Optimal adaptive wavelet methods

for linear operator equations

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Overview

- Linear operator equation $Au = g$ with $A : \mathcal{H} \to \mathcal{H}'$
- Riesz basis $\Psi = \{\psi_\lambda\}$ of $\mathcal{H}$, e.g. $u = \sum_\lambda \mathbf{u}_\lambda \psi_\lambda$
- Infinite dimensional matrix-vector system $\mathbf{Au} = \mathbf{g}$, with $\mathbf{u} = (\mathbf{u}_\lambda)_\lambda$ and $\mathbf{A} : \ell_2 \to \ell_2$
- Convergent iterations such as $\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \alpha[\mathbf{g} - \mathbf{Au}^{(i)}]$
- We can approximate $\mathbf{Au}^{(i)}$ by a finitely supported vector
- How cheap can we compute this approximation?
- The answer will depend on A and Ψ

Outline

Continuous problem, discretization, and convergent iterations

Linear operator equations

Discretization

Convergent iterations in discrete space

Complexity analysis

Uniform methods - convergence, complexity

Nonlinear approximation

Optimal complexity

Computability

An adaptive Galerkin method

Optimal complexity with coarsening

Optimal complexity without coarsening

Linear Operator Equations

- Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{H}'$ be its dual
- $A : \mathcal{H} \to \mathcal{H}'$ is boundedly invertible
- $g \in \mathcal{H}'$ is a linear functional

Problem

$$u \in \mathcal{H} \text{ is such that } Au = g$$

- For $v \in \mathcal{H}$ and $h \in \mathcal{H}'$, $\langle h, v \rangle = h(v)$ the duality pairing

Sobolev Spaces

- Let Ω be an n-dimensional domain or smooth manifold
- $\mathcal{H} = H^t \subset H^t(\Omega)$ is a closed subspace
- $\mathcal{H}' = H^{-t}$ the dual space

Linear Differential Operators

- Partial differential operators of order $2t$

$$\langle Au, v\rangle = \sum_{|\alpha|,|\beta|\le t} \langle a_{\alpha\beta}\partial^{\beta} u, \partial^{\alpha} v\rangle,$$

- Example: The reaction-diffusion equation ($t = 1$)

$$\langle Au, v\rangle = \int_{\Omega} \nabla u \cdot \nabla v + \kappa^2 uv,$$

Boundary Integral Operators

- Boundary integral operators

$$\langle Au, v\rangle = \int_\Omega \int_\Omega v(x)K(x,y)u(y)d\Omega_y d\Omega_x$$

  with the kernel $K(x,y)$ singular at $x = y$

- Example: The single layer operator for the Laplace BVP in 3-d domain ($t = -\frac{1}{2}$)

$$K(x,y) = \frac{1}{4\pi|x-y|}$$

Convergent Iterations in Continuous Space

Gradient Iterations

$$u^{(i+1)} = u^{(i)} + B_i(g - Au^{(i)}), \qquad B_i : \mathcal{H}' \to \mathcal{H}$$

- $u - u^{(i+1)} = u - u^{(i)} - B_iA(u - u^{(i)}) = (I - B_iA)(u - u^{(i)})$
- $\|u - u^{(i+1)}\|_{\mathcal{H}} \leq \|I - B_iA\|_{\mathcal{H}\to\mathcal{H}}\|u - u^{(i)}\|_{\mathcal{H}}$

Convergence

$$\rho_i := \|I - B_iA\|_{\mathcal{H}\to\mathcal{H}} < 1$$

Normal Equations

Observation

Let $R : \mathcal{H}' \to \mathcal{H}$ be self-adjoint: $\langle Rf, h\rangle = \langle f, Rh\rangle$ for $f, h \in \mathcal{H}'$

and $\mathcal{H}'$-elliptic: with some $\alpha > 0$ $\langle Rf, f\rangle \geq \alpha \|f\|_{\mathcal{H}}^2$ for $f \in \mathcal{H}'$.

