Adaptive wavelet algorithms

with truncated residuals

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Contents

Elliptic boundary value problem

- $\mathcal{H} := H_0^1(\Omega)$
- $A : \mathcal{H} \to \mathcal{H}'$ linear, self-adjoint, $\mathcal{H}$-elliptic
  $(\langle Av, v\rangle \geq c\|v\|_{\mathcal{H}}^2 \qquad v \in \mathcal{H})$

Find $u \in \mathcal{H}$ s.t. $\quad Au = f \qquad (f \in \mathcal{H}')$

- Example: Reaction-diffusion equation $\mathcal{H} = H_0^1(\Omega)$

$$\langle Au, v\rangle = \int_\Omega \nabla u \cdot \nabla v + \kappa^2 uv$$

Equivalent discrete problem

[Cohen, Dahmen, DeVore '01, '02]

- Wavelet basis $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$ of $\mathcal{H}$
- Stiffness $\mathbf{A} = \langle A\psi_\lambda, \psi_\mu \rangle_{\lambda,\mu}$ and load $\mathbf{f} = \langle f, \psi_\lambda \rangle_\lambda$

Linear equation in $\ell_2(\nabla)$

$$\mathbf{Au} = \mathbf{f}, \qquad \mathbf{A} : \ell_2(\nabla) \to \ell_2(\nabla) \text{ SPD and } \mathbf{f} \in \ell_2(\nabla)$$

- $u = \sum_\lambda \mathbf{u}_\lambda \psi_\lambda$ is the solution of $Au = f$
- $\|\mathbf{u} - \mathbf{v}\|_{\ell_2} \eqsim \|u - v\|_{\mathcal{H}}$ with $v = \sum_\lambda \mathbf{v}_\lambda \psi_\lambda$

Galerkin solutions

- $\|\!|\cdot|\!\| := \langle \mathbf{A}\cdot, \cdot \rangle^{\frac{1}{2}}$ is a norm on ℓ_2
- $\Lambda \subset \nabla$
- $\mathbf{I}_\Lambda : \ell_2(\Lambda) \to \ell_2(\nabla)$ incl., $\quad \mathbf{P}_\Lambda := \mathbf{I}_\Lambda^*$
- $\mathbf{A}_\Lambda := \mathbf{P}_\Lambda \mathbf{A} \mathbf{I}_\Lambda : \ell_2(\Lambda) \to \ell_2(\Lambda)$ SPD
- $\mathbf{f}_\Lambda := \mathbf{P}_\Lambda \mathbf{f} \in \ell_2(\Lambda)$

Lemma

A unique solution $\mathbf{u}_\Lambda \in \ell_2(\Lambda)$ to $\mathbf{A}_\Lambda \mathbf{u}_\Lambda = \mathbf{f}_\Lambda$ exists, and

$$\|\!|\mathbf{u} - \mathbf{u}_\Lambda|\!\| = \inf_{\mathbf{v} \in \ell_2(\Lambda)} \|\!|\mathbf{u} - \mathbf{v}|\!\|$$

Galerkin orthogonality

- $\operatorname{supp} \mathbf{w} \subset \Lambda, \qquad \mathbf{A}_\Lambda \mathbf{u}_\Lambda = \mathbf{f}_\Lambda$
- $\langle \mathbf{f} - \mathbf{A}\mathbf{u}_\Lambda, \mathbf{v}_\Lambda \rangle = 0 \quad$ for $\mathbf{v}_\Lambda \in \ell_2(\Lambda)$

$$|\!|\!|\mathbf{u} - \mathbf{w}|\!|\!|^2 = |\!|\!|\mathbf{u} - \mathbf{u}_\Lambda|\!|\!|^2 + |\!|\!|\mathbf{u}_\Lambda - \mathbf{w}|\!|\!|^2$$

-

Error reduction

$$\|\mathbf{u} - \mathbf{u}_\Lambda\|^2 = \|\mathbf{u} - \mathbf{w}\|^2 - \|\mathbf{u}_\Lambda - \mathbf{w}\|^2$$

