Convergence rates of adaptive boundary element methods

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Boundary integral equations

Boundary integral equations are rooted in the works of Gauss, August Beer, and Carl Neumann on reformulations of the Dirichlet problem as integral equations involving the single- and double layer potentials.

There are many ways to convert (interior or exterior) boundary value problems for Ω into an integral equation

$$Au = f \quad \text{on} \quad \Gamma = \partial\Omega.$$

Typically, A has a singular kernel, $A : H^t(\Gamma) \to H^{-t}(\Gamma)$ is self-adjoint and bounded, and satisfies

$$\langle Au, u \rangle \geq \alpha \|u\|_t^2,$$

with $\alpha > 0$ and $t \in \{0, \pm\frac{1}{2}\}$. In particular, A is invertible.

Adaptive boundary element methods

For a triangulation T of Γ, let $S = S(T)$ be the space of piecewise constant functions on Γ subordinate to T. Then the Galerkin approximation $u_T \in S$ of u from the subspace $S \subset H^t$ $\left(t < \frac{1}{2}\right)$ is the solution of

$$\langle A u_T, v\rangle = \langle f, v\rangle, \qquad \forall v \in S.$$

Local *a posteriori* error indicators, $\eta(T, \tau)$, are supposed to measure how much error the triangle τ contains, e.g., $\|u - u_T\|_{t,\tau}$. We need a parameter $0 < \theta < 1$, and an initial triangulation T_0. Then we repeat the following for $k = 0, 1, \ldots$.

- Compute $u_k = u_{T_k}$, and the error indicators $\eta(T_k, \tau)$, $\tau \in T_k$.
- Choose a minimal subset $R \subset T_k$, such that

$$\sum_{\tau \in R} \eta(T_k, \tau)^2 \geq \theta \sum_{\tau \in T_k} \eta(T_k, \tau)^2.$$

- Refine (at least) all triangles in R, to get T_{k+1}.

The questions

Important questions that should be addressed:

- Quality of the error indicators: $\sum_{\tau \in T} \eta(T,\tau)^2 \sim \|u_T - u\|^2$?
- Efficient solution of the linear system, including the matrix assembly
- Convergence to the true solution: $u_k \to u$?
- Geometric error reduction: $\|u_k - u\| \le c\rho^k$ with some $\rho < 1$?
- Convergence rate: Largest s for which $\|u_k - u\| \le c(\#T_k)^{-s}$
- Optimality: Would s stay the same if T_k was replaced by the best possible triangulation with the same $\#T_k$?

Along these lines, a very satisfactory theory has been developed for adaptive finite element methods.

Analysis of boundary element methods is more involved because of nonlocality.

Some prior work on *a posteriori* error indicators

Residual is equivalent to error: $\|r_T\|_{-t} \equiv \|f - Au_T\|_{-t} \sim \|u - u_T\|_t$. There is a localization issue for t fractional. Recall the Slobodeckij norm

$$|v|^2_{s,\omega} = \int_{\omega\times\omega} \frac{|v(x)-v(y)|^2}{|x-y|^{2+2s}} \mathrm{d}x\mathrm{d}y.$$

- Faermann '00-'02: for $-1 < t \leq 0$, global equivalence

$$\|r_T\|^2_{-t} \sim \sum_{z\in N_T} |r_T|^2_{-t,\omega(z)}.$$

- Carstensen, Maischak, Stephan '01: for $-1 < t \leq 0$, global upper bound

$$\|r_T\|^2_{-t} \lesssim \sum_{\tau\in T} h^{2(1-t)} |r_T|^2_{1,\tau}.$$

- Carstensen, Maischak, Praetorius, Stephan '04, Nochetto, von Petersdorff, Zhang '10: for $t > 0$, global upper bound

$$\|r_T\|^2_{-t} \lesssim \sum_{\tau\in T} h^{2t} |r_T|^2_{0,\tau}.$$

Results on *a posteriori* error indicators

Gantumur '11: Lower bounds and local results. Similar results were independently obtained for $t = -\frac{1}{2}$ by Feischl, Karkulik, Melenk, and Praetorius. Example of a local result for $t = 0$:

Lemma

Let T' be a refinement of T, and let $\gamma = \bigcup_{\tau \in T \setminus T'} \tau$. Then we have

$$\alpha \| u_T - u_{T'} \| \leq \| r_T \|_\gamma \leq \beta \| u_T - u_{T'} \| + 2 \| r_T - v \|_\gamma$$

for any function $v \in S_{T'}$.