Then $A'RA : \mathcal{H} \to \mathcal{H}'$ is self-adjoint and $\mathcal{H}$-elliptic.

Normal Equation

$$Au = g \qquad \Longrightarrow \qquad A'RAu = A'Rg$$

Assumption

A is self-adjoint and $\mathcal{H}$-elliptic.

Riesz bases

$\Psi = \{\psi_\lambda : \lambda \in \nabla\}$ is a Riesz basis for $\mathcal{H}$
– each $v \in \mathcal{H}$ has a unique expansion

$$v = \sum_{\lambda\in\nabla} d_\lambda(v)\psi_\lambda \quad \text{s.t.} \quad c\|v\|_{\mathcal{H}}^2 \le \sum_{\lambda\in\nabla} |d_\lambda(v)|^2 \le C\|v\|_{\mathcal{H}}^2$$

- $d_\lambda \in \mathcal{H}'$ and $d_\lambda(\psi_\mu) = \delta_{\lambda\mu}$
- $\{d_\lambda : \lambda \in \nabla\}$ is a Riesz basis for $\mathcal{H}'$
- $\tilde{\Psi} = \{\tilde{\psi}_\lambda\} := \{d_\lambda\}$ is the dual basis: $\langle \tilde{\psi}_\lambda, \psi_\mu \rangle = \delta_{\lambda\mu}$

For $v \in \mathcal{H}$, we have $\mathbf{v} = \{\mathbf{v}_\lambda\} := \{d_\lambda(v)\} \in \ell_2(\nabla)$

Wavelet bases

- Ψ Riesz basis for $\mathcal{H} = H^t$
- Nested index sets $\nabla_0 \subset \nabla_1 \subset \ldots \subset \nabla_j \subset \ldots \subset \nabla$,
- $\mathcal{S}_j = \mathrm{span}\{\psi_\lambda : \lambda \in \nabla_j\} \subset \mathcal{H}$ and
  $\tilde{\mathcal{S}}_j = \mathrm{span}\{\tilde{\psi}_\lambda : \lambda \in \nabla_j\} \subset \mathcal{H}'$

Locality, Polynomial exactness and Vanishing moments

$$\mathrm{diam}(\mathrm{supp}\,\psi_\lambda) = \mathcal{O}(2^{-j}) \text{ if } \lambda \in \nabla_j \setminus \nabla_{j-1}$$

All polynomials of degree $d-1$, $P_{d-1} \subset \mathcal{S}_0$

$$P_{\tilde{d}-1} \subset \tilde{\mathcal{S}}_0 \text{ more precisely, } \langle P_{\tilde{d}-1}, \cdot \rangle_{L_2} \subset \tilde{\mathcal{S}}_0$$

- $\{\mathcal{S}_j\}$ has a good approximation property
- If $\lambda \in \nabla \setminus \nabla_0$, we have $\langle P_{\tilde{d}-1}, \psi_\lambda \rangle_{L_2} = 0 \rightsquigarrow$ cancellation property

Equivalent Discrete Problem

[Cohen, Dahmen, DeVore '02]

- Wavelet basis $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$
- Stiffness $\mathbf{A} = \langle A\psi_\lambda, \psi_\mu\rangle_{\lambda,\mu}$ and load $\mathbf{g} = \langle g, \psi_\lambda\rangle_\lambda$

Linear equation in $\ell_2(\nabla)$

$$\mathbf{Au} = \mathbf{g}, \qquad \mathbf{A} : \ell_2(\nabla) \to \ell_2(\nabla) \text{ SPD and } \mathbf{g} \in \ell_2(\nabla)$$

- $u = \sum_\lambda \mathbf{u}_\lambda \psi_\lambda$ is the solution of $Au = g$
- $\|\mathbf{u} - \mathbf{v}\|_{\ell_2(\nabla)} \eqsim \|u - v\|_{\mathcal{H}}$ with $v = \sum_\lambda \mathbf{v}_\lambda \psi_\lambda$
- A good approx. of $\mathbf{u}$ induces a good approx. of u
- Ψ defines a topological isomorphism between $\mathcal{H}$ and $\ell_2(\nabla)$