Lemma [CDD01]

Let $\mu \in (0,1)$, and Λ be s.t.

$$\|\mathbf{P}_\Lambda(\mathbf{f} - \mathbf{A}\mathbf{w})\| \geq \mu \|\mathbf{f} - \mathbf{A}\mathbf{w}\|$$

Then we have

$$\|\mathbf{u} - \mathbf{u}_\Lambda\| \leq \sqrt{1 - \kappa(\mathbf{A})^{-1}\mu^2}\, \|\mathbf{u} - \mathbf{w}\|$$

Ideal algorithm

SOLVE$[\varepsilon] \to \mathbf{u}_k$
$k := 0$; $\Lambda_0 := \emptyset$
`do`
  `Solve` $\mathbf{A}_{\Lambda_k}\mathbf{u}_k = \mathbf{f}_{\Lambda_k}$
  $\mathbf{r}_k := \mathbf{f} - \mathbf{A}\mathbf{u}_k$
  `determine a set` $\Lambda_{k+1} \supset \Lambda_k$`, with minimal`
  `cardinality, such that` $\|\mathbf{P}_{\Lambda_{k+1}}\mathbf{r}_k\| \geq \mu\|\mathbf{r}_k\|$
  $k := k + 1$
`while` $\|\mathbf{r}_k\| > \varepsilon$

Approximate Iterations

Approximate right-hand side

$$\mathbf{RHS}[\varepsilon] \to \mathbf{f}_\varepsilon \text{ with } \|\mathbf{f} - \mathbf{f}_\varepsilon\|_{\ell_2} \le \varepsilon$$

Approximate application of the matrix

$$\mathbf{APPLY_A}[\mathbf{v}, \varepsilon] \to \mathbf{w}_\varepsilon \text{ with } \|\mathbf{Av} - \mathbf{w}_\varepsilon\|_{\ell_2} \le \varepsilon$$

Approximate residual

$$\mathbf{RES}[\mathbf{v}, \varepsilon] := \mathbf{RHS}[\varepsilon/2] - \mathbf{APPLY_A}[\mathbf{v}, \varepsilon/2]$$

Best N-term approximation

Given $u \in \mathcal{H}$, approximate u using N wavelets

$$\Sigma_N := \left\{ \sum_{\lambda \in \Lambda} a_\lambda \psi_\lambda : \#\Lambda \leq N, a_\lambda \in \mathbb{R} \right\}$$

- Σ_N is a nonlinear manifold

Nonlinear vs. linear approximation in $H^t(\Omega)$

Using wavelets of order d

Nonlinear approximation

If $u \in B_p^{t+ns}(L_p)$ with $\frac{1}{p} = \frac{1}{2} + s$ for some $s \in (0, \frac{d-t}{n})$

$$\varepsilon_N = \mathrm{dist}(u, \Sigma_N) \lesssim N^{-s}$$

Linear approximation

If $u \in H^{t+ns}$ for some $s \in (0, \frac{d-t}{n}]$, uniform refinement

$$\varepsilon_j = \|u_j - u\| \lesssim N_j^{-s}$$

- [Dahlke, DeVore]: $u \in B_p^{t+ns}(L_p) \backslash H^{t+ns}$ "often"

Approximation spaces

- Approximation space $\mathcal{A}^s := \{v \in \mathcal{H} : \mathrm{dist}(v, \Sigma_N) \lesssim N^{-s}\}$
- Quasi-norm $|v|_{\mathcal{A}^s} := \|v\|_{\mathcal{H}} + \sup_{N\in\mathbb{N}} N^s \mathrm{dist}(v, \Sigma_N)$
- $B_p^{t+ns}(L_p) \subset \mathcal{A}^s$ with $\frac{1}{p} = \frac{1}{2} + s$ for $s \in (0, \frac{d-t}{n})$