Proof of the first inequality.

Let $v = u_{T'} - u_T$, and let $v_T \in S_T$ be the L^2-orthogonal projection of v onto S_T. Then we have

$$\langle Av, v \rangle = \langle r_T, v \rangle = \langle r_T, v - v_T \rangle \leq \| r_T \|_\gamma \| v - v_T \|_\gamma \leq \| r_T \|_\gamma \| v \|_\gamma$$

where we have used that $v = v_T$ outside γ. □

Oscillation

The second inequality $\|r_T\|_\gamma \le \beta\|u_T - u_{T'}\| + 2\|r_T - v\|_\gamma$.

Let $v \in S_{T'}$ be supported in γ. Then we have

$$\|v\|_\gamma^2 = \langle v, v\rangle = \langle v - r_T, v\rangle + \langle A(u_{T'} - u_T), v\rangle \le \left(\|v - r_T\|_\gamma + \|A(u_{T'} - u_T)\|_\gamma\right)\|v\|_\gamma$$

implying that $\|r_T\|_\gamma \le \|r_T - v\|_\gamma + \|v\|_\gamma \le 2\|r_T - v\|_\gamma + \|A(u_{T'} - u_T)\|$. $\square$

Suppose r_T is piecewise H^r. Then

$$\inf_{v \in S_{T'}} \|r_T - v\|_\gamma^2 \le C_J^2 \sum_{\tau \in T \setminus T'} h_\tau^{2r} |r_T|_{r,\tau}^2.$$

Define

$$\operatorname{osc}(T, \omega) := \left(\sum_{\tau \in T, \tau \subset \omega} h_\tau^{2r} |f - Au_T|_{r,\tau}^2\right)^{\frac{1}{2}},$$

for $\omega \subseteq \Gamma$ and $v \in S_T$, so that we have

$$\alpha\|u_T - u_{T'}\| \le \|r_T\|_\gamma \le \beta\|u_T - u_{T'}\| + 2C_J \operatorname{osc}(T, \gamma).$$

Some other works on convergence analysis

Symm's integral equation ($t = -\frac{1}{2}$).

- Ferraz-Leite, Ortner, Praetorius '10: With $\tilde{T}$ the uniform refinement of T, use error estimators of the type

  $$\eta(T,\tau) = h_\tau^{1/2} \| u_T - u_{\tilde{T}} \|_\tau .$$

  Assume saturation (1985-):

  $$\| u - u_{\tilde{T}} \| \le \alpha \| u - u_T \|, \qquad (\alpha < 1).$$

  Then $\| u - u_k \| \le C\rho^k$ with $\rho < 1$.

- Aurada, Ferraz-Leite, Praetorius '11: Estimator convergence $\sum_\tau \eta(T_k, \tau) \to 0$ without saturation.
- Feischl, Karkulik, Melenk, Praetorius '11: Weighted residual estimator from [CMS01], geometric error reduction and convergence rate, without saturation.

