# (In)consistency of the combinatorial codifferential


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# The question

Recall the coderivative

$$\langle \mathrm{d}^* u, v\rangle_{L^2\Lambda^{k-1}} = \langle u, \mathrm{d}v\rangle_{L^2\Lambda^k}, \qquad \forall v \in H\Lambda^{k-1}.$$

Let $(\Lambda_h^k)_k$ constitute some subcomplex of the Hilbert-deRham complex, associated to a triangulation of an $n$-manifold. Then we can define a discrete coderivative by

$$\langle \mathrm{d}_h^* u, v\rangle_{L^2\Lambda^{k-1}} = \langle u, \mathrm{d}v\rangle_{L^2\Lambda^k}, \qquad u \in \Lambda_h^k, \quad v \in \Lambda_h^{k-1}.$$

We are interested in the consistency of $\mathrm{d}_h^*$, i.e., the question

$$\lim_{h\to 0} \|\mathrm{d}_h^* \pi_h u - \mathrm{d}^* u\| = 0, \qquad u \in \Lambda^k \cap \mathrm{Dom}(\mathrm{d}^*)$$

where $\pi_h : \Lambda^k \to \Lambda_h^k$ is some projection operator. The most interesting cases occur when $\Lambda_h^k$ are spanned by the Whitney forms and $\pi_h$ are the canonical projections. We will mostly be concerned with these cases.

# Historical digression: *R*- and combinatorial torsions

The torsion of a simplicial complex was introduced by W. Franz and K. Reidemeister in the 1930's, and it is today known as the *R*-torsion or the *Reidemeister-Franz torsion*. A closely related quantity is the combinatorial torsion

$$\tau_h^2 = \prod_k (\det \Delta_{k,h}^+)^{(-1)^{k+1}k} := \prod_k \left( \prod_{\lambda_{k,j}>0} \lambda_{k,j} \right)^{(-1)^{k+1}k},$$

where $(\lambda_{k,j})_j$ are the eigenvalues of the discrete Laplacian $\Delta_{k,h} = \mathrm{d}_h^* \mathrm{d} + \mathrm{d}\mathrm{d}_h^*$ on $\Lambda_h^k \otimes V$, with $V$ some vector space. We have

$$\log\det \Delta_{k,h}^+ = \sum_{\lambda_{k,j}>0} \log \lambda_{k,j} = -\frac{\mathrm{d}}{\mathrm{d}s} \left( \sum_{\lambda_{k,j}>0} \lambda_{k,j}^{-s} \right)_{s=0} =: -\zeta'_{k,h}(s)\big|_{s=0},$$

so

$$\log \tau_h = \sum_k \frac{(-1)^k k}{2} \zeta'_{k,h}(0).$$

# Historical digression: analytic torsion

Let $(\mu_{k,j})_j$ be the eigenvalues of (the Friedrichs extension of) the Laplacian $\Delta_k = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$ on $\Lambda^k \otimes V$. The difficulty with extending the definition of the torsion to the Hodge-deRham complex is that the series

$$\zeta_k(s) = \sum_{\mu_{k,j}>0} \mu_{k,j}^{-s},$$

converges only in the half-plane $\mathrm{Re}(s) > \frac{n}{2}$. Nevertheless, it is known that $\zeta_k$ can be analytically continued to a meromorphic function on $\mathbb{C}$, that is analytic at $s=0$ provided $n$ is odd. So the definition

$$\log\tau = \sum_k \frac{(-1)^k k}{2} \zeta_k'(0),$$

of the Ray-Singer analytic torsion, introduced by R.B. Ray and I.M. Singer makes sense if $n$ is odd.

# Historical digression: Cheeger-Müller theorem

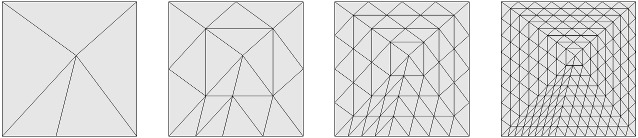
Ray and Singer conjectured in 1970 that the analytic and the *R*-torsions are one and the same, and it was proved independently by J. Cheeger and W. Müller around 1978.

The main step of Müller's proof consists of showing that the combinatorial torsion $\tau_h$ converges to the analytic torsion $\tau$ as $h \to 0$. For this, he used results previously proven in 1976 by J. Dodziuk and V.K. Patodi on convergence of the eigenvalues of the discrete Laplacian $\Delta_{k,h}$ to the eigenvalues of the continuous Laplacian $\Delta_k$.

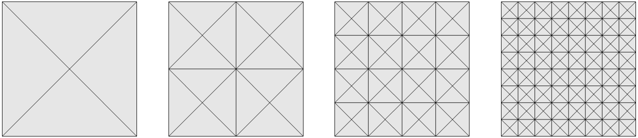
In the same paper Dodziuk and Patodi raised the question $\mathrm{d}_h^* \pi_h u \to \mathrm{d}^* u$? They offered a counterexample, but it was not valid as noted by L. Smits in 1991. Moreover, Smits answered the question in the affirmative for 1-forms in 2D, under regular standard subdivision (red refinement).