Convergent Iterations in Discrete Space

Richardson's iterations

$\mathbf{u}^{(0)} = \mathbf{0}$

$\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \alpha[\mathbf{g} - \mathbf{A}\mathbf{u}^{(i)}] \qquad i = 0, 1, \ldots$

- $\mathbf{u} - \mathbf{u}^{(i+1)} = \mathbf{u} - \mathbf{u}^{(i)} - \alpha\mathbf{A}(\mathbf{u} - \mathbf{u}^{(i)}) = (\mathbf{I} - \alpha\mathbf{A})(\mathbf{u} - \mathbf{u}^{(i)})$
- $\|\mathbf{u} - \mathbf{u}^{(i+1)}\|_{\ell_2} \leq \|\mathbf{I} - \alpha\mathbf{A}\|_{\ell_2 \to \ell_2} \|\mathbf{u} - \mathbf{u}^{(i)}\|_{\ell_2}$

Convergence

$$\rho := \|\mathbf{I} - \alpha\mathbf{A}\|_{\ell_2 \to \ell_2} < 1$$

- $\mathbf{g}$ and $\mathbf{A}\mathbf{u}^{(i)}$ are infinitely supported
- Approximate them by finitely supported sequences

Approximate Iterations

Approximate right-hand side

$$\mathbf{RHS}[\mathbf{g}, \varepsilon] \rightarrow \mathbf{g}_\varepsilon \text{ satisfies } \|\mathbf{g} - \mathbf{g}_\varepsilon\|_{\ell_2} \leq \varepsilon$$

Approximate application of the matrix

$$\mathbf{APPLY}[\mathbf{A}, \mathbf{v}, \varepsilon] \rightarrow \mathbf{w}_\varepsilon \text{ satisfies } \|\mathbf{A}\mathbf{v} - \mathbf{w}_\varepsilon\|_{\ell_2} \leq \varepsilon$$

Approximate Richardson's iterations

$$\tilde{\mathbf{u}}^{(0)} = \mathbf{0}$$

$$\tilde{\mathbf{u}}^{(i+1)} = \tilde{\mathbf{u}}^{(i)} + \alpha \left(\mathbf{RHS}[\mathbf{g}, \varepsilon_i] - \mathbf{APPLY}[\mathbf{A}, \tilde{\mathbf{u}}^{(i)}, \varepsilon_i]\right) \qquad i = 0, 1, \ldots$$

- Choose ε_i such that $\|\mathbf{u}^{(i)} - \tilde{\mathbf{u}}^{(i)}\| \eqsim \|\mathbf{u} - \mathbf{u}^{(i)}\|$

Convergence

$\mathbf{RICHARDSON}[\tilde{\mathbf{u}}^{(0)}, \varepsilon_{\text{fin}}] \to \tilde{\mathbf{u}}^{(i)}$

`for` $i = 0, 1, \ldots$

  $\varepsilon_i := C\rho^i$`;` $\tilde{\mathbf{r}}^{(i)} := \mathbf{RHS}[\mathbf{g}, \varepsilon_i] - \mathbf{APPLY}[\mathbf{A}, \tilde{\mathbf{u}}^{(i)}, \varepsilon_i]$

  `if` $\|\tilde{\mathbf{r}}^{(i)}\|_{\ell_2} + 2\varepsilon_i \leq \varepsilon_{\text{fin}}$ `then terminate;`

  $\tilde{\mathbf{u}}^{(i+1)} := \tilde{\mathbf{u}}^{(i)} + \alpha\mathbf{r}^{(i)}$

`endfor`

Lemma

$\mathbf{RICHARDSON}[\tilde{\mathbf{u}}^{(0)}, \varepsilon] \to \tilde{\mathbf{u}}$ terminates with $\|\mathbf{g} - \mathbf{A}\tilde{\mathbf{u}}\|_{\ell_2} \leq \varepsilon$

- Computational cost of $\mathbf{RICHARDSON}[\tilde{\mathbf{u}}^{(0)}, \varepsilon]$ depending on ε?