Complexity of the problem

- $U : f \mapsto \tilde{u}$ algorithm for solving $Au = f$
- $\mathrm{cost}(U, F) := \sup_{f \in F} \mathrm{cost}(U, f)$
- $e(U, F) := \sup_{f \in F} \|U(f) - u\|_{\mathcal{H}}$
- $\mathrm{comp}(\varepsilon, F) := \inf\{\mathrm{cost}(U, F) : \text{over all } U \text{ s.t. } e(U, F) \leq \varepsilon\}$
- $B_r^s := \{v \in \mathcal{A}^s : |v|_{\mathcal{A}^s} \leq r\}$
- $U(f)$ lin. comb. of N wavs. $\Rightarrow$ $\mathrm{cost}(U, f) \gtrsim N$

Since $v \in \mathcal{A}^s \Leftrightarrow \mathrm{dist}(v, \Sigma_N) \lesssim N^{-s}|v|_{\mathcal{A}^s}$, we have

$$\mathrm{comp}(\varepsilon, A(B_r^s)) \gtrsim r^{1/s}\varepsilon^{-1/s}$$

Requirements on the subroutines

Assume: $u \in \mathcal{A}^s$ for some $s \in (0, \frac{d-t}{n})$

Complexity of $\mathbf{RHS}$

$\mathbf{RHS}[\varepsilon] \to \mathbf{f}_\varepsilon$ terminates with $\|\mathbf{f} - \mathbf{f}_\varepsilon\|_{\ell_2} \leq \varepsilon$

- $\#\operatorname{supp} \mathbf{f}_\varepsilon \lesssim \varepsilon^{-1/s} |u|_{\mathcal{A}^s}^{1/s}$
- $\text{cost} \lesssim \varepsilon^{-1/s} |u|_{\mathcal{A}^s}^{1/s} + 1$

Complexity of $\mathbf{APPLY_A}$

For $\#\operatorname{supp} \mathbf{v} < \infty$

$\mathbf{APPLY_A}[\mathbf{v}, \varepsilon] \to \mathbf{w}_\varepsilon$ terminates with $\|\mathbf{A}\mathbf{v} - \mathbf{w}_\varepsilon\|_{\ell_2} \leq \varepsilon$

- $\#\operatorname{supp} \mathbf{w}_\varepsilon \lesssim \varepsilon^{-1/s} |\mathbf{v}^T \Psi|_{\mathcal{A}^s}^{1/s}$
- $\text{cost} \lesssim \varepsilon^{-1/s} |\mathbf{v}^T \Psi|_{\mathcal{A}^s}^{1/s} + \#\operatorname{supp} \mathbf{v} + 1$

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The subroutine $\mathbf{APPLY_A}$

- Ψ is piecewise polynomial wavelets that are sufficiently smooth and have sufficiently many vanishing moments
- A is either differential or singular integral operator

Then we can construct $\mathbf{APPLY_A}$ satisfying the requirements.
Ref: [CDD01], [Stevenson '04], [Gantumur, Stevenson '05,'06], [Dahmen, Harbrecht, Schneider '05]

Optimal expansion

Lemma [Gantumur, Harbrecht, Stevenson '05]
Let $u \in \mathcal{B}^s_r$ and $\mu \in (0, \kappa(\mathbf{A})^{-\frac{1}{2}})$. Then the smallest set $\Lambda \supset \operatorname{supp} \mathbf{w}$ with

$$\|\mathbf{P}_\Lambda(\mathbf{f} - \mathbf{A}\mathbf{w})\| \geq \mu \|\mathbf{f} - \mathbf{A}\mathbf{w}\|$$

satisfies

$$\#\Lambda - \#\operatorname{supp} \mathbf{w} \lesssim r^{1/s} \|\mathbf{f} - \mathbf{A}\mathbf{w}\|^{-1/s}$$

Optimal complexity

Theorem [GHS05]

SOLVE$[\varepsilon] \to \mathbf{w}$ terminates with $\|\mathbf{f} - \mathbf{A}\mathbf{w}\|_{\ell_2} \leq \varepsilon$. Whenever $u \in B^s_r$ with $s \in (0, \frac{d-t}{n})$, we have