Geometric error reduction

Assume

$$\sum_{\tau \in T} h_\tau^{2r} |Av|_{r,\tau}^2 \le C_A \|v\|^2, \qquad v \in S_T,$$

for all admissible T. Let T, T' be admissible partitions with T' being a refinement of T, and let $\gamma = \bigcup_{\tau \in T \setminus T'} \tau$. Suppose, for some $\theta \in (0,1]$ that

$$\|r_T\|_\gamma^2 + \operatorname{osc}(T,\gamma)^2 \ge \theta \left(\|r_T\|_\Gamma^2 + \operatorname{osc}(T,\Gamma)^2 \right).$$

Then there exist constants $\delta \ge 0$ and $\rho \in (0,1)$ such that

$$\|u - u_{T'}\|^2 + \delta \operatorname{osc}(T',\Gamma)^2 \le \rho \left(\|u - u_T\|^2 + \delta \operatorname{osc}(T,\Gamma)^2 \right).$$

Proof sketch:

$$\|u - u_T\| \lesssim \|r\|_\Gamma \lesssim \|r\|_\gamma \lesssim \|u_T - u_{T'}\|.$$

$$\|u - u_T\|^2 = \|u_T - u_{T'}\|^2 + \|u - u_{T'}\|^2.$$

Convergence rates

We know $\|u-u_k\| \le C\rho^k$ with $\rho<1$. How fast does $\#T_k$ grow?
Define approximation classes

$$\mathscr{A}_s = \{u \in L^2 : \inf_{\#T \le N} \inf_{v \in S_T} \|u-v\| \le CN^{-s}\}.$$

It is known that $W^{2s,p} \subset \mathscr{A}_s$ with $\frac{1}{p} = s + \frac{1}{2}$, and that $W^{2s,p}$ is much larger than H^{2s}, and friendlier to solutions of BVP and BIE.

Define $\mathscr{A}_{r,s}$ by replacing $\|u-v\|$ with $\|u-v\| + \mathrm{osc}$. We expect $\mathscr{A}_{r,s}$ to be close to $\mathscr{A}_s$.

Assume

$$\sum_{\tau \in T} h_\tau^{2r} |Av|_{r,\tau}^2 \le C_A \|v\|^2, \qquad v \in S_T,$$

for all admissible T. Let $\theta \in (0,\theta^*)$. Let f be piecewise H^r in the initial triangulation, and $u \in \mathscr{A}_{r,s}$ for some $s>0$. Then

$$\|u-u_k\| \le C|u|_{A_{r,s}} (\#T_k)^{-s}.$$

Inverse-type inequalities

$$\sum_{\tau\in T} h_\tau^{2r} |Av|_{r,\tau}^2 \le C_A \|v\|^2, \qquad v \in S_T.$$

If $A = I$ or multiplication by a smooth function, then it is the standard inverse inequality. Validity of this inequality depends on how A shifts low frequencies to high frequencies locally, and how it moves frequencies around in space. We decompose $L^2 = S_T \oplus H_T$ and correspondingly, $Av = (Av)_S + (Av)_H$. The low frequency component poses no problem:

$$\sum_{\tau\in T} h_\tau^{2r} |(Av)_S|_{r,\tau}^2 \lesssim \|(Av)_S\|^2 \le \|Av\|^2 \lesssim \|v\|^2.$$

For each triangle $\tau \in T$, we decompose v as $v = v_\tau + (v - v_\tau)$, where v_τ is the part of v near τ. Then the high frequency component of Av locally decomposes into near-field interactions and far-field interactions:

$$(Av)_H|_\tau = (Av_\tau)_H|_\tau + (A(v - v_\tau))_H|_\tau.$$

For boundary integral operators, the far-field part is harmless, and the near-field part is ok if the underlying surface is regular (e.g., $C^{1,1}$).

Further developments

The inverse-type inequalities for polyhedral surfaces and for the 4 standard BIOs have been proved by Aurada, Feischl, Führer, Karkulik, Melenk, and Praetorius in 2012.

I speculate that wavelet techniques can be adapted to prove the same result.

It should also be possible to characterize the approximation classes.

Open problems

- to characterize the approximation classes associated to the proposed adaptive BEMs
- to extend the analysis to transmission problems, and adaptive FEM-BEM coupling
- complexity analysis, i.e., the problem of quadrature and linear algebra solvers
- convergence rate for adaptive BEMs based on non-residual type error estimators