Uniform Refinement Galerkin Methods

- Wavelet basis $\Psi_j := \{\psi_\lambda : \lambda \in \nabla_j\}$ of $\mathcal{S}_j$
- Stiffness $\mathbf{A}_j = \langle A\psi_\lambda, \psi_\mu \rangle_{\lambda,\mu\in\nabla_j}$
- Load $\mathbf{g}_j = \langle g, \psi_\lambda \rangle_{\lambda\in\nabla_j}$

Linear equation in $\ell_2(\nabla_j)$

$$\mathbf{A}_j\mathbf{u}_j = \mathbf{g}_j, \qquad \mathbf{A}_j : \ell_2(\nabla_j) \to \ell_2(\nabla_j) \text{ SPD and } \mathbf{g}_j \in \ell_2(\nabla_j)$$

- $u_j = \sum_\lambda [\mathbf{u}_j]_\lambda \psi_\lambda \in \mathcal{S}_j$ approximates the solution of $Au = g$
- With the orthogonal projector $\mathbf{P}_j : \ell_2(\nabla) \to \ell_2(\nabla_j)$, the above equation is equivalent to $\mathbf{P}_j\mathbf{A}\mathbf{u}_j = \mathbf{P}_j\mathbf{g}$

Convergence and Complexity

If $u \in H^{t+ns}$ for some $s \in (0, \frac{d-t}{n}]$

$$\varepsilon_j := \|u - u_j\|_{H^t} \leq C \inf_{v \in \mathcal{S}_j} \|u - v\|_{H^t} \leq \mathcal{O}(2^{-jns})$$

- $N_j = \dim \mathcal{S}_j = \mathcal{O}(2^{jn})$
- $\varepsilon_j \leq \mathcal{O}(N_j^{-s})$
- Solve $\mathbf{A}_j \mathbf{u}_j = \mathbf{g}_j$ with Cascadic CG $\rightsquigarrow$ complexity $\mathcal{O}(N_j)$
- Similar estimates for FEM

Best N-term Approximation

Given $\mathbf{u} = (\mathbf{u}_\lambda)_\lambda \in \ell_2$, approximate $\mathbf{u}$ using N nonzero coeffs

$$\aleph_N := \bigcup_{\Lambda \subset \nabla : \#\Lambda = N} \ell_2(\Lambda)$$

- $\aleph_N$ is a nonlinear manifold
- Let $\mathbf{u}_N$ be such that $\|\mathbf{u} - \mathbf{u}_N\|_{\ell_2} \leq \|\mathbf{u} - \mathbf{v}\|_{\ell_2}$ for $\mathbf{v} \in \aleph_N$
- $\mathbf{u}_N$ is a best approximation of $\mathbf{u}$ with $\#\operatorname{supp} \mathbf{u}_N \leq N$
- $\mathbf{u}_N$ can be constructed by picking N largest in modulus coeffs from $\mathbf{u}$

Nonlinear vs. linear approximation

Nonlinear approximation

If $u \in B_{\tau}^{t+ns}(L_{\tau})$ with $\frac{1}{\tau} = \frac{1}{2} + s$ for some $s \in (0, \frac{d-t}{n})$

$$\varepsilon_N = \|\mathbf{u}_N - \mathbf{u}\| \leq \mathcal{O}(N^{-s})$$

Linear approximation

If $u \in H^{t+ns}$ for some $s \in (0, \frac{d-t}{n}]$, uniform refinement

$$\varepsilon_j = \|\mathbf{u}_j - \mathbf{u}\| \leq \mathcal{O}(N_j^{-s})$$

- H^{t+ns} is a proper subset of $B_{\tau}^{t+ns}(L_{\tau})$
- [Dahlke, DeVore]: $u \in B_{\tau}^{t+ns}(L_{\tau})$ much milder than $u \in H^{t+ns}$