# The main results

We extend Smits' result to arbitrary dimensions, i.e., for 1-forms, $\mathrm{d}_h^* \pi_h u \to \mathrm{d}^* u$ as $h \to 0$ under a special type of refinements.



We also provide a counterexample, suggesting that the consistency does not hold unless the mesh possesses a special type of symmetry.



Numerical experiments suggest that the consistency is not true for $1 < k < n$, regardless of the mesh.

We reveal an equivalence between the consistency and a form of superconvergence. Our theoretical results depend on this equivalence.

# Upper bound on the consistency error

We have

$$\|\mathrm{d}^* u - \mathrm{d}_h^* \pi_h u\| \le \|\mathrm{d}^* u - P_h \mathrm{d}^* u\| + \|P_h \mathrm{d}^* u - \mathrm{d}_h^* \pi_h u\|,$$

where $P_h : L^2\Lambda^{k-1} \to \Lambda_h^{k-1}$ is the $L^2$-orthogonal projection. Defining $w = P_h \mathrm{d}^* u - \mathrm{d}_h^* \pi_h u \in \Lambda_h^k$, we have

$$\|w\|^2 = \langle P_h \mathrm{d}^* u - \mathrm{d}_h^* \pi_h u, w\rangle = \langle u - \pi_h u, \mathrm{d}w\rangle,$$

hence

$$\|w\| = \frac{\langle u - \pi_h u, \mathrm{d}w\rangle}{\|w\|} \le \sup_{v_h \in \Lambda_h^{k-1}} \frac{\langle u - \pi_h u, \mathrm{d}v_h\rangle}{\|v_h\|} =: A_h(u),$$

implying that

$$\|\mathrm{d}^* u - \mathrm{d}_h^* \pi_h u\| \le \mathrm{dist}(\mathrm{d}^* u, \Lambda_h^{k-1}) + A_h(u).$$

We can try to bound $A_h(u)$ as

$$|\langle u - \pi_h u, dv_h\rangle| \le \|u - \pi_h u\| \|dv_h\| \le Ch^\alpha \|u\|_{H^\ell\Lambda^k} \|dv_h\| \le Ch^{\alpha-1} \|u\|_{H^\ell\Lambda^k} \|v_h\|,$$

but we have $\alpha = 1$ for Whitney forms. Note $k = 0$ and $k = n$ cases are ok.

# Lower bound on the consistency error

We can also get a lower bound in terms of the same quantity

$$A_h(u) = \sup_{v_h \in \Lambda_h^{k-1}} \frac{\langle u - \pi_h u, \mathrm{d} v_h \rangle}{\| v_h \|}.$$

For any $v_h \in \Lambda_h^{k-1}$, we have

$$\frac{\langle u - \pi_h u, \mathrm{d} v_h \rangle}{\| v_h \|} = \frac{\langle \mathrm{d}^* u - \mathrm{d}_h^* \pi_h u, v_h \rangle}{\| v_h \|} \le \| \mathrm{d}^* u - \mathrm{d}_h^* \pi_h u \|,$$

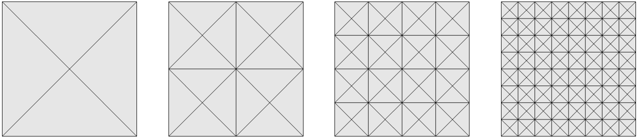
implying that $A_h(u) \le \| \mathrm{d}^* u - \mathrm{d}_h^* \pi_h u \|$.

To summarize, suppose that $\mathrm{dist}(\mathrm{d}^* u, \Lambda_h^{k-1}) \to 0$ as $h \to 0$. Then

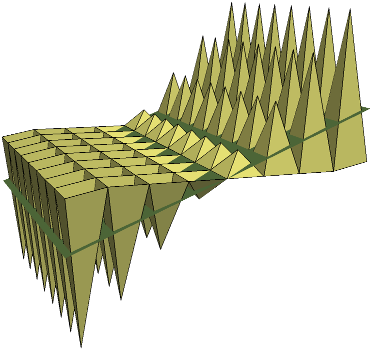
$$\| \mathrm{d}^* u - \mathrm{d}_h^* \pi_h u \| \to 0 \qquad \Longleftrightarrow \qquad A_h(u) \to 0.$$

# A counterexample

Take $M = [-1,1]^2$, and $u = (1-x^2)\mathrm{d}x$. We have $\mathrm{d}^* u = 2x$. Mesh sequence:

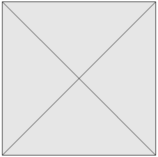


Green: $\mathrm{d}^* u$. Yellow: $\mathrm{d}_h^* \pi_h u$ for $h = 1/4$.



