Contents

Introduction

The main topic of this paper will the study of the differential equation:

$$\begin{cases} \Delta u + g(u) = 0 & \text{in } \Omega & (0.1.1) \\ u = 0 & \text{on } \partial\Omega, & (0.1.2) \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^N : |x| < R\}$.

If $R = \infty$, the boundary condition naturally becomes $\lim\limits_{|x| \to \infty} u(x) = 0$. We will always assume that the function $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, and we of course look for nontrivial solutions.

We study this equation as an introduction to elliptical partial differential equations. This equation has connection with physics, and I will give several occurrences of this connection with examples from classical mechanics. We will study this equation in a different number of dimensions, and for different radii. We will discuss the properties of the solutions, the necessary and sufficient conditions for a solution to exist, and other interesting aspects. As another topic, we will also get into the existence and uniqueness of the Dirichlet energy problem towards the end, including a description of Sobolev spaces.

Differential Equation

1.1 The line

To start, consider the case of (0.1.1), (0.1.2) with N=1 ($N \equiv$ number of dimensions), and $R = \infty$. The problem becomes

$$\begin{cases} u''(x) + g(u(x)) = 0 \quad \text{for all x} \in \mathbb{R} & (1.1.1) \\ \lim\limits_{|x| \to \infty} u(x) = 0 & (1.1.2) \end{cases}$$

Comments 1.1.1 :

(i) Given $x_0,\ u_0,\ u_1 \in \mathbb{R}$, such that $u(x_0) = u_0,\ u'(x_0) = u_1$, there exists a unique solution to (1.1.1) on a maximal interval (a, b), a, b $\in [-\infty, \infty]$ ($x_0 \in (a, b)$ clearly). Also note that if $a, b \in (-\infty, \infty)$, then $|u(x)|+|u'(x)| \to \infty$ as $x \to a^+$, or $x \to b^-$, otherwise we could extend the maximal interval.

(ii) If u satisfies (1.1.1), and $u'(x_0) = 0$ and $g(u(x_0)) = 0$ for some $x_0 \in (a, b)$, then $u = u(x_0)\ \forall\ x \in (a, b)$.

(iii) If u satisfies (1.1.1) on (a, b), then

$$\frac{1}{2}u'(x)^2 + G(u(x)) = C \qquad (1.1.3)$$

where C is a constant, and

$$G(s) = \int_0^s g(t)\,dt\,, \quad s \in \mathbb{R}.$$

(iv) Given $x_0 \in \mathbb{R}$ and h>0, if u satisfies (1.1.1) on $(x_0 - h, x_0 + h)$, and $u'(x_0) = 0$, then u is symmetric about x_0.

(v) If u satisfies (1.1.1) on (a,b), and u' vanishes at least twice at some points $x_0,\ x_1 \in (a,b)$, then u exists on all of $\mathbb{R}$, and u is periodic with period $2|x_1 - x_0|$.

Proof : To prove (i), we use Banach fixed point theorem. First note that by rearranging (1.1.1) (and letting $g(u(t)) = g(t, u(t))$) and integrating, we get

$$u(x) = u_0 + (x - x_0)u_1 - \int_{x_0}^x \int_{x_0}^s g(t, u(t))\,dt\,ds\,.$$

Suppose $g : [x_0 - \epsilon, x_0 + \epsilon] \times [u_0 - \upsilon, u_0 + \upsilon] \to \mathbb{R}$, and let the bound be M. Now let $\Omega \equiv C([x_0 - \alpha, x_0 + \alpha]) \to C([u_0 - \upsilon, u_0 + \upsilon])$, where we will choose $\alpha \leq \epsilon$ later. Now let

$$T(u)(x) = u_0 + (x - x_0)u_1 - \int_{x_0}^x \int_{x_0}^s g(t, u(t))\,dt\,ds\,.$$

The first step is to show that if $u \in \Omega$, then $T(u) \in \Omega$. It is clear that if $u \in \Omega$, then $T(u)$ is differentiable, so also let $x \in [x_0 - \alpha, x_0 + \alpha]$. Then

$$|T(u)(x) - u_0| = |\int_{x_0}^x \int_{x_0}^s g(t, u(t))\,dt\,ds - (x - x_0)u_1|$$

$$\leq \left| \int_{x_0}^{x} \int_{x_0}^{s} g(t, u(t))\, dt\, ds \right| + |(x - x_0)u_1| \leq M\frac{\alpha^2}{2} + \alpha|u_1|.$$

Now choose $\alpha > 0$ such that $\alpha \leq \epsilon$ and $M\frac{\alpha^2}{2} + \alpha|u_1| \leq \upsilon$, then we find that $|T(u)(x) - u_0| \leq \upsilon$, and so $T(u)(x) \in [u_0 - \upsilon, u_0 + \upsilon]$, and thus $T(u) \in \Omega$.

We must now prove that T is a contraction mapping. Let $u, v \in \Omega, x \in [x_0 - \alpha, x_0 + \alpha]$.

Then

$$d(T(u), T(v)) = \sup_{x\in[x_0-\alpha,x_0+\alpha]} |T(u(x) - T(v(x))| = \sup_{x\in[x_0-\alpha,x_0+\alpha]} \left| \int_{x_0}^{x} \int_{x_0}^{s} g(u(t)) - g(v(t))\, dt\, ds \right|$$

$$\leq \sup_{x\in[x_0-\alpha,x_0+\alpha]} \int_{x_0}^{x} \int_{x_0}^{s} |g(u(t)) - g(v(t))|\, dt\, ds \;\leq \sup_{x\in[x_0-\alpha,x_0+\alpha]} \int_{x_0}^{x} \int_{x_0}^{s} k|u(t) - v(t)|\, dt\, ds$$

where the last step follows because g is locally Lipschitz continuous, and k is the Lipschitz constant. Now

$$\sup_{x\in[x_0-\alpha,x_0+\alpha]} \int_{x_0}^{x} \int_{x_0}^{s} k|u(t) - v(t)|\, dt\, ds \leq \sup_{x\in[x_0-\alpha,x_0+\alpha]} \int_{x_0}^{x} \int_{x_0}^{s} k\, d(u, v)\, dt\, ds = k\, d(u, v)\, \frac{\alpha^2}{2}.$$

If $k\frac{\alpha^2}{2} < 1$ we are done, if not, choose $\alpha > 0$ smaller so that it is. Then

$$d(T(u), T(v)) < \; d(u, v),$$

and so T is a contraction. Thus by Banach fixed point theorem, $T(u)$ admits a unique fixed point, and so we have a unique $u \in \Omega$ such that

$$u(x) = u_0 + (x - x_0)u_1 - \int_{x_0}^{x} \int_{x_0}^{s} g(u(t))\, dt\, ds\,,$$

and that proves (i).

Now for (ii), since $g(u(x_0)) = 0$, we have $u(x_0)'' + g(u(x_0)) = 0 + 0 = 0$, and since $u(x_0) = u_0,\ u'(x_0) = 0 = u(x_0)'$, we see that $u = u(x_0)$ satisfies (1.1.1) and all the given conditions, so from the uniqueness in (i), $u = u(x_0)\ \ \forall\, x \in (a, b)$.