Approximation spaces

- Approximation space $\mathcal{A}^s := \{\mathbf{v} \in \ell_2 : \|\mathbf{v} - \mathbf{v}_N\|_{\ell_2} \leq \mathcal{O}(N^{-s})\}$
- Quasi-semi-norm $|\mathbf{v}|_{\mathcal{A}^s} := \sup_{N \in \mathbb{N}} N^s \|\mathbf{v} - \mathbf{v}_N\|_{\ell_2}$
- $u \in B_\tau^{t+ns}(L_\tau)$ with $\frac{1}{\tau} = \frac{1}{2} + s$ for some $s \in (0, \frac{d-t}{n}) \Rightarrow \mathbf{u} \in \mathcal{A}^s$

Assumption

$$\mathbf{u} \in \mathcal{A}^s \text{ for some } s \in (0, \tfrac{d-t}{n})$$

Best approximation

$$\|\mathbf{u} - \mathbf{v}\| \leq \varepsilon \text{ satisfies } \#\operatorname{supp} \mathbf{v} \leq \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s}$$

Requirements on the Subroutines

Complexity of **RHS**

$\mathbf{RHS}[\mathbf{g}, \varepsilon] \to \mathbf{g}_\varepsilon$ terminates with $\|\mathbf{g} - \mathbf{g}_\varepsilon\|_{\ell_2} \leq \varepsilon$

- $\#\mathrm{supp}\, \mathbf{g}_\varepsilon \lesssim \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s}$
- flops, memory $\lesssim \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s} + 1$

Complexity of **APPLY**

For $\#\mathrm{supp}\, \mathbf{v} < \infty$

$\mathbf{APPLY}[\mathbf{A}, \mathbf{v}, \varepsilon] \to \mathbf{w}_\varepsilon$ terminates with $\|\mathbf{A}\mathbf{v} - \mathbf{w}_\varepsilon\|_{\ell_2} \leq \varepsilon$

- $\#\mathrm{supp}\, \mathbf{w}_\varepsilon \lesssim \varepsilon^{-1/s} |\mathbf{v}|_{\mathcal{A}^s}^{1/s}$
- flops, memory $\lesssim \varepsilon^{-1/s} |\mathbf{v}|_{\mathcal{A}^s}^{1/s} + \#\mathrm{supp}\, \mathbf{v} + 1$

Complexity of **RICHARDSON**

RICHARDSON$[\tilde{\mathbf{u}}^{(0)}, \varepsilon_{\text{fin}}] \to \tilde{\mathbf{u}}^{(i)}$
`for` $i = 0, 1, \ldots$
  $\varepsilon_i := C\rho^i$`;` $\tilde{\mathbf{r}}^{(i)} :=$ **RHS**$[\mathbf{g}, \varepsilon_i] -$ **APPLY**$[\mathbf{A}, \tilde{\mathbf{u}}^{(i)}, \varepsilon_i]$
  `if` $\|\tilde{\mathbf{r}}^{(i)} + 2\varepsilon_i\|_{\ell_2} \leq \varepsilon_{\text{fin}}$ `then terminate;`
  $\tilde{\mathbf{u}}^{(i+1)} := \tilde{\mathbf{u}}^{(i)} + \alpha \mathbf{r}^{(i)}$
`endfor`

Lemma

RICHARDSON$[\tilde{\mathbf{u}}^{(0)}, \varepsilon] \to \tilde{\mathbf{u}}$ terminates with $\|\mathbf{g} - \mathbf{A}\tilde{\mathbf{u}}\|_{\ell_2} \leq \varepsilon$.