# A class of examples

Consider the following square $Q$, with center $p$ and edge length $2\ell$.



For $\psi$ a quadratic polynomial, and $\phi_p$ the hat function at $p$, we have

$$\langle \mathrm{d}\psi - \pi_h \mathrm{d}\psi, \mathrm{d}\phi_p \rangle = \frac{2}{3}\ell^2 \Delta\psi = \frac{1}{6}\int_Q \Delta\psi.$$

Let $N_h'$ be the set of vertices that are centres of the cubes of type $Q$.

$$\langle \mathrm{d}\psi - \pi_h \mathrm{d}\psi, \mathrm{d}v \rangle \equiv \sum_{p\in N_h'} \langle \mathrm{d}\psi - \pi_h \mathrm{d}\psi, \mathrm{d}\phi_p \rangle = \frac{1}{6}\sum_{p\in N_h'} \int_{\mathrm{supp}\phi_p} \Delta\psi \geq Ch^2 |N_h'| \Delta\psi.$$

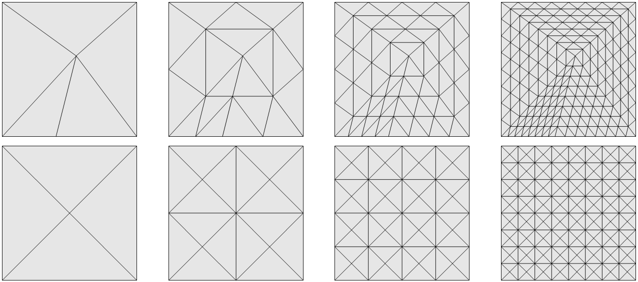
This argument is due to [R. Durán, M.A. Muschietti, R. Rodríguez 1991]

# Uniform triangulations

## Definition (Brandts and Křížek 2003)

A triangulation $T$ on $M$ is called uniform if there exist $n$ linearly independent vectors $e_1, \ldots, e_n$, such that

1. Every simplex in $T$ contains an edge parallel to each $e_j$.
2. If an edge $e$ is parallel to one of the $e_j$ and is not contained in $\partial M$, then the union $P_e$ of simplices containing $e$ is invariant under reflection through the midpoint $m_e$ of $e$, i.e., $2m_e - x \in P_e$ for all $x \in P_e$.



# Application of superconvergence theory

### Theorem (Brandts and Křížek 2003)

*Let $\{T_h\}$ be a shape regular family of uniform triangulations of $M$, and let $u$ be a smooth $1$-form. Then there exists a constant $C > 0$ such that*

$$|\langle \pi_h u - u, \mathrm{d}v_h \rangle| \le Ch^2 \|u\|_{H^\ell \Lambda^1} \|\mathrm{d}v_h\|,$$

*for all $v_h \in \Lambda_h^0 \cap H_0^1(M)$ and $h > 0$.*

### Corollary

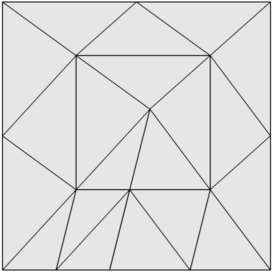
*If in addition $\{T_h\}$ is quasiuniform, then we have $A_h(u) \le Ch\|u\|_{H^\ell \Lambda^1}$.*

### Corollary

*We can relax uniformity to piecewise uniformity, and still get $A_h(u) \sim \sqrt{h}$ as $h \to 0$. So if $u \in H^\ell \Lambda^1(M)$ is a 1-form in the domain of $\mathrm{d}^*$, then*

$$\|\mathrm{d}^* u - \mathrm{d}_h^* \pi_h u\| \to 0 \qquad \text{as} \quad h \to 0.$$

# Superconvergence



$$\langle \pi_h u - u, \nabla v_h \rangle = \sum_{j=1}^{n} \langle \pi_h u - u, \nabla_j v_h \rangle = \sum_{j=1}^{n} \sum_{e \in E_j} \alpha_e \langle \pi_h u - u, \phi_e \rangle$$

$$F_e(u) := \langle \pi_h u - u, \phi_e \rangle = \int_{\mathrm{supp}\phi_e} (\pi_h u - u) \cdot \phi_e$$

$F_e(u) = 0$ for constant vector fields, and for linear vector fields.

# Conclusion

- We have consistency for 1-forms for mesh sequences satisfying certain symmetry properties.
- There is a large class of mesh sequences for which the consistency does not hold for 1-forms.
- Numerical experiments suggest that the consistency is not true for $1 < k < n$, regardless of the mesh.
- We reveal an equivalence between the consistency and a form of superconvergence.

Paper:


- D. Arnold, R. Falk, J. Guzmán, G. Tsogtgerel. On the consistency of the combinatorial codifferential. arXiv:1212.4472