Now for (iii), after multiplying (1.1.1) by u', we get

$$\frac{d}{dx}\Big(\frac{1}{2}u'(x)^2 + G(u(x))\Big) = u''u' + g(u)u' = 0,$$

$$\implies \frac{1}{2}u'(x)^2 + G(u(x)) = C.$$

For (iv), let $v(s) = u(x_0 + s),\ w(s) = u(x_0 - s),\ s \in (-h, h)$. Then it is easily seen that both v and w satisfy (1.1.1) on $(-h, h)$, that $v(0) = w(0)$, and that $v'(0) = 0 = w'(0)$. Therefore by uniqueness, v(s)=w(s), $\implies$ $u(x_0 - s) = u(x_0 + s)$, **ie**. u is symmetric about x_0.

Finally for (v), it follows from (iv) that u is symmetric about both x_0 and x_1. Suppose without loss of generality that $x_1 > x_0$ and take s small enough such that $x_0 - s,\ x_0 + s \in (a, b)$. Then we have

$$u(x_0{-}s) = u(x_0{+}s) = u(x_1{-}(x_1{-}(x_0{+}s))) = u(x_1{+}(x_1{-}(x_0{+}s))) = u(2x_1{-}(x_0{+}s))$$

$$\implies u(x_0 - s) = u(2x_1 - (x_0 + s)) \implies u(w) = u(w + 2|x_1 - x_0|).$$

Thus the period $T = 2|x_1 - x_0|$, and it follows that since u exists on (a, b) and is symmetric about $x_0,\ x_1$, u exists on $\mathbb{R}$. This completes the proofs. $\square$

Properties 1.1.2 : If $u \neq 0$ identically, and satisfies (1.1.1), (1.1.2), then the following hold:
(i) g(0)=0
(ii) Either u is always positive, or u is always negative.
(iii) u is symmetric about some $x_0 \in \mathbb{R}$, and $u'(x) \neq 0$ if $x \neq x_0$. Also, $|u(x-x_0)|$ is symmetric about 0 clearly, and is increasing for $x < 0$ and decreasing for $x > 0$.
(iv) For all $y \in \mathbb{R}$, $u(\cdot - y)$ satisfies (1.1.1), (1.1.2).

Proof : First off, suppose $g(0) \neq 0$. Then from (1.1.1) and (1.1.2), we see that

$$\lim_{|x|\to\infty} u''(x) = -\lim_{|x|\to\infty} g(u(x)) = -g(0) = C \neq 0$$

$$\implies \lim_{|x|\to\infty} u''(x) \neq 0$$

but then we can't have $u(x) \to 0$ as $|x| \to \infty$, so we must have g(0)=0, this proves (i).
Since $\lim\limits_{|x|\to\infty} u(x) = 0$, u can't be periodic, so by comment 1.1.1 (v), u' can have at most one zero on $\mathbb{R}$. It follows from intermediate value theorem, and mean value theorem, that u' must have a zero on on $\mathbb{R}$. This is because $u(x) \to 0$ as $x \to \pm\infty$, so by intermediate value theorem, there will be some $\epsilon > 0$ such that all points in $(0, \epsilon] \in \text{Ran}(u)$ for $x < 0$ (let $a \in (-\infty, 0)$ be such that $u(a) = \epsilon$), and all points in $(0, \epsilon] \in \text{Ran}(u)$ for $x > 0$ (let $b \in (0, \infty)$ be such that $u(b) = \epsilon$). So by mean value theorem

$$\frac{u(b) - u(a)}{b - a} = u'(\gamma), \quad \gamma \in (a, b)$$

$$\implies u'(\gamma) = 0.$$

So, from comment 1.1.1 (iv), u is symmetric about γ, and since $u'(x) \neq 0$ for $x \neq \gamma$, u must be either always positive on $\mathbb{R}$, or always negative on $\mathbb{R}$, because if it crossed the x-axis there would have to be another point where $u'(x) = 0$ since $\lim_{|x| \to \infty} u(x) = 0$. The same reasoning also prevents $u(x)$ from ever equaling zero. This proves (ii).
Since $u'(x)$ has only one zero at some γ and is symmetric about γ, if $u > 0$ on $\mathbb{R}$, then since $\lim_{x \to -\infty} u(x) = 0$, u must be increasing on the left side of of γ, which means it is decreasing to the right of it. If $u < 0$ on $\mathbb{R}$, then u must be decreasing to the left of γ, and increasing to the right of it. This implies (iii), and (iv) is obvious since clearly $u(x-y)$ still satisfies (1.1.1) and (1.1.2) if $u(x)$ does. $\square$

As a consequence of these properties, we only need to study even, positive or negative solutions, and we must assume $g(0) = 0$.

Preliminary result : Before we move on, we will prove a proposition that we will use in the next theorem.

Proposition 1.1.3 : Suppose g is (as always) locally Lipschitz continuous, that $g(0) = 0$, and that $\lim_{x \to \infty} u(x) = l \in \mathbb{R}$. Then $\lim_{x \to \infty} u'(x) = 0$.

Proof : First write

$$u'(s) = u'(x) + \int_x^s u''(t)\, dt$$

and by writing u in the same fashion, we get

$$u(x+1) - u(x) = \int_x^{x+1} u'(s)\, ds = u'(x) + \int_x^{x+1} \int_x^s u''(t)\, dt\, ds. \quad (1.1.4)$$

We have that $\lim_{x \to \infty} u(x) = l$, so by taking the limit of the left hand side of (1.1.4), we see that it goes to zero. Since $g(0) = 0$, it also follows from (1.1.1) that $\lim_{x\to\infty} u(''x) = 0$, and we can write

$$\int_x^{x+1}\int_x^s u''(t)\,dt\,ds \le \int_x^{x+1}\int_x^s (\sup_{t\in[x,s]} u''(t))\,dt\,ds = \frac{1}{2}\sup_{t\in[x,x+1]} u''(t).$$

where $\lim_{x\to\infty} \frac{1}{2}\sup_{t\in[x,x+1]} u''(t) = 0$, so it follows from (1) that $\lim_{x\to\infty} u'(x) = 0$.

Theorem 1.1.4 : Let $g : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous with g(0)=0. Then there exisits an even, positive solution of (1.1.1), (1.1.2) if and only if there exists a $u_0 > 0$ such that

$$g(u_0) > 0,\ G(u_0) = 0 \text{ and } G(u) < 0 \text{ for } 0 < u < u_0, \quad (1.1.5)$$

and such a solution is unique. Similary there exists an even, negative solutions of (1.1.1), (1.1.2) if and only if there exists a $v_0 < 0$ such that

$$g(v_0) < 0,\ G(v_0) = 0 \text{ and } G(u) < 0 \text{ for } v_0 < v < 0,$$

and such a solution is also unique.

Proof : We will prove the first statement, the second one is directly analogous.

