Partial regularity results for generalized alpha models of turbulence

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Outline

Generalized alpha models

Basic results

Katz-Pavlović result and its extension

Further directions

Consider a closed manifold and the Leray projector P on it. Let $s - \Delta s = u$.

NS	$\partial_t u = \Delta u - P(u \cdot \nabla u)$
hyperviscous	$\partial_t u = \Delta^2 u - P(u \cdot \nabla u)$
Leray-α	$\partial_t u = \Delta u - P(s \cdot \nabla u)$
modified Leray-α	$\partial_t u = \Delta u - P(u \cdot \nabla s)$
Simplified Bardina	$\partial_t u = \Delta u - P(s \cdot \nabla s)$
NS-Voight	$\partial_t u = \Delta s - P(s \cdot \nabla s)$
NS-α	$\partial_t u = \Delta u - P(s \times \nabla \times u)$
NS-ω	$\partial_t u = \Delta u - P(u \times \nabla \times s)$
Clark-α	$\partial_t u = \Delta u - P(s \cdot \nabla u + u \cdot \nabla s - s \cdot \nabla s - \nabla(\nabla s \cdot \nabla s^T))$
MHD	$\partial_t \begin{pmatrix} u \\ h \end{pmatrix} = \Delta \begin{pmatrix} u \\ h \end{pmatrix} + P \begin{pmatrix} u \\ h \end{pmatrix}^T \begin{pmatrix} - \cdot \nabla & \cdot \nabla \\ \cdot \nabla & - \cdot \nabla \end{pmatrix} \begin{pmatrix} u \\ h \end{pmatrix}$

Generalized model:

$$\partial_t u = Au + B(u, u) \qquad \text{with} \quad B(u, v) = B_0(Mu, Nv), \qquad \langle B_0(u, v), v \rangle = 0$$

General model

$$\partial_t u = Au + B(u,u) \qquad \text{with} \quad B(u,v) = B_0(M_1 u, M_2 v), \qquad \langle B_0(u,v), v\rangle = 0$$

V a linear space of smooth (tensor) fields, e.g. divergence free fields.
$A : V \to V$ dissipation operator, e.g. $A = \Delta^\theta$
$B_0 : V \times V \to V$ bilinear structure, e.g. $B_0(u,v) = u \cdot \nabla v$ or $u \times \nabla \times v$
$M_i : V \to V$ smoothing operators, e.g. $M_i = (I - \Delta)^{-\theta_i}$

Under certain conditions

- existence of a global weak solution (e.g. $\theta + \theta_1 > \frac{1}{2}$)
- global regularity (e.g. $4\theta + 4\theta_1 + 2\theta_2 > n + 2$)
- inviscid limits, and $\alpha \to 0$ limits
- finite dimensionality of the flow

Partial regularity for $4\theta + 4\theta_1 + 2\theta_2 < n + 2$?

Katz-Pavlović idea

In $\mathbb{R}^n$ consider

$$\partial_t u = Au + B(u,u)$$

where $A = \Delta^\theta$, and B is a bilinear Fourier multiplier.
Let P_j be Littlewood-Paley projectors, and let

$$\tilde{P}_j = \sum_{k=j-2}^{j+2} P_k, \qquad \text{so that} \qquad P_j \tilde{P}_j = P_j.$$

Fix $\varepsilon > 0$. If λ is a ball of radius $2^{\varepsilon j} 2^{-j}$, let $\phi_\lambda \in C_0^\infty(2\lambda, [0,1])$ satisfy $\phi_\lambda \equiv 1$ on λ, and define $P_\lambda = \phi_\lambda P_j$. We have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|P_\lambda u\|^2 = \langle P_\lambda A u, P_\lambda u\rangle + \langle P_\lambda B(u,u), P_\lambda u\rangle$$

Growth estimates

For all large j, and a sufficiently large "neighbourhood" Λ of λ

$$\int_0^t \sum_{\mu\in\Lambda} \langle P_\mu A u, P_\mu u\rangle \lesssim -2^{2\theta j} \int_0^t \sum_{\mu\in\Lambda} \|P_\mu u\|^2 + O(2^{-Nj})$$

Thinking of a Leray-α type model, we have

$$\|P_\lambda B(u,u)\| \lesssim \sum_{k\geq j} 2^{(1-2\theta_1)k+\frac{n}{2}j} \|\phi_\lambda P_k u\|^2 + \sum_{\delta j\leq k\leq j+2} 2^{(\frac{n}{2}-2\theta_1)k+j} \|\phi_\lambda P_k u\|^2$$
$$+ 2^{\delta' j} \sum_{k\leq j} \|P_k u\|^2$$

Corresponding estimates for dyadic models are

$$\langle P_\lambda A u, P_\lambda u\rangle \lesssim -2^{2\theta j} \|P_\lambda u\|^2, \qquad \|P_\lambda B(u,u)\| \lesssim 2^{(\frac{n}{2}+1-2\theta_1)j} \sum_{j-1\leq k\leq j+1} \|\phi_\lambda P_k u\|^2$$

The "critical regularity" is $P_\lambda u \sim 2^{(2\theta+2\theta_1-\frac{n}{2}-1)j}$

Dyadic model

Fix a constant $h>0$, and call λ *hopeless* if

$$\|P_\lambda u\| > h2^{(2\theta+2\theta_1-\frac{n}{2}-1-\varepsilon)j}$$

A point $x\in\mathbb{R}^n$ is hopeless at level j if it is in some hopeless ball of radius $2^{\varepsilon j}2^{-j}$. Let E be the set of points that are hopeless at infinitely many levels. Then the Hausdorff dimension of E is at most $n+2+\varepsilon-4\theta-4\theta_1$.
($n+2+\varepsilon-4\theta-4\theta_1-2\theta_2$ for the general model)
If λ is not hopeless

$$\frac{\mathrm{d}}{\mathrm{d}t}\|P_\lambda u\|^2 \lesssim -2^{2\theta j}\|P_\lambda u\|^2 + 2^{(2\theta-\varepsilon)j}\sum_{j-1\le k\le j+1}\|\phi_\lambda P_k u\|^2,$$

and one can show that u is regular inside λ (roughly speaking).

Further directions

Ladyzhenskaya's μ model

$$\partial_t u = \operatorname{div} F(D) + B(u,u), \qquad D = \nabla u + \nabla u^T, \qquad F(D) \sim |D|^{2\mu+1}.$$

- Weak solution for $\mu \geq \frac{n-2}{2(n+2)}$, global regularity for $\mu \geq \frac{n-2}{4}$ [Ladyzhenskaya].
- For some models in 3D, weak solution for $\mu \leq \frac{1}{2}$, global regularity for $\mu \geq \frac{1}{10}$ [Malek et al].
- For a dyadic model in 3D, Hausdorff dim of the space singular set is at most $\frac{1-10\mu}{1-2\mu}$ [Friedlander-Pavlović].

Space-time singular set

- For NS, the parabolic Hausdorff dim < 1 [CKN]
- For μ model, it is ≤ 3 [Seregin]