- $\varepsilon_0 := \|\mathbf{u} - \tilde{\mathbf{u}}^{(0)}\|_{\ell_2}$
- $\#\text{supp}\, \tilde{\mathbf{u}} \lesssim$ $\varepsilon^{-1/s}|\mathbf{u}|_{\mathcal{A}^s}^{1/s} + \varepsilon_0^{-1/s}(\varepsilon_0/\varepsilon)^C|\mathbf{u}|_{\mathcal{A}^s}^{1/s} + \varepsilon^{-1/s}(\varepsilon_0/\varepsilon)^C|\tilde{\mathbf{u}}^{(0)}|_{\mathcal{A}^s}^{1/s}$
- flops, memory $\lesssim$ the same expression

Coarsening

$\mathbf{COARSE}[\mathbf{v}, \varepsilon] \to \mathbf{w}$
$\|\mathbf{v} - \mathbf{w}\| \leq \varepsilon$ and $\#\operatorname{supp} \mathbf{v}$ is minimal

Lemma

$$\theta < 1/2. \text{ Let } \|\mathbf{u} - \mathbf{v}\| \leq \theta\varepsilon. \ \mathbf{w} = \mathbf{COARSE}[\mathbf{v}, (1-\theta)\varepsilon] \text{ satisfies}$$
$$\#\operatorname{supp} \mathbf{w} \lesssim \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s} \text{ and } \|\mathbf{u} - \mathbf{w}\| \leq \varepsilon$$

Complexity with coarsening

$\mathbf{SOLVE}[\varepsilon_{\text{fin}}] \to \tilde{\mathbf{u}}^{(i)}$
$\tilde{\mathbf{u}}^{(0)} := 0$; $\varepsilon_0 := \|\mathbf{f}\|$
for $i = 0, 1, \ldots$
  $\varepsilon_{i+1} := \varepsilon_i/2$
  $\mathbf{v} := \mathbf{RICHARDSON}[\tilde{\mathbf{u}}^{(i)}, \theta\varepsilon_{i+1}]$
  $\tilde{\mathbf{u}}^{(i+1)} := \mathbf{COARSE}[\mathbf{v}, (1-\theta)\varepsilon_{i+1}]$
until $\varepsilon_{i+1} \leq \varepsilon_{\text{fin}}$

Theorem [Cohen, Dahmen, DeVore '02]

$\mathbf{SOLVE}[\varepsilon] \to \tilde{\mathbf{u}}$ terminates with $\|\mathbf{g} - \mathbf{A}\tilde{\mathbf{u}}\|_{\ell_2} \leq \varepsilon$.

- $\#\text{supp}\, \tilde{\mathbf{u}} \lesssim \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s}$
- flops, memory $\lesssim$ the same expression

Computing the Right Hand Side

Complexity of **RHS**

$\mathbf{RHS}[\mathbf{g}, \varepsilon] \to \mathbf{g}_\varepsilon$ terminates with $\|\mathbf{g} - \mathbf{g}_\varepsilon\|_{\ell_2} \leq \varepsilon$

- $\#\mathrm{supp}\, \mathbf{g}_\varepsilon \lesssim \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s}$
- flops, memory $\lesssim \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s} + 1$

A naive approach:

- Compute $\tilde{\mathbf{g}} = \langle g, \psi_\lambda \rangle_{\lambda \in \Lambda}$ for some $\Lambda \subset \nabla$ s.t. $\|\mathbf{g} - \tilde{\mathbf{g}}\| \leq \delta$
- Arrange the coeffs in $\tilde{\mathbf{g}}$ in modulus beforehand
- $\mathbf{RHS}[\mathbf{g}, \varepsilon] := \mathbf{COARSE}[\tilde{\mathbf{g}}, \varepsilon - \delta]$

The Subroutine **APPLY**

Computability

Matrix $\mathbf{A}$ is called q^*-computable, when for each N one can construct an infinite matrix $\mathbf{A}_N$ s.t.

- for any $q < q^*$, $\|\mathbf{A}_N - \mathbf{A}\| \leq \mathcal{O}(N^{-q})$
- having in each column $\mathcal{O}(N)$ non-zero entries
- whose computation takes $\mathcal{O}(N)$ operations

Theorem [Cohen, Dahmen, DeVore '01]

Recall $s \in (0, \frac{d-t}{n})$. Let $\mathbf{A}$ be q^*-computable with $q^* > s$. Then we can construct **APPLY** satisfying the requirements.

