MATH 170C SPRING 2007 FINAL EXAM

SOLUTIONS

Problem 1: Numerical differentiation

> Use the three-point centered difference formula for the second derivative to approximate $f''(1)$, where $f(x) = x^{-1}$, for $h = 0.1$, 0.01, and 0.001. Find the approximation error and verify the error estimate predicted by theory.

Solution. We have

$$f''(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} + \frac{h^2}{12} f^{(iv)}(\xi),$$

for some $\xi \in [x-h, x+h]$. Let us write it in the form

$$f''(x) = D(x,h) + E(\xi, h),$$

where $D(x,h)$ denotes the approximate derivative and $E(\xi,h)$ denotes the error term. Note that ξ may depend on both x and h. The exact derivative is $f''(x) = 2x^{-3}$, so $f''(1) = 2$. Direct calculation gives for the approximate derivative

h	$D(1,h)$	$f''(1) - D(1,h)$
0.1	2.020202	0.020202
0.01	2.000200	0.000200
0.001	2.000002	0.000002

By using $f^{(iv)}(x) = 24x^{-5}$, we derive

$$E(\xi, h) = 2h^2\xi^{-5}.$$

Since ξ^{-5} is a decreasing monotone function of ξ when $\xi > 0$, for any closed positive interval the function ξ^{-5} takes its maximum at the left end of the interval and its minimum at the right end. For given h, let us calculate the values of $E(\xi, h)$ with ξ having values on the end points of the interval $[1-h, 1+h]$.

h	$E(1+h,h)$	$E(1-h,h)$
0.1	0.01242	0.03387
0.01	0.00019	0.00021
0.001	0.0000019	0.000002

We see that the computed error $f''(1) - D(1,h)$ lies between the above calculated values of $E(1+h,h)$ and $E(1-h,h)$.

Problem 2: Deriving numerical differentiation formula

Develop a first-order method for approximating $f''(x)$ that uses the data $f(x-h)$, $f(x)$, and $f(x+3h)$ only. Find the error term.

Solution. From Taylor's theorem we have

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(\xi_1), \qquad \xi_1 \in [x-h, x]$$
$$f(x+3h) = f(x) + 3hf'(x) + \frac{9h^2}{2}f''(x) + \frac{9h^3}{2}f'''(\xi_2), \qquad \xi_2 \in [x, x+3h].$$

In order to eliminate the terms involving $f'(x)$, we multiply the first equality by 3 and add to the second:

$$3f(x-h) + f(x+3h) = 4f(x) + 6h^2 f''(x) + \frac{h^3}{2}[9f'''(\xi_2) - f'''(\xi_1)].$$

Now we find $f''(x)$

$$f''(x) = \frac{3f(x-h) + f(x+3h) - 4f(x)}{6h^2} - \frac{h}{12}[9f'''(\xi_2) - f'''(\xi_1)].$$

Problem 3: One-step method

Show that Euler's method fails to approximate the solution $u(x) = (\frac{2}{3}x)^{3/2}$ of the initial value problem $u' = u^{1/3}$, $u(0) = 0$. Explain this failure.

Solution. The function $f(x,u) = u^{1/3}$ in the right hand side of the differential equation does not satisfy the Lipschitz condition at $u = 0$. Therefore there may be multiple solutions and consequently we cannot guarantee the convergence of the Euler method. Indeed, we can see that apart from $u(x) = (\frac{2}{3}x)^{3/2}$, the zero function $u(x) \equiv 0$ is also a solution of the initial value problem. So the problem has at least two solutions. The Euler method gives the solution $u(x) \equiv 0$, failing to "see" the other solutions.

Problem 4: Explicit method

Consider the approximate solution of the initial value problem

$$u' + 100u = 100, \qquad u(0) = 2,$$

by the Euler method. Explain why for an accurate approximation the step-size h has to be chosen smaller than $h < 0.02$ despite the fact that the solution is almost constant for x not too small, say, for $x > 0.1$.

