Adaptive wavelet algorithms

for solving operator equations

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Overview

Ideal benchmark: Nonlinear approximation

Optimal adaptive wavelet algorithm

Numerical illustration

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Elliptic operator equation

$$Au = f$$

- $u \in \mathcal{H}$ (separable Hilbert space), $f \in \mathcal{H}'$
- $A : \mathcal{H} \to \mathcal{H}'$ linear, self-adjoint, $\mathcal{H}$-elliptic

$$\langle Av, v \rangle \geq c\|v\|_{\mathcal{H}}^2 \qquad v \in \mathcal{H}$$

- Example: Reaction-diffusion equation $\mathcal{H} = H_0^1(\Omega)$

$$\langle Au, v \rangle = \int_\Omega \nabla u \cdot \nabla v + \kappa^2 uv$$

Adaptive wavelet algorithms

- Wavelet basis $\Psi = (\psi_i)_{i \in \mathbb{N}}$ of $\mathcal{H}$ $\quad (\|\sum_i a_i \psi_i\|_{\mathcal{H}} \eqsim \|(a_i)_i\|_{\ell_2})$
- $U_\varepsilon : f \mapsto \tilde{u} = \sum_{i \in E} a_i \psi_i \quad (E \subset \mathbb{N}, \quad \|\tilde{u} - u\|_{\mathcal{H}} \leq \varepsilon)$
- Non-adaptive: $E = \{1, 2, \ldots, k\}$ for some k
- Adaptive: no (or mild) constraint

Computational model 1

Complexity measure: $\#E$ as a function of ε

Best N-term approximation

Given $u \in \mathcal{H}$, approximate u using N wavelets

$$\Sigma_N := \left\{ \sum_{i \in E} a_i \psi_i : \#E \leq N, a_i \in \mathbb{R} \right\}$$

Linear

$$S_N := \left\{ \sum_{i=1}^{N} a_i \psi_i : a_i \in \mathbb{R} \right\}$$

Nonlinear vs. linear approximation in $H^t(\Omega)$

Using wavelets of order d

Nonlinear approximation

If $u \in B_p^{t+ns}(L_p)$ with $\frac{1}{p} = \frac{1}{2} + s$ for some $s \in (0, \frac{d-t}{n})$

$$\operatorname{dist}(u, \Sigma_N) \leq cN^{-s}$$

Linear approximation

If $u \in H^{t+ns}$ for some $s \in (0, \frac{d-t}{n}]$, uniform refinement

$$\operatorname{dist}(u, S_N) \leq cN^{-s}$$

Poisson on polygon: $u \in H^{1+2s}$ for only $s < \frac{\pi}{\alpha}$, $u \in B_p^{1+ns}(L_p)$ for $\forall s > 0$

Approximation spaces

- Approximation space $\mathcal{A}^s := \{v \in \mathcal{H} : \mathrm{dist}(v, \Sigma_N) \leq cN^{-s}\}$
- Quasi-norm $|v|_{\mathcal{A}^s} := \|v\|_{\mathcal{H}} + \sup_{N \in \mathbb{N}} N^s \mathrm{dist}(v, \Sigma_N)$
- $B_p^{t+ns}(L_p) \subset \mathcal{A}^s$ with $\frac{1}{p} = \frac{1}{2} + s$ for $s \in (0, \frac{d-t}{n})$

Model of computation

With unit cost:

- Real number model: $+, -, \ldots$ in $\mathbb{R}$, function evaluations
- multiplication by a scalar, addition in $\mathcal{H}$, e.g., $a_i\psi_i$
  $U_\varepsilon(f)$ lin. comb. of N wavs. $\quad \Rightarrow \quad \mathrm{cost}(U_\varepsilon, f) \geq N$
- Availability of certain subroutine(s)

Complexity of the problem

- $U_\varepsilon : F \ni f \mapsto \tilde{u}$ algorithm for solving $Au = f$
- $\text{cost}(U_\varepsilon, F) := \sup_{f \in F} \text{cost}(U_\varepsilon, f)$
- $\text{comp}(\varepsilon, F) := \inf\{\text{cost}(U_\varepsilon, F) : \text{over all } U_\varepsilon\}$

Since $v \in \mathcal{A}^s \Leftrightarrow \text{dist}(v, \Sigma_N) \leq cN^{-s}$, we have

$$\text{comp}(\varepsilon, A(\mathcal{A}^s)) \geq C\varepsilon^{-1/s}$$

Equivalent problem in ℓ_2

[Cohen, Dahmen, DeVore '01, '02]