- $\mathbf{A}$ needs to be approximated well by computable sparse matrices

Compressibility

- Assume $A, A' : H^{t+\sigma} \to H^{-t+\sigma}$
- Level $|\lambda| := j$ such that $\lambda \in \nabla_j \setminus \nabla_{j-1}$
- $\|\psi_\lambda\|_{H^r} \eeqsim 2^{|\lambda|(r-t)}$ for $r \in [-\tilde{d}, \gamma)$, $\gamma := \sup\{q : \psi_\lambda \in H^q\}$
- $r \leq \min\{t + \tilde{d}, \sigma\}$ and $r < \gamma - t$, $|\mu| \geq |\lambda|$

$$\begin{aligned} |\langle A\psi_\lambda, \psi_\mu \rangle| &\leq \|A\psi_\lambda\|_{H^{-t+r}} \|\psi_\mu\|_{H^{t-r}} \lesssim \|\psi_\lambda\|_{H^{t+r}} \|\psi_\mu\|_{H^{t-r}} \\ &\lesssim 2^{-r(|\mu|-|\lambda|)} \end{aligned}$$

Theorem [Stevenson '04]

- $\{\psi_\lambda\}$ are piecewise polynomial wavelets that are sufficiently smooth and have sufficiently many vanishing moments
- A is either differential or singular integral operator
- any entry of $\mathbf{A}$ can be computed spending $\mathcal{O}(1)$ operations

then $\mathbf{A}$ is q^*-computable for some $q^* \geq \frac{d-t}{n}$ $\quad (> s)$

Computability

Unit cost assumption

Any entry of $\mathbf{A}$ can be computed spending $\mathcal{O}(1)$ operations

- Only satisfied for very special cases: differential operators with constant coefficients, single layer potential operator on $\mathbb{R}$
- Numerical quadrature is needed

Theorem [Gantumur, Stevenson '04, '05]

- $\{\psi_\lambda\}$ are piecewise polynomial wavelets that are sufficiently smooth and have sufficiently many vanishing moments
- A is either differential or singular integral operator

then $\mathbf{A}$ is q^*-computable for some $q^* \geq \frac{d-t}{n}$ $\quad (> s)$

Galerkin solutions

- $\langle\!\langle \cdot, \cdot \rangle\!\rangle := \langle \mathbf{A} \cdot, \cdot \rangle$ is an inner product on ℓ_2, $|\!|\!| \cdot |\!|\!| := \langle\!\langle \cdot, \cdot \rangle\!\rangle^{\frac{1}{2}}$ is a norm
- Let $\tilde{\mathbf{u}} \in \ell_2(\Lambda)$ be an approx. to $\mathbf{u}$ inside **SOLVE**
- $\mathbf{A}_\Lambda := \mathbf{P}_\Lambda \mathbf{A}|_{\ell_2(\Lambda)} : \ell_2(\Lambda) \to \ell_2(\Lambda)$, and $\mathbf{g}_\Lambda := \mathbf{P}_\Lambda \mathbf{g} \in \ell_2(\Lambda)$
- $\mathbf{u}_\Lambda \in \ell_2(\Lambda)$ is the solution to $\mathbf{A}_\Lambda \mathbf{u}_\Lambda = \mathbf{g}_\Lambda$

$$|\!|\!| \mathbf{u} - \mathbf{u}_\Lambda |\!|\!| = \inf_{\mathbf{v} \in \ell_2(\Lambda)} |\!|\!| \mathbf{u} - \mathbf{v} |\!|\!|$$

- In a sense, $\mathbf{u}_\Lambda$ is the best approx. from $\ell_2(\Lambda)$
- The next set $\tilde{\Lambda}$ generated by **SOLVE** can be too big, not optimal