First for the forward direction (showing (1.1.5) is necessary). Notice that if u solves (1.1.1), then $u \in C^2[(-\infty.\infty)]$, this follows because u must be twice differentiable, and since g is locally Lipschitz continuous, g is continous, so u'' is continuous. So from before, we have

$$\frac{1}{2}u'(x)^2+G(u(x)) = C \implies \frac{1}{2}u'(x)^2+G(u(x)) = \frac{1}{2}u'(x_0)^2+G(u(x_0)) \text{ for any } x_0 \in \mathbb{R}.$$

Taking $x_0 \to \infty$ and using that $u'(x_0) \to 0$ (Proposition (1.1.2)), and that

$G(u(x_0)) \to G(0) = 0$ as $x_0 \to \infty$, we see that

$$\frac{1}{2}u'(x)^2 + G(u(x)) = 0 \quad (1.1.6)$$

$\big($note that from the definition of G, it is clear that G(0)=0$\big)$. Since we are looking for even solutions u, $u'(0) = 0$, and so it follows that $G(u(0)) = 0$. Now $\sup\limits_{x \in \mathbb{R}} u(x) = u(0)$ clearly, since $u'(x)$ only has a zero at $x = 0$ (no other zeros from previous property), and since $\lim\limits_{|x| \to \infty} u(x) \to 0$ and u is positive by assumption, u(0) can't be a minumum.
Now we claim that g satisfies (1.1.5) with $u_0 = u(0)$. With this u_0, we have $G(u_0)$=0. Since $u'(x) \neq 0$ for $x \neq 0$, from (2) we have $G(u(x)) < 0 \; \forall x \neq 0$, so $G(u(x)) < 0$ for $0 < u < u_0$ (remember u is positive by assumption). Finally, since $u(x)$ is decreasing for $x > 0$, we have $u'(x) < 0$ for $x > 0$. Since $u'(0) = 0$, and $u'(x) < 0$ for $x > 0$, we must have

$$u''(0) \leq 0 \implies g(u_0) \geq 0 \text{ by (1.1.1).}$$

Now if $g(u_0) = 0$, then $u(x) = u_0$ solves (1.1.1) and satifies both of the initial conditions ($u(0) = u_0, \; u'(0) = 0$), so that $u(x) = u_0$ by uniqueness, but then (1.1.2) can't be met unless $u_0 = 0$, but this is the uninteresting trivial solution.

Now for the other direction, asssume g satisfies (1.1.5). Let $u_0 = u(0)$. Since $g(u(0)) > 0$, we have $u''(0) < 0$. So since $\frac{1}{2}u'(x)^2 + G(u(x)) = 0$ and $G(u(0) = 0$, we have $u'(0)$=0, so since $u''(0) < 0$, $u'(x) < 0$ for small $x > 0$. u' can't vanish for $x \neq 0$ while $u > 0$, otherwise we would have G(u(x))=0 for some $x \neq 0$ such that $0 < u(x) < u(0) = u_0$ by (1.1.5). Also there can't be an $x_0 > 0$ such that $u(x_0) = 0$, because then we would have $u'(x_0) = 0$, but then by property (ii), $u(x) = u(x_0) = 0$, an uninteresting solution. Therefore u is positive and decreasing for $x > 0$, and thus $\lim\limits_{x \to \infty} u(x) = l \in [0, u(0))$

exists. Now since, by taking the limit of (1.1.1), $u''(x) \to g(l)$ as $x \to \infty$, we must have $g(l) = 0$ for the limit of u to exist. From the prelimary result, this means that $u'(x) \to 0$ as $x \to \infty$. Taking $x \to \infty$ in (1.1.6), we see that $G(l) = 0$, and since $G(u) < 0$ for $0 < u < u(0)$ by assumption and $l \in [0, u(0))$, we see that $l = 0$. Also, by property (iv), u is even.
To conclude, we prove uniqueness of such a solution. Let u, v be positive, even solutions of (1.1.1), (1.1.2). From previous argument, g satisfies (1.1.4) with both $u_0 = u(0)$ and $u_0 = v(0)$. Since there exists only one such u_0 by (1.1.5), we have $u(0) = v(0)$, and since u and v are even, $0 = u'(0) = v'(0)$, so by uniqueness, $u = v$. $\square$

1.2 The case of the finite interval

Consider now the case N=1, $\Omega = (-R, R)$, $R < \infty$. The problem becomes

$$\begin{cases} u'' + g(u) = 0 & \text{in } \Omega \\ u(-R) = u(R) = 0 \end{cases}$$

As we shall see, the problem isn't as simple as in the case of the real line. We will state the new properties of the solution, and the necessary and sufficient conditions for existence and uniqueness of the solution. Note that everything from comments 1.1.1 still holds.

Comments 1.2.1 :

(i) If u satisfies (1.1.1) on some interval (a, b), $u > 0$ on (a, b), and $u(a) = u(b) = 0$, then u symmetric about $\frac{a+b}{2}$, and $u' > 0$ for all $x \in (a, \frac{a+b}{2})$. If instead $u < 0$ on (a, b), then u still is symmetric about $\frac{a+b}{2}$, but $u' < 0$ for all $x \in (a, \frac{a+b}{2})$.

(ii) If u satisfies (1.1.1) on (a, b), $u(a) = u(b) = 0$, and $u > 0$ on (a, b), then $g(\frac{a+b}{2}) > 0$. If instead $u < 0$ on (a, b), then $g(\frac{a+b}{2}) < 0$.

Note that it is implied that $a, b \in \mathbb{R}$.

Proof : We will prove it for the $u > 0$ on (a, b) case, the other case is directly analgous.
First off, $u(a) = u(b) = 0$, so Rolle's theorem dictates that there exists at least one $x_0 \in (a, b)$ such that $u'(x_0) = 0$, and by property (iv), u is symmetric about x_0. Suppose that $x_0 < \frac{a+b}{2}$, then $u(2x_0 - a) = u(a) = 0$ by symmetry about x_0, but $2x_0 - a \in (a, b)$ and we assumed $u > 0 \in (a, b)$, a contradiction. If $x_0 > \frac{a+b}{2}$ then similarly we find $u(2x_0 - b) = u(b) = 0$, but $2x_0 - b \in (a, b)$, a contradiction. So x_0 must equal $\frac{a+b}{2}$, and thus x_0 is the only zero of $u'(x)$, and u is symmetric about x_0. Since u' has only one zero which is at $x_0 = \frac{a+b}{2}$, $u(x_0)$ is either a global minumum or maximum. Since $u(x_0) > u(a) = u(b) = 0$, $u(x_0)$ is a global maximum, so that $u'(x) > 0 \ \forall \ x \in (a, \frac{a+b}{2})$. That proves (i). Now for (ii), we use the fact that $u(\frac{a+b}{2}) = \sup_{x \in (a,b)} u(x)$, which implies that $u''(\frac{a+b}{2}) \leq 0 \implies g(\frac{a+b}{2}) \geq 0$ from (1.1.1). Now if $g(\frac{a+b}{2}) = 0$, then due to uniqueness (property (ii)), $u(x) = u(\frac{a+b}{2}) \ \forall \ x \in (a, b)$. However since $u(a) = u(b) = 0$, we then must have $u = 0$ everywhere since u is constant, but this is the trivial solution. So, $g(\frac{a+b}{2}) > 0$, and this completes the proof. $\square$

Properties 1.2.2 : From comments 1.1.1, 1.2.1, we see that any nontrivial solution of (1.1.1), (1.2.1) must have the following properties:
(i) If u is always positive (or negative), then u is even and $|u(x)|$ is decreasing for $x \in (0, R)$ (by comment 1.2.1 (i)).