- Wavelet basis $\Psi = (\psi_i)_{i\in\mathbb{N}}$ of $\mathcal{H}$
- Stiffness $\mathbf{A} = (\langle A\psi_i, \psi_k\rangle)_{i,k}$ and load $\mathbf{f} = (\langle f, \psi_i\rangle)_i$

Linear equation in ℓ_2

$$\mathbf{Au} = \mathbf{f}, \qquad \mathbf{A} : \ell_2 \to \ell_2 \text{ SPD and } \mathbf{f} \in \ell_2$$

- $u = \sum_i \mathbf{u}_i \psi_i$ is the solution of $Au = f$
- $\|\mathbf{u} - \mathbf{v}\|_{\ell_2} \eqsim \|u - v\|_{\mathcal{H}}$ with $v = \sum_i \mathbf{v}_i \psi_i$

Galerkin solutions

- $\|\!|\cdot|\!\| := \langle \mathbf{A}\cdot, \cdot\rangle^{\frac{1}{2}}$ is a norm on ℓ_2
- $E \subset \mathbb{N}$
- $\mathbf{I}_E : \ell_2(E) \to \ell_2$ incl., $\qquad \mathbf{P}_E := \mathbf{I}_E^*$
- $\mathbf{A}_E := \mathbf{P}_E \mathbf{A} \mathbf{I}_E : \ell_2(E) \to \ell_2(E)$ SPD
- $\mathbf{f}_E := \mathbf{P}_E \mathbf{f} \in \ell_2(E)$

Lemma

A unique solution $\mathbf{u}_E \in \ell_2(E)$ to $\mathbf{A}_E \mathbf{u}_E = \mathbf{f}_E$ exists, and

$$\|\!|\mathbf{u} - \mathbf{u}_E|\!\| = \inf_{\mathbf{v} \in \ell_2(E)} \|\!|\mathbf{u} - \mathbf{v}|\!\|$$

Galerkin orthogonality

$$\mathbf{A}_E \mathbf{u}_E = \mathbf{f}_E$$

- for $\mathbf{v}_E \in \ell_2(E)$:

$$0 = \langle \mathbf{f} - \mathbf{A}\mathbf{u}_E, \mathbf{v}_E \rangle = \langle \mathbf{A}(\mathbf{u} - \mathbf{u}_E), \mathbf{v}_E \rangle$$

$$|\!|\!|\mathbf{u} - \mathbf{u}_E - \mathbf{v}_E|\!|\!|^2 = |\!|\!|\mathbf{u} - \mathbf{u}_E|\!|\!|^2 + |\!|\!|\mathbf{v}_E|\!|\!|^2$$

Error reduction

$E_0 \subset E_1 \subset E_2 \subset \ldots \subset \mathbb{N}$

$$\mathbf{A}_{E_0}\mathbf{u}_{E_0} = \mathbf{f}_{E_0}, \quad \mathbf{A}_{E_1}\mathbf{u}_{E_1} = \mathbf{f}_{E_1}$$

$$|\!|\!|\mathbf{u} - \mathbf{u}_{E_1}|\!|\!|^2 = |\!|\!|\mathbf{u} - \mathbf{u}_{E_0}|\!|\!|^2 - |\!|\!|\mathbf{u}_{E_1} - \mathbf{u}_{E_0}|\!|\!|^2$$

Lemma [CDD01]

Let $\mu \in (0, 1)$, and $E_1 \supset E_0$ be s.t.

$$\|\mathbf{P}_{E_1}(\mathbf{f} - \mathbf{A}\mathbf{u}_{E_0})\|_{\ell_2} \geq \mu \|\mathbf{f} - \mathbf{A}\mathbf{u}_{E_0}\|_{\ell_2}$$

Then we have

$$|\!|\!|\mathbf{u} - \mathbf{u}_{E_1}|\!|\!| \leq \sqrt{1 - \kappa(\mathbf{A})^{-1}\mu^2}\, |\!|\!|\mathbf{u} - \mathbf{u}_{E_0}|\!|\!|$$

Ideal algorithm

SOLVE$[\varepsilon] \to \mathbf{u}_k$
$k := 0$; $E_0 := \varnothing$
do
  Solve $\mathbf{A}_{E_k}\mathbf{u}_k = \mathbf{f}_{E_k}$
  $\mathbf{r}_k := \mathbf{f} - \mathbf{A}\mathbf{u}_k$
  determine a set $E_{k+1} \supset E_k$, with minimal cardinality, such that $\|\mathbf{P}_{E_{k+1}}\mathbf{r}_k\|_{\ell_2} \geq \mu\|\mathbf{r}_k\|_{\ell_2}$
  $k := k+1$
while $\|\mathbf{r}_k\| > \varepsilon$