Saturation

Galerkin orthogonality

$$\mathbf{u} - \mathbf{u}_\Lambda \perp_{\mathbf{A}} \ell_2(\Lambda)$$

Lemma

$\mu \in (0,1)$, $\mathbf{w} \in \ell_2$, *and* $\Lambda \supset \operatorname{supp} \mathbf{w}$ *s.t.*

$$\|\mathbf{P}_\Lambda(\mathbf{g} - \mathbf{A}\mathbf{w})\| \geq \mu \|\mathbf{g} - \mathbf{A}\mathbf{w}\|$$

Then we have

$$\|\mathbf{u} - \mathbf{u}_\Lambda\| \leq [1 - \kappa(\mathbf{A})^{-1}\mu^2]^{\frac{1}{2}} \|\mathbf{u} - \mathbf{w}\|$$

Adaptive Galerkin Method

$\mathbf{GROW}[\mathbf{w}] \to [\Lambda, \nu]$:
$\mathbf{r} := \mathbf{RHS}[\mathbf{g}, \zeta] - \mathbf{APPLY}[\mathbf{A}, \mathbf{w}, \zeta]$
$\nu := \|\mathbf{r}\| + 2\zeta$
determine a set $\Lambda \supset \operatorname{supp} \mathbf{w}$, with minimal cardinality, such that $\|\mathbf{P}_\Lambda \mathbf{r}\| \geq \mu \|\mathbf{r}\|$

$\mathbf{GALSOLVE}[\varepsilon] \to \mathbf{w}_k$:
$k := 0$; $\mathbf{w}_k := 0$
while *with* $[\Lambda_{k+1}, \nu_k] := \mathbf{GROW}[\mathbf{w}_k]$, $\nu_k > \varepsilon$ do
  Solve $\mathbf{A}_{\Lambda_{k+1}} \mathbf{w}_{k+1} = \mathbf{g}_{\Lambda_{k+1}}$
  $k := k + 1$
  if $k = 0 \pmod{K}$ then $\mathbf{w}_{k+1} = \mathbf{COARSE}[\mathbf{w}_{k+1}, \xi]$
enddo

Complexity

Theorem [Cohen, Dahmen, DeVore '01]

Let $k < \infty$ suitably chosen. $\mathbf{GALSOLVE}[\varepsilon] \to \mathbf{w}$ terminates with $\|\mathbf{g} - \mathbf{A}\mathbf{w}\|_{\ell_2} \leq \varepsilon$.

- $\#\mathrm{supp}\,\mathbf{w} \lesssim \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s}$
- flops, memory $\lesssim$ the same expression

Optimal expansion

Lemma [Gantumur, Harbrecht, Stevenson '05]

$\mu \in (0, \kappa(\mathbf{A})^{-\frac{1}{2}})$, $\mathbf{w} \in \ell_2$. Then the smallest set $\Lambda \supset \operatorname{supp} \mathbf{w}$ with

$$\|\mathbf{P}_\Lambda(\mathbf{g} - \mathbf{A}\mathbf{w})\| \geq \mu \|\mathbf{g} - \mathbf{A}\mathbf{w}\|$$

satisfies

$$\#(\Lambda \setminus \operatorname{supp} \mathbf{w}) \lesssim \|\mathbf{g} - \mathbf{A}\mathbf{w}\|^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s}$$

Optimal Complexity without Coarsening

Theorem [Gantumur, Harbrecht, Stevenson '05]

Let $K = \infty$. **GALSOLVE**$[\varepsilon] \to \mathbf{w}$ terminates with $\|\mathbf{g} - \mathbf{A}\mathbf{w}\|_{\ell_2} \leq \varepsilon$.

- $\#\operatorname{supp}\mathbf{w} \lesssim \varepsilon^{-1/s}|\mathbf{u}|_{\mathcal{A}^s}^{1/s}$
- flops, memory $\lesssim$ the same expression

Summary

- There exist asymptotically optimal fully discrete adaptive wavelet algorithms for solving linear operator equations.
- There exist adaptive Galerkin methods without coarsening of the intermediate iterands.

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