(ii) If u isn't either positive always, or negative always, then u' vanishes at least twice in Ω, and thus u is a periodic function in Ω (comment 1.1.1).

Theorem 1.2.3 : There exists a solution $u > 0$ of (1.1.1), (1.2.1) if and only if there exists a $u_0 > 0$ such that

(i) $g(u_0) > 0$
(ii) $G(u) < G(u_0)$ for all $0 < u < u_0$
(iii) either $G(u_0) > 0$ or else $G(u_0) = 0$ and $g(0) < 0$
(iv) $\int_0^{u_0} \frac{ds}{\sqrt{2(G(u_0)-G(s))}} = R$, (1.2.1)
and $u > 0$ implicitly defined by
$\int_{u(x)}^{u_0} \frac{ds}{\sqrt{2(G(u_0)-G(s))}} = |x|$ (1.2.2)
satisfies (1.1.1), (1.2.1), and any positive solution has this form for some $u_0 > 0$ satisfying (i), (ii).
A similar statement holds for $u < 0$.

Proof : We prove only the case $u > 0$, the proof of the second case is directly analogous.
Let's start with the forward direction, assume u is a solution to (1.1.1), (1.2.1). Let $u_0 = u(0)$. Then from comment 1.2.1 (ii), we have $g(u(\frac{a+b}{2})) > 0$, and in this case $\frac{a+b}{2} = \frac{-R+R}{2} = 0$, so we have $g(u_0) = g(u(0)) > 0$. So we have (i). By comment 1.2.1 (i), u is symmetric about 0, so $u'(0) = 0$. So we get that

$$\frac{1}{2}u'(x)^2 + G(u(x)) = G(u_0), \ \forall\, x \in (-R, R). \quad (1.2.3)$$

Since $u'(x) \neq 0$ for $x \neq 0$ (by comment 1.2.1 (i)), we get that

$$G(u(x)) = G(u_0) - \frac{1}{2}u'(x)^2 < G(u_0)\ \forall\, x \in (-R, R) \implies G(u) < G(u_0)\ \forall\, 0 < u < u_0.$$

Note that u_0 is a maxiumum of u on $(-R, R)$, and that we are assuming u is positive.

So we have (ii). Note that $u(R) = 0$, so $G(u(R)) = G(0) = 0$, so that from (1.2.3) we have $G(u_0) = \frac{1}{2}u'(R)^2 \geq 0$. Suppose that $G(u_0) = 0$. If $g(0) > 0$, then by continuity we can find an $\epsilon > 0$ such that $g(x) > 0$ on $(0, u(R-\epsilon))$ (note that $u(R-\epsilon)$ can be made arbituarily close to 0 by taking ϵ small), but then we have

$$G(u(R-\epsilon)) = \int_0^{u(R-\epsilon)} g(t)\, dt > 0 = G(u_0)$$

for $0 < u < u_0$, so (ii) can't hold. Now if $g(0) = 0$,, then by Theorem 1.1.4, u would solve (1.1.1), (1.1.2), and u couldn't vanish. So $g(0) < 0$, which proves (iii). Now from (1.2.3), we get

$$u'(x) = -\sqrt{2(G(u_0) - G(x)}$$

on $(0, R)$, (we take the negative root since $u' < 0$ on $(0, R)$, for the interval $(-R, 0)$, we take the positive root). Therefore, if we define a function F such that

$$F(x) = \int_x^{u_0} \frac{ds}{\sqrt{2(G(u_0) - G(s))}},$$

we get that $F'(u(x)) = \frac{-u'(x)}{\sqrt{2(G(u_0)-G(u(x))}} = 1$,

$$\implies \int_{u(x)}^{u_0} \frac{ds}{\sqrt{2(G(u_0) - G(s))}} = \int_0^x \frac{-du(x')}{\sqrt{2(G(u_0) - G(u(x'))}} = x.$$

For the interval $(-R, 0)$, we end up with

$$\int_{u(x)}^{u_0} \frac{ds}{\sqrt{2(G(u_0) - G(s))}} = \int_0^x \frac{-du(x')}{\sqrt{2(G(u_0) - G(u(x'))}} = -x \quad (x \in (-R, 0)),$$

so we have

$$\int_{u(x)}^{u_0} \frac{ds}{\sqrt{2(G(u_0) - G(s))}} = |x| \implies \int_0^{u_0} \frac{ds}{\sqrt{2(G(u_0) - G(s))}} = R$$

by letting $x = \pm R$, which gives us (iv).
Now for the other direction, suppose (i)-(iv) hold, and define u by (1.2.2). then from (iv), $u(R) = u(-R) = 0$. Now differetiating (1.2.2) and rearranging, we get

$$-u'(x) = \sqrt{2(G(u_0) - G(u(x)))} \implies \frac{1}{2}u'(x)^2 = G(u_0) - G(u(x)),$$

and after differentiating this last equation, we get

$$u'u'' = -u'g(u(x)),$$

and since $u' \neq 0$ everywhere, or else u would be constant, and thus $u = u(R) = 0$, which is the trivial solution. So we deduce that $u'' + g(u(x)) = 0$, and that any solution u has the form of (1.2.2) follows from step 1. $\square$

A Paradigm :

The differential equation representing an orbit of two spherical bodies (perhaps a comet orbiting a planet) is

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu k}{l^2}, \quad (1.2.4)$$

where $u = \frac{1}{r}$, $\mu = \frac{M_1M_2}{M_1+M_2}$, l is the angular momentum of the system (which is constant), and $k = GM_1M_2$ where r is the distance separating the centers of the spherical bodies, G is the gravitational constant, and M_1, M_2 are the masses of the two spherical bodies. Notice that this can be written as $u'' + g(u) = 0$, where $g(u) = u - \frac{\mu k}{l^2}$.
The solution to (1.2.4) is

$$u = \frac{1}{r} = \frac{\mu k}{l^2} + Acos\,\phi$$

for some constant A. Upon inspection of the solution, it is clear that the orbit is unbounded if and only if $\frac{\mu k}{l^2} \leq A$ (for then r must go to infinity). We will let $A = \frac{\mu k}{l^2}$, so $\frac{1}{r} = u(\phi) = \frac{\mu k}{l^2}(1 + cos\,\phi)$. Letting $p = \frac{l^2}{\mu k}$, we get that

$$pu = \frac{p}{r} = 1 + cos\,\phi \implies p = \sqrt{x^2 + y^2} + x \quad (r = \sqrt{x^2 + y^2},\ rcos\,\phi = x)$$

$$\implies p^2 - 2px - y^2 = 0$$

which is the equation of a parabola, and so our orbit is parabolic. Notice that $u(-\pi) = u(\pi) = 0$, and that $u(\phi) > 0$ for all $\phi \in (-\pi, \pi)$.

Comments 1.2.1 tell us that u is symmetric about 0, as it clearly is, and that $u' > 0$ for all $\phi \in (-\pi, 0)$, as it clearly is. Comments 1.2.1 also tell us that $g(0) > 0$, and since our $g(0) = \frac{\mu k}{l^2} > 0$, what the comments 1.2.1 tell us about this equation hold.