Solution. Let us write the differential equation in the form

$$u' = f(x,u) := 100 - 100u.$$

From here we can see that the equilibrium solution is $u \equiv 1$, and the solution tends to this equilibrium solution exponentially as x increases. The problem is stiff if $|\partial f/\partial u| = 100$ is considered large. Let us examine the Euler method applied to the above problem more closely. We can write

$$w_{i+1} = w_i + h(100 - 100w_i) = 100h + (1 - 100h)w_i,$$

with $w_0 = 2$. Since we know that the equilibrium solution is $u \equiv 1$, let us subtract this from the both sides of the equation to see how the difference between the Euler solution and the equilibrium solution evolves.

$$w_{i+1} - 1 = 100h - 1 + (1 - 100h)w_i = (1 - 100h)(w_i - 1).$$

So the Euler solution converges to 1 if $|1 - 100h| < 1$ or equivalently, $h \in (0, 0.02)$. For $h = 0.02$ the Euler solution will oscillate about 1 without damping, and for $h > 0.02$ the oscillation amplitude will even increase with iteration.

Note: Instead of subtracting the equilibrium solution, one could also use the fixed point iteration theory directly.

PROBLEM 5: MULTI-STEP METHOD

> The Milne-Simpson method is a weakly stable fourth-order, two-step method. Are there any weakly stable third-order, two step methods?

SOLUTION. A general two-step method has the form

$$w_{i+1} = a_1 w_i + a_2 w_{i-1} + h[b_0 f_{i+1} + b_1 f_i + b_2 f_{i-1}],$$

and its characteristic polynomial is

$$p(x) = x^2 - a_1 x - a_2.$$

We are looking for a third-order method, so at least we have to impose the condition (cf. (6.77) and (6.91) in the textbook)

$$a_1 + a_2 = 1.$$

By finding a_2 from this and substituting into the characteristic polynomial, we have

$$p(x) = x^2 - a_1 x - (1 - a_1).$$

We immediately see that $x_1 = 1$ is a root; the other root is $x_2 = a_1 - 1$.

Suppose that the method we are looking for is an explicit method. In this case, from (6.79) in the textbook we infer that $a_1 = -4$ because the method should be third-order. This gives

$$x_2 = -5,$$

implying that the method is not stable. We conclude that there exists no weakly stable third-order two-step explicit method.

Now consider implicit methods. For a weakly stable method it should hold that $|x_2| = |a_1 - 1| = 1$, meaning that either $a_1 = 0$ or $a_1 = 2$. The case $a_1 = 2$ gives a double root at 1 so it is by definition not stable. The case $a_1 = 0$ is excluded because it is the Milne-Simpson method and so fourth-order. We conclude that there exists no weakly stable third-order two-step method.

PROBLEM 6: SHOOTING METHOD

Find the solution of the boundary value problem

$$y'' = 100y, \qquad y(0) = 1,\ y(3) = e^{-30}.$$

Explain why the shooting method would have to be applied with great care, assuming you solve the associated initial value problems numerically.

SOLUTION. Temporarily neglecting the boundary conditions, the general solution of the differential equation is

$$y(x) = Ae^{10x} + Be^{-10x},$$

where A and B are constants. Now by imposing the boundary conditions at $x = 0$ and $x = 3$, we find the constants as $A = 0$ and $B = 1$, so the solution of the boundary value problem is

$$y(x) = e^{-10x}.$$

Now assume we solve the problem by the shooting method. For a given parameter s, denote by $Y_s(x)$ the solution of the initial value problem

$$Y'' = 100Y, \qquad Y(0) = 1,\ Y'(0) = s.$$

For the analysis purpose, let us find the solution

$$Y_s(x) = \tfrac{10+s}{20}e^{10x} + \tfrac{10-s}{20}e^{-10x} = \tfrac{e^{10x}+e^{-10x}}{2} + \tfrac{s}{10}\tfrac{e^{10x}-e^{-10x}}{2} = \cosh(10x) + \tfrac{s}{10}\sinh(10x).$$

In a computational scheme, one would compute $Y_s(x)$ numerically and try to find a value of s such that $Y_s(3) = e^{-30}$. From the above formula we see that for not too small x,

$$Y_s(x) \approx \tfrac{e^{10x}}{2} + \tfrac{e^{10x}}{20}s,$$

and so in particular $Y_s(3) \approx \frac{e^{30}}{2} + \frac{e^{30}}{20}s$. Since the exponential e^{10x} is a rapidly growing function, the solution of the BVP found by the numerical shooting method depends on the parameter s extremely sensitively. This requires us to compute s with an extreme accuracy.