Approximate iterations

Assume: $u \in \mathcal{A}^s$ for some $s \in (0, \frac{d-t}{n})$

$\mathbf{RHS}[\varepsilon] \to \mathbf{f}_\varepsilon$ with $\|\mathbf{f} - \mathbf{f}_\varepsilon\|_{\ell_2} \leq \varepsilon$

- $\#\mathrm{supp}\, \mathbf{f}_\varepsilon \leq C\varepsilon^{-1/s}$
- $\mathrm{cost} \leq C(\varepsilon^{-1/s} + 1)$

$\mathbf{APPLY_A}[\mathbf{v}, \varepsilon] \to \mathbf{w}_\varepsilon$ with $\|\mathbf{Av} - \mathbf{w}_\varepsilon\|_{\ell_2} \leq \varepsilon$

- $\#\mathrm{supp}\, \mathbf{w}_\varepsilon \leq C\varepsilon^{-1/s}$
- $\mathrm{cost} \leq C(\varepsilon^{-1/s} + \#\mathrm{supp}\, \mathbf{v} + 1)$

$\mathbf{RES}[\mathbf{v}, \varepsilon] := \mathbf{RHS}[\varepsilon/2] - \mathbf{APPLY_A}[\mathbf{v}, \varepsilon/2]$

The subroutine $\mathbf{APPLY_A}$

- $(\psi_i)_i$ are piecewise polynomial wavelets that are sufficiently smooth and have sufficiently many vanishing moments
- A is either differential or singular integral operator

Then we can construct $\mathbf{APPLY_A}$ satisfying the requirements.

Ref: [CDD01], [Stevenson '04], [Gantumur, Stevenson '05,'06], [Dahmen, Harbrecht, Schneider '05]

Optimal expansion

Lemma [Gantumur, Harbrecht, Stevenson '05]
Let $\mu \in (0, \kappa(\mathbf{A})^{-\frac{1}{2}})$. Then the smallest set $E \supset \operatorname{supp} \mathbf{w}$ with

$$\|\mathbf{P}_E(\mathbf{f} - \mathbf{A}\mathbf{w})\|_{\ell_2} \geq \mu \|\mathbf{f} - \mathbf{A}\mathbf{w}\|_{\ell_2}$$

satisfies

$$\#(E \setminus \operatorname{supp} \mathbf{w}) \leq C \|\mathbf{u} - \mathbf{w}\|_{\ell_2}^{-1/s}$$

Optimal complexity

Theorem [GHS05]

SOLVE$[\varepsilon] \rightarrow \mathbf{w}$ terminates with $\|\mathbf{f} - \mathbf{A}\mathbf{w}\|_{\ell_2} \leq \varepsilon$. Whenever $u \in \mathcal{A}^s$ with $s \in (0, \frac{d-t}{n})$, we have

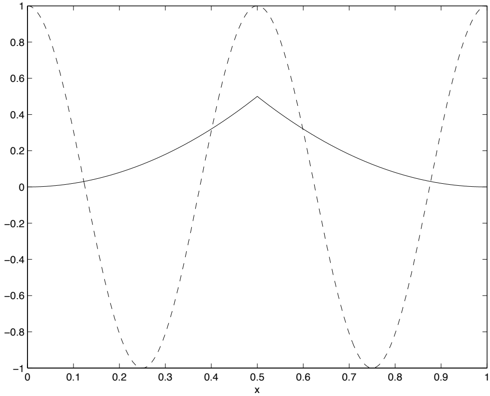
- $\#\mathrm{supp}\,\mathbf{w} \leq C\varepsilon^{-1/s}$
- $\mathrm{cost} \leq C\varepsilon^{-1/s}$

Further result

- Can be extended to mildly nonsymmetric and indefinite problems [Gantumur '06]

Numerical illustration

- The problem: $-\Delta u + u = f$ on $\mathbb{R}/\mathbb{Z}$ $\quad (t = 1)$
- $u \in H^{1+s}$ only for $s < \frac{1}{2}$; $\quad u \in B^{1+s}_{\tau,\tau}$ for any $s > 0$



Convergence histories

- B-spline wavelets of order d=3 with 3 vanishing moments from [Cohen, Daubechies, Feauveau '92] $\Rightarrow$ $u \in \mathcal{A}^s$ for any $s < \frac{d-t}{n} = \frac{3-1}{1} = 2$