From properties 1.2.2 we see that since u is always positive in $(-\pi, \pi)$, then u must be even and $|u|$ decreasing in $(0, \pi)$, as it clearly is.

Now for Theorem (1.2.3). Note that $u > 0 \in (-\pi, \pi)$, and take $\frac{\mu k}{l^2} = 1$ for simplicity. Since we know the solution exists, everything in the theorem should hold. Take $u_0 = u(0) = 2$. Then $g(u_0) = 1 > 0$, and $G(u) < G(u_0)$ for $0 < u < u_0$ clearly, since $g(t) = t - 1$ is strictly increasing. Now $G(u_0) = \int_0^2 t - 1\,dt = 0$, and $g(0) = -1 < 0$. Now looking at (1.2.2) and making the substitution $s = 1 + cos\,\phi'$, we get

$$\int_{1+cos\,\phi}^{2} \frac{ds}{\sqrt{2(s - \frac{s^2}{2})}} = \int_0^{|\phi|} \frac{sin\,\phi'}{\sqrt{1 - cos^2\,\phi'}}\,d\phi' = \int_0^{|\phi|} d\phi' = |\phi|$$

and so (iv) holds, expectedly.

A Preliminary Physical Interpretation :

Consider a damped harmonic oscillator (a swinging pendulum, perhaps) with a damping force $-\frac{\dot{x}}{Q}$, where Q is the quality factor of the oscillator. The Energy is

$$E = T + V = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2$$

and the Lagrangian is

$$L = T - V = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2.$$

The equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = -\frac{\dot{x}}{Q}.$$

$$\Longrightarrow \ \ddot{x} + x = -\frac{\dot{x}}{Q}. \quad (1.2.5)$$

Now,

$$\frac{dE}{dt} = \dot{x}\ddot{x} + x\dot{x} = -\frac{\dot{x}^2}{Q},$$

as can be seen from (1.2.5), so the energy is monotone decreasing as would be expected due to the damping term.

1.3 The case of $\mathbb{R}^N, \ N \geq 2$

We now consider the case where $\Omega = \mathbb{R}^N, \ N \geq 2$. The problem becomes

$$\begin{cases} u'' + \frac{N-1}{r}u' + g(u) = 0 \quad \text{in } \mathbb{R}^N \quad (1.3.1) \\ \lim\limits_{r\to\infty} u(r) = 0 \end{cases}$$

for $r > 0$. We consider a general nonlinear case:

$$g(u) = -\lambda u + \mu |u|^{p-1} u,$$

now (1.3.1) becomes

$$u'' + \frac{N-1}{r} u' - \lambda u + \mu |u|^{p-1} u = 0. \quad (1.3.2)$$

Now after multiplying this last equation by r^{N-1}, we can write it as

$$(r^{N-1} u'(r))' = r^{N-1} (\lambda u(r) - \mu |u(r)|^{p-1} u(r)).$$

With the initial conditions $u(0) = u_0$, $u'(0) = 0$, we can integrate this last equation to get

$$u(r) = u_0 + \int_0^r s^{1-N} \int_0^s t^{N-1} (\lambda u(t) - \mu |u(t)|^{p-1} u(t)) \, dt \, ds. \quad (1.3.3)$$

which can be solved using the Banach fixed point method. As we can see, given the initial conditions $u(0) = u_0$, $u'(0) = 0$, there exists a unique maximal solution $u \in C^2([0, R_m))$ such that either $R_m = \infty$, or $|u(r)| + |u'(r)| \to \infty$ as $r \to R_m^-$. Now write

$$E(u) = \frac{1}{2} u'(r)^2 - \lambda u(r)^2 + \frac{\mu}{p+1} |u(r)|^{p+1}. \quad (1.3.4)$$

Notice this looks just like $E = T + V$ where V is a non-quadratic potential. After differentiating (1.3.4) with respect to r and multiplying (1.3.2) by u' and comparing the two, we see that

$$\frac{dE}{dr} = -\frac{N-1}{r} u'(r)^2, \quad (1.3.5)$$

so E(u) is a decreasing quantity due to the damping term $\frac{N-1}{r} u'$ (notice the strong analogy with the prelimanry physical interpretation).

Proposition 1.3.1 : If u is a solution of

$$\begin{cases} u'' + \frac{N-1}{r}u' + |u|^{p-1}u = 0 \quad (1.3.6) \\ u(0) = 1, \quad u'(0) = 0 \end{cases}$$

then the following properties hold:
(i) If $N \geq 3$ and $(N-2)p \geq N+2$, then $u(r) > 0$ and $u'(r) < 0$ for all $r > 0$, and $\lim_{r\to\infty} u(r) = 0$.
(ii) If $(N-2)p < N+2$, then u oscillates indefinitely, **ie**. for any $r_0 \geq 0$ such that $u(r_0) \neq 0$, $\exists\, r_1 > r_0$ such that $u(r_0)u(r_1) < 0$.

Proof : First off, note that it must be true that $u' < 0$ for small r, otherwise if $u' \geq 0$, this would mean that $u'' \geq 0$ (since $u'(0) = 0$ and $u'(r) \geq 0$ for small r), and then in (1) we would have 3 nonnegative terms for small r, with at least one of which is nonzero, so the three can't add to zero. Now suppose $u(r) > 0$. Suppose there exists a first r_1 such that $u'(r_1) = 0$, we would get $u''(r_1) < 0$ from (1), but this is impossible since for $r < r_1$, $u'(r) < 0$ ($u'(0^+) < 0$, and u' is continuous, and this is the first point where u' vanishes), so if $u'(r_1) = 0$, u' would have increased, so u'' would be positive. So $u' < 0$ while $u > 0$. Next, after multiplying (1) by u', we can see that

$$\left(\frac{u'^2}{2} + \frac{|u|^{p+1}}{p+1}\right)' = -\frac{N-1}{r}u'^2 \ \ (1.3.7).$$

Also, after multiplying (1.3.6) by r^{N-1}, we can see that

$$\left(r^{N-1}uu'\right)' + r^{N-1}|u|^{p+1} = r^{N-1}u'^2. \ \ (1.3.8)$$

Lastly, after multiplying (1) by $r^N u'$, we can see that

$$\left(\frac{r^N}{2}u'^2 + \frac{r^N}{p+1}|u|^{p+1}\right)' + \frac{N-2}{2}r^{N-1}u'^2 = \frac{N}{p+1}r^{N-1}|u|^{p+1}. \quad (1.3.9)$$

We first prove (i). Assume that u has a first zero r_0. By uniqueness we must have $u'(r_0) \neq 0$, otherwise $u = 0$ everywhere, but $u(0) = 1$ by assumption. So integrating (1.3.8) from 0 to r_0, we obtain

$$\int_0^{r_0} r^{N-1}u^{p+1}\,dr = \int_0^{r_0} r^{N-1}u'^2\,dr \quad (1.3.10)$$