- $\#\mathrm{supp}\,\mathbf{w} \lesssim r^{1/s}\varepsilon^{-1/s}$
- $\mathrm{cost} \lesssim r^{1/s}\varepsilon^{-1/s}$

Further result

- Can be extended to mildly nonsymmetric and indefinite problems [Gantumur '06]

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Sketch of a proof

$$
\begin{aligned}
\#\Lambda_{K+1} &= \sum_{k=0}^{K} \#\Lambda_{k+1} - \#\Lambda_k \\
&\lesssim r^{1/s} \sum_{k=0}^{K} \|\mathbf{f} - \mathbf{A}\mathbf{u}_k\|^{-1/s} \\
&\lesssim r^{1/s} \|\mathbf{f} - \mathbf{A}\mathbf{u}_K\|^{-1/s} \\
&< r^{1/s} \varepsilon^{-1/s}
\end{aligned}
$$

Algorithm with truncated residuals

[Harbrecht, Schneider '02], [Berrone, Kozubek '04]

SOLVE$[\varepsilon] \to \mathbf{u}_k$
$k := 0$; $\Lambda_0 := \emptyset$
do
  Solve $\mathbf{A}_{\Lambda_k}\mathbf{u}_k = \mathbf{f}_{\Lambda_k}$
  $\mathbf{r}_k^\star := \mathbf{P}_{\Lambda_k^\star}(\mathbf{f} - \mathbf{A}\mathbf{u}_k)$
  determine a set $\Lambda_{k+1} \supset \Lambda_k$, with minimal
  cardinality, such that $\|\mathbf{P}_{\Lambda_{k+1}}\mathbf{r}_k^\star\| \geq \mu\|\mathbf{r}_k^\star\|$
  $k := k + 1$
while $\|\mathbf{r}_k^\star\| > \varepsilon$

Error reduction

- $\mathbf{r}_k^\star = \mathbf{P}_{\Lambda_k^\star}(\mathbf{f} - \mathbf{A}\mathbf{u}_k)$ truncated residual
- $\mathbf{r}_k = \mathbf{f} - \mathbf{A}\mathbf{u}_k$ full residual

Suppose $\Lambda_k^\star = \mathcal{V}(\Lambda_k)$ is such that

$$\|\mathbf{P}_{\Lambda_k^\star}(\mathbf{f} - \mathbf{A}\mathbf{u}_k)\| \geq \eta \|\mathbf{f} - \mathbf{A}\mathbf{u}_k\|$$

then we have

$$\|\mathbf{P}_{\Lambda_{k+1}}\mathbf{r}_k\| = \|\mathbf{P}_{\Lambda_{k+1}}\mathbf{r}_k^\star\| \geq \mu \|\mathbf{r}_k^\star\| \geq \mu\eta \|\mathbf{r}_k\|$$

$\rightarrow$ error reduction

Cardinality of expansion

$\tilde{\Lambda} = \mathcal{V}(\Lambda, \bar{\Lambda})$, $\Lambda \subset \bar{\Lambda}$ trees

- $\|\mathbf{u}_{\tilde{\Lambda}} - \mathbf{u}_{\Lambda}\| \geq \eta \|\mathbf{u}_{\bar{\Lambda}} - \mathbf{u}_{\Lambda}\|$
- $\Lambda \subset \tilde{\Lambda} \subseteq \mathcal{V}(\Lambda, \nabla)$
- $\#\mathcal{V}(\Lambda, \nabla) \lesssim \#\Lambda$
- $\#(\tilde{\Lambda} \setminus \Lambda) \lesssim \#(\bar{\Lambda} \setminus \Lambda)$

Lemma

Let $u \in B^s_r$ and $\mu \in (0, \eta\kappa(\mathbf{A})^{-\frac{1}{2}})$. Then with $\Lambda^\star = \mathcal{V}(\Lambda, \nabla)$, the smallest tree $\breve{\Lambda} \supset \Lambda$ with

$$\|\mathbf{P}_{\breve{\Lambda}} \mathbf{r}^\star\| \geq \mu \|\mathbf{r}^\star\|$$

satisfies

$$\#(\breve{\Lambda} \setminus \Lambda) \lesssim r^{1/s} \|\mathbf{u} - \mathbf{u}_\Lambda\|^{-1/s}$$

