdot p_1(x_1) = w_0[a + \tfrac{1}{3}(b-a)] + w_1[a + \tfrac{2}{3}(b-a)] = \tfrac{1}{2}(b-a)^2.$$

The solution is $w_0 = w_1 = \frac{b-a}{2}$.