(the first term in (1.3.8) vanishes when integrated since $u(r_0) = 0$ by assumption, and we can remove the absolute value sign around u since $u \geq 0$ in $[0, r_0]$). Integrating (1.3.9), we get

$$\frac{r_0^N}{2}u'(r_0)^2 + \frac{N-2}{2}\int_0^{r_0} r^{N-1}u'^2\,dr = \frac{N}{p+1}\int_0^{r_0} r^{N-1}u^{p+1}\,dr,$$

so we get

$$0 < \frac{r_0^N}{2}u'(r_0)^2 = \frac{N}{p+1}\int_0^{r_0} r^{N-1}u^{p+1}\,dr - \frac{N-2}{2}\int_0^{r_0} r^{N-1}u'^2\,dr = \left(\frac{N}{p+1} - \frac{N-2}{2}\right)\int_0^{r_0} r^{N-1}u'^2\,dr,$$

where the last step follows from (1.3.10). Now, by assumption $(N-2)p \geq N+2$,

$$\implies (N-2)(p+1) \geq 2N \implies \frac{N-2}{2} \geq \frac{N}{p+1},$$

so we get

$$0 < \frac{r_0^N}{2}u'(r_0)^2 = \left(\frac{N}{p+1} - \frac{N-2}{2}\right)\int_0^{r_0} r^{N-1}u'^2 \leq 0 \implies 0 < 0,$$

a contradiction. Thus $u > 0$ for all r, and so $u' < 0$ for all r, which means $\lim_{r\to\infty} u(r) = l \geq 0$ exists. Now since u' is negative always, it is bounded, otherwise a finite limit of u wouldn't exist. So from (1.3.6) we can see that $u'' \to l^p$ as $r \to \infty$, which implies that $l = 0$, otherwise the limit of u wouldn't exist. This proves (i).

Now for (ii), assume that $u > 0$ always, which means $u' < 0$ always. Then $u(r) \to l$ as $r \to \infty$, and as before, $l = 0$. Now we can multiply (1.3.8) by r^{N-1} and rewrite is as

$$(r^{N-1}u'(r))' = -r^{N-1}u(r)^p \implies r^{N-1}u'(r) = -\int_0^r s^{N-1}u(s)^p\,ds.$$

Now the right hand side of the last equation is clearly negative and decreasing (just look at the derivative), so we have

$$r^{N-1}u'(r) \leq -c < 0 \;\; \forall\, r \geq 1,$$

thus

$$u(r) \leq u(1) - c\int_1^r s^{-(N-1)}\,ds.$$

If $N = 2$, we have

$$0 < u(r) \leq u(1) - c\int_1^r \frac{ds}{s} = u(1) - c\ln r,$$

but this can't be true for large r (the right hand side will become negative since $\ln r$ is unbounded). Now suppose $N \geq 3$. We have

$$-r^{N-1}u'(r) = \int_0^r s^{N-1}u(s)^p\,ds \geq \inf_{s\in[0,r]} u(s)^p \int_0^r s^{N-1}\,ds = u(r)^p \int_0^r s^{N-1}\,ds = \frac{r^N}{N}u(r)^p,$$

then we see that

$$-u'(r)u(r)^{-p} \geq \frac{r}{N} \implies \left(\frac{1}{(p-1)u(r)^{p-1}}\right)' \geq \left(\frac{r^2}{2N}\right)'.$$

This last equality can be seen by simply carrying out the differentiation. So we have that

$$\frac{1}{(p-1)u(r)^{p-1}} \geq k\frac{r^2}{2N} \text{ for some } k \geq 1 \implies u(r) \leq \left(\frac{k}{2N}(p-1)r^2\right)^{\frac{-1}{p-1}}.$$

Now after absorbing the constants together, we get

$$u(r) \leq Cr^{-\frac{2}{p-1}} \text{ for some } C > 0.$$

From this, we see that

$$r^{N-1}u^{p+1} \leq Cr^{-\frac{2(p+1)}{p-1}}r^{N-1}$$

Now note that by assumption,

$$N+2 > (N-2)p \implies 2(p+1) > N(p-1) \implies 2\frac{p+1}{p-1} > N$$

$$\implies -2\frac{p+1}{p-1} < -N \implies -2\frac{p+1}{p-1} + (N-1) < -1.$$

So we get that

$$r^{N-1}u^{p+1} \leq Cr^{-\frac{2(p+1)}{p-1}+(N-1)} < Cr^{-1}$$

which means that

$$\int_\epsilon^\infty r^{N-1}u^{p+1}\,dr \leq C\int_\epsilon^\infty r^{-\frac{2(p+1)}{p-1}+(N-1)}\,dr$$

with $\epsilon > 0$, which converges since the exponent is less than -1. Now we can integrate (1.3.9) on (ϵ, r) and get

$$\frac{r^N}{2}u'(r)^2 + \frac{r^N}{p+1}|u(r)|^{p+1} - \epsilon^N\frac{u'(\epsilon)^2}{2} - \epsilon^N\frac{|u(\epsilon)|^{p+1}}{p+1} + \frac{N-2}{2}\int_\epsilon^r s^{N-1}u'^2\,ds$$

$$= \frac{N}{p+1}\int_{\epsilon}^{r} s^{N-1}u^{p+1}\,ds, \quad (1.3.11)$$

since the right hand side is finite, the left hand side is finite, and since the first two terms on the left are positive, this implies that

$$\int_{\epsilon}^{\infty} r^{N-1}u'(r)^2\,dr < \infty. \;\; (1.3.12)$$

Now(1.3.11) and (1.3.12) imply that there exists a sequence (r_n) with $r_n \to \infty$ such that

$$r_n^N u'(r_n)^2 + r_n^N u(r_n)^{p+1} \to 0, \quad (1.3.13)$$

this is because the integrands of (1.3.11), (1.3.12) are positive and since the integrals converge, for some (r_n) the integrands of (1.3.11), (1.3.12) must converge to zero faster than $\frac{1}{r_n}$ (since $\int_0^\infty \frac{1}{r}\,dr$ diverges), which implies (1.3.13). Letting $r = r_n$ in (1.3.11) and taking $n \to \infty$, we see that

$$-\epsilon^N\frac{u'(\epsilon)^2}{2} - \epsilon^N\frac{|u(\epsilon)|^{p+1}}{p+1} + \frac{N-2}{2}\int_{\epsilon}^{\infty} s^{N-1}u'^2\,ds = \frac{N}{p+1}\int_{\epsilon}^{\infty} s^{N-1}u^{p+1}\,ds. \;\; (1.3.14)$$

Now integrating (1.3.8) on (ϵ, r), we get that

$$r^{N-1}u(r)u'(r) - \epsilon^{N-1}u(\epsilon)u'(\epsilon) + \int_{\epsilon}^{r} s^{N-1}|u(s)|^{p+1}\,ds = \int_{\epsilon}^{r} s^{N-1}u'(s)^2\,ds. \;\; (1.3.14)$$

Now from (1.3.13), we can see that

$$|u'(r_n)| < \alpha r_n^{-\frac{N}{2}}, \;\; u(r_n) < \beta r_n^{-\frac{N}{p+1}}$$

for large n, where $\alpha, \beta > 0$ are constants. The above line must be true, otherwise clearly (1.3.13) wouldn't be true (the line would not go to zero). Therefore,

$$r_n^{N-1}u(r_n)|u'(r_n)| < \alpha\beta r_n^{N-1-\left(\frac{N}{2}+\frac{N}{p+1}\right)} \quad (1.3.15).$$