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Optimal convergence rate

Theorem

SOLVE$[\varepsilon] \to \mathbf{w}$ terminates with $\|\mathbf{u} - \mathbf{w}\|_{\ell_2} \lesssim \varepsilon$. Whenever $u \in B_r^s$ with $s \in (0, \frac{d-t}{n})$, we have

- $\#\mathrm{supp}\,\mathbf{w} \lesssim r^{1/s}\varepsilon^{-1/s}$
- $\mathrm{cost} \lesssim r^{1/s}\varepsilon^{-1/s}$

Activable sets

$\Lambda^\star = \mathcal{V}(\Lambda, \nabla)$:

- $\#\Lambda^\star \lesssim \#\Lambda$
- $\|\mathbf{P}_{\Lambda^\star}(\mathbf{f} - \mathbf{A}\mathbf{u}_\Lambda)\| \geq \eta \|\mathbf{f} - \mathbf{A}\mathbf{u}_\Lambda\|$

[Berrone, Kozubek '04]:

- For $\lambda \in \Lambda$, add μ to $\Lambda^\star$ if ψ_μ intersects with a contracted support of ψ_λ and $|\mu| = |\lambda| + 1$

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FEM error estimators

[Verfürth], [Stevenson '04], [Dahmen, Schneider, Xu '00], [Bittner, Urban '05]

- $f \in L_2(\Omega)$
- $\mathcal{S} := \operatorname{span}\{\psi_\lambda : \lambda \in \Lambda\}$
- $\mathcal{T}$ mesh corresponding to $\mathcal{S}$

$$\mathcal{E}_{\mathcal{T}}(w) \gtrsim \|f - Aw\|_{H^{-1}(\Omega)} \qquad \text{for } w \in \mathcal{S}$$

if

- Λ is a graded tree
- Duals $\tilde{\Psi}$ are compactly supported

Saturation

[Verfürth], [Morin, Nochetto, Siebert '00], [Stevenson '04], [Mekchay, Nochetto '04]
With $\mathcal{S}^\star = \operatorname{span}\{\psi_\lambda : \lambda \in \Lambda^\star \supset \Lambda\}$

$$\mathcal{E}_\mathcal{T}(w) \lesssim \|u_{\mathcal{S}^\star} - w\|_{H^1(\Omega)} \qquad \text{for } w \in \mathcal{S}$$

if

- f is a piecewise polynomial w.r.t. $\mathcal{T}$
- "Bubble functions" are in $\mathcal{S}^\star$, i.e., duals $\tilde{\Psi}$ are compactly supported

$$\begin{aligned} \|\mathbf{P}_{\Lambda^\star}(\mathbf{f} - \mathbf{A}\mathbf{u}_\Lambda)\| &\gtrsim \|u_{\mathcal{S}^\star} - u_\mathcal{S}\|_{H^1(\Omega)} \gtrsim \mathcal{E}_\mathcal{T}(u_\mathcal{S}) \\ &\gtrsim \|f - Au_\mathcal{S}\|_{H^{-1}(\Omega)} \\ &\gtrsim \|\mathbf{f} - \mathbf{A}\mathbf{u}_\Lambda\| \end{aligned}$$

Activable sets

$\Lambda^\star = \mathcal{V}(\Lambda, \nabla)$:

- For $\Delta \in \mathcal{T}$, add μ to $\Lambda^\star$ if $\tilde{\psi}_\mu$ intersects with Δ and $|\mu| \leq |\lambda| + N$

References

[GHS05] Ts. Gantumur, H. Harbrecht, R.P. Stevenson. An optimal adaptive wavelet method without coarsening of the iterands. Technical Report 1325, Utrecht University, March 2005. To appear in *Math. Comp.*.

[Gan06] Ts. Gantumur. Adaptive wavelet algorithms for solving operator equations. PhD thesis. Utrecht University. To appear.

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