Now, $(N-2)p < N+2$ by assumption,

$$\implies -(N+2) < -(N+2)p \implies 2Np + 2N - 2p - 2 < Np + 3N$$

$$\implies N - 1 < \frac{Np+3N}{2(p+1)} = \frac{N}{2} + \frac{N}{p+1} \implies N - 1 - \frac{N}{2} + \frac{N}{p+1} < 0,$$

so from (1.3.15), we see that as $n \to \infty$, the right side goes to zero, so the left side also goes to zero.
So we see from (1.3.14) that

$$-\epsilon^{N-1}u(\epsilon)u'(\epsilon) + \int_\epsilon^\infty s^{N-1}|u(s)|^{p+1}\,ds = \int_\epsilon^\infty s^{N-1}u'(s)^2\,ds$$

$$0 = \frac{N}{p+1}\int_\epsilon^\infty s^{N-1}u^{p+1}\,ds + \epsilon^N\frac{u'(\epsilon)^2}{2} + \epsilon^N\frac{|u(\epsilon)|^{p+1}}{p+1} - \frac{N-2}{2}\int_\epsilon^\infty s^{N-1}u'^2\,ds \quad \text{(from (1.3.11))}$$

$$= \left(\frac{N}{p+1} - \frac{N-2}{2}\right)\int_\epsilon^\infty s^{N-1}u'^2\,ds + \epsilon^{N-1}u(\epsilon)u'(\epsilon)\frac{N}{p+1} + \epsilon^N\frac{u'(\epsilon)^2}{2} + \epsilon^N\frac{|u(\epsilon)|^{p+1}}{p+1}. \quad (1.3.16)$$

Now, by assumption $N+2 > (N-2)p$,

$$\implies 2N - (N-2) > (N-2)p \implies 2N > (N-2)(p+1) \implies \frac{N}{p+1} > \frac{N-2}{2},$$

so we have that

$$\frac{N}{p+1} - \frac{N-2}{2} = a$$

for some $a > 0$. So from (1.3.16), we have that

$$0 = \lim_{\epsilon\to 0}\left(a\int_\epsilon^\infty s^{N-1}u'^2\,ds + \epsilon^{N-1}u(\epsilon)u'(\epsilon)\frac{N}{p+1} + \epsilon^N\frac{u'(\epsilon)^2}{2} + \epsilon^N\frac{|u(\epsilon)|^{p+1}}{p+1}\right) \quad (1.3.17)$$

$$= \lim_{\epsilon\to 0} a\int_\epsilon^\infty s^{N-1}u'^2\,ds$$

since the other terms clearly go to zero. So finally, we have that

$$0 = \lim_{\epsilon \to 0} a \int_{\epsilon}^{\infty} s^{N-1} u'^2 \, ds > 0$$

since $a > 0$ and the integrand is positive, so we have that $0 > 0$, a contradiction. Actually, from (1.3.17) we can see that if r_0 is such that $u(r) \neq 0$ and $u'(r_0) = 0$, then there exists a $\gamma > r_0$ such that $u(\gamma) = 0$. We can see this from (1.3.17) by taking $\epsilon \to r_0$ instead of to zero as so:

$$0 = \lim_{\epsilon \to r_0} \left(a \int_{\epsilon}^{\infty} s^{N-1} u'^2 \, ds + \epsilon^{N-1} u(\epsilon) u'(\epsilon) \frac{N}{p+1} + \epsilon^N \frac{u'(\epsilon)^2}{2} + \epsilon^N \frac{|u(\epsilon)|^{p+1}}{p+1} \right)$$

$$= a \int_{r_0}^{\infty} s^{N-1} u'^2 \, ds \, + \, r_0^N \frac{|u(r_0)|^{p+1}}{p+1} > 0,$$

and we get the same $0 > 0$ contradiction. Lastly, we must show that if $\gamma > 0$ is such that $u(\gamma) = 0$, then there exists an $r > \gamma$ such that $u(r) \neq 0$, and $u'(r) = 0$. To see this, note that $u'(\gamma) \neq 0$ (otherwise $u \equiv 0$ by uniqueness), and suppose without loss of generality that $u'(\gamma) > 0$. If $u'(r) > 0 \;\; \forall \, r \geq \gamma$, then since u is bounded, $u(r) \to l > 0$ as $r \to \infty$, and so by the equation, $u''(r) \to -l^p$ as $r \to \infty$, which is impossible because then u wouldn't converge. So we have shown that, given any $u(r_0) \neq 0$, there exists a $r_1 > r_0$ such that $u(r_1) = 0$, and since u' can't vanish when u vanishes by uniqueness, u must cross the x-axis after each of its zeros, of which there are infinitely many. Thus u oscillates indefinitely, and this completes the proof. $\square$

Corollary 1.3.2 : Assume λ, $\mu > 0$ and $(N-2)p < N+2$. For any $\gamma > 0$ and any $n \in \mathbb{N}$, there exists $M_{n,\gamma}$ such that if $x_0 > M_{n,\gamma}$, then the solution u of (1.3.1) with the initial conditions $u(0) = x_0$, $u'(0) = 0$, has at least n zeroes on $(0, \gamma)$.

Proof : We would like to get the equation into a simpler form, namely

$$y'' + \frac{N-1}{r} y' - y + |y|^{p-1} y = 0\,. \ (1.3.18)$$

To do so, let $y(r) = \alpha u(\beta r)$. Then from (1.3.18), we have

$$\alpha\beta^2 u''(\beta r) + \alpha\beta \frac{N-1}{r} u'(\beta r) - \alpha u(\beta r) + \alpha^{p-1}\alpha |u(\beta r)|^{p-1} u(\beta r) = 0$$

$$\implies \alpha\beta^2 u''(\beta r) + \alpha\beta^2 \frac{N-1}{\beta r} u'(\beta r) - \alpha u(\beta r) + \alpha^{p-1}\alpha |u(\beta r)|^{p-1} u(\beta r) = 0$$

$$\implies u''(\beta r) + \frac{N-1}{\beta r} u'(\beta r) - \frac{1}{\beta^2} u(\beta r) + \frac{\alpha^{p-1}}{\beta^2} |u(\beta r)|^{p-1} u(\beta r) = 0.$$

We need to get this in the form $u''(\sigma) + \frac{N-1}{\sigma} u'(\sigma) - \lambda u(\sigma) + \mu |u(\sigma)|^{p-1} u(\sigma) = 0$ for (1.3.18) to be possible, and for this we see that we need to have

$$\beta = \lambda^{-\frac{1}{2}},\ \alpha = \left(\frac{\mu}{\lambda}\right)^{\frac{1}{p-1}},$$

and thus $y(r) = \left(\frac{\mu}{\lambda}\right)^{\frac{1}{p-1}} u(\lambda^{-\frac{1}{2}} r)$. Now, let $R > 0$ be such that the solution v of (1.3.6) has n zeroes on (0,R). Let $x > 0$ and let y be the solution of (1.3.18) such that $y(0) = x$, $y'(0) = 0$. Now note that

$$y''\Big(\frac{r}{x^{\frac{p-1}{2}}}\Big) + \frac{N-1}{\big(r/x^{\frac{p-1}{2}}\big)} y'\Big(\frac{r}{x^{\frac{p-1}{2}}}\Big) - y\Big(\frac{r}{x^{\frac{p-1}{2}}}\Big) + |y\Big(\frac{r}{x^{\frac{p-1}{2}}}\Big)|^{p-1} y\Big(\frac{r}{x^{\frac{p-1}{2}}}\Big) = 0$$

This implies that

$$\frac{1}{x}\frac{1}{x^{p-1}} y'' + \frac{1}{x}\frac{1}{x^{p-1}} \frac{N-1}{\big(r/x^{\frac{p-1}{2}}\big)} y' - \frac{1}{x}\frac{1}{x^{p-1}} y + \frac{1}{x}\frac{1}{x^{p-1}} |y|^{p-1} y = 0$$

$$\implies \frac{1}{x}\frac{1}{x^{p-1}} y'' + \frac{1}{x}\frac{1}{x^{\frac{p-1}{2}}} \frac{N-1}{r} y' - \frac{1}{x}\frac{1}{x^{p-1}} y + \frac{1}{x}\frac{1}{x^{p-1}} |y|^{p-1} y = 0$$

where the argument of y is $\frac{r}{x^{\frac{p-1}{2}}}$. Letting $w(r) = \frac{1}{x} y\left(\frac{r}{x^{\frac{p-1}{2}}}\right)$, we have

$$w''(r) + \frac{N-1}{r} w'(r) - \frac{1}{x^{p-1}} w(r) + |w(r)|^{p-1} w(r) = 0.$$

We see that as we take $x \to \infty$, our equation becomes (1.3.6), and so $w \to v$ in $C^1([0, R])$. Now since $v' \neq 0$ whenever $v = 0$, v crosses the x-axis and so for x large enough, say $x \geq x_n$, w has n zeros on $(0, R)$ which means that y has n zeros on $\left(0, \frac{R}{x^{\frac{p-1}{2}}}\right)$, and thus u has n zeros on $\left(0, \frac{\lambda^{-\frac{1}{2}}}{x^{\frac{p-1}{2}}} R\right)$. The result follows with $M_{n,\gamma} = \sup\left\{x_n, \left(\frac{\lambda^{-\frac{1}{2}}}{\gamma} R\right)^{\frac{2}{p-1}}\right\}$, since then $\left(0, \frac{\lambda^{-\frac{1}{2}}}{x^{\frac{p-1}{2}}} R\right) \subset (0, \gamma)$. $\square$

Lemma 1.3.3 : For every $c > 0$, there exists an $\alpha(c) > 0$ with the following property. If u is a solution of (1.3.1), and if $E(u) = -c < 0$, and $u(R) > 0$ for some $R \geq 0$, then $u(r) \geq \alpha(c)$ for all $r \geq R$.

Proof : Let $f(x) = \mu \frac{|x|^{p+1}}{p+1} - \lambda \frac{x^2}{2}$, $x \in \mathbb{R}$, $\lambda > 0$, $p > 1$. Notice that $f(0) = 0$, $f'(0) = 0$, $f''(0) < 0$, so $\inf f < 0$. Let $-m = \inf f$. Now, let $f(x) = -c$, $-c \in (-m, 0)$, so that

$$\mu \frac{|x|^{p+1}}{p+1} - \lambda \frac{x^2}{2} = -c. \quad (1.3.19)$$

The graph has the following form:

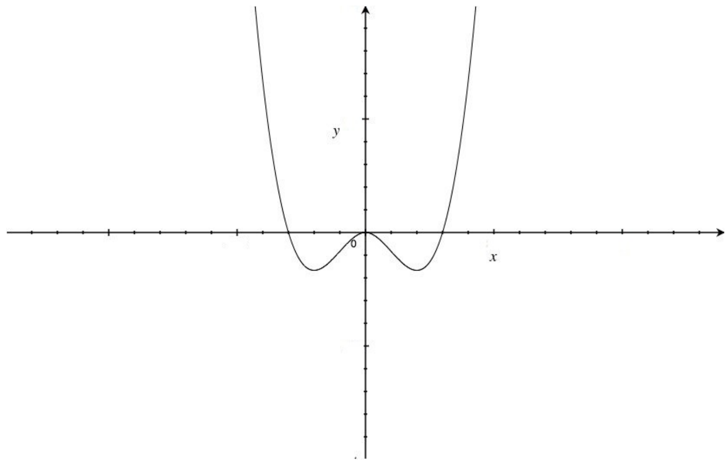

figure 1.3.20

Let the two positive solutions to (1.3.19) be $0 < \alpha(c) \leq \beta(c)$. It is clear from figure 1.3.20 that if $f(x) < -c$, then $x \in [-\beta(c), -\alpha(c)] \cup [\alpha(c), \beta(c)]$. Now, it is clear from the definitions that $f(u(r)) \leq E(u(r))$ for all r, and since by (1.3.5) $E(u(r))$ is always decreasing with respect to r and $E(u(R)) = -c$, it follows that $f(u(r)) \leq E(u(r)) \leq -c$ for all $r \geq R$. Since $u(R) > 0$, and $f(u(R)) \leq -c$, we must have $\alpha(c) \leq u(R) \leq \beta(c)$. Then since $f(u(r)) \leq -c$ for all $r \geq R$ and u is continuous, we must have $u(r) \geq \alpha(c)$ for all $r \geq R$. $\square$

Theorem 1.3.4 : Assume $\lambda,\ \mu > 0$ and $(N-2)p < N+2$. There exists $x_0 > 0$ such that the solution u of (1.3.6) with the initial conditions $u(0) = x_0$ and $u'(0) = 0$ is defined for all $r > 0$, is positive and decreasing. Moreover, there exists C such that

$$u(r)^2 + u'(r)^2 \leq Ce^{-2\sqrt{\lambda}r},$$

for all $r > 0$.

Proof : Let

$$A_0 = \{x > 0 : u > 0 \text{ on } (0, \infty)\},$$

where u is the solution of (1.3.6) with the initial values $u(0) = x$, u'(0)=0 Claim: $I = \left(0, \left(\frac{\lambda(p+1)}{2\mu}\right)^{\frac{1}{p-1}}\right) \subset A_0$, so that $A_0 \neq \emptyset$. Suppose $x \in I$, then $E(u(0)) < 0$ (which is clear by plugging these x into (1.3.5)), so by Lemma (1.3.3), $\inf_{r \geq 0} u(r) > 0$. Also, $A_0 \subset (0, M_{1,1})$ by Corollary 1.3.2 (because u doesn't have any zeros). Thus we may consider $x_0 = \sup A_0$; we will show that x_0 has the desired properties. Let u be the solution such that $u(0) = x_0$. Note that $x_0 \in A_0$, otherwise u has a first zero at some $r_0 > 0$. By uniqueness, $u'(r_0) \neq 0$, so that u takes on negative values. By continuous dependence, this is also the case for solutions with initial values close to x_0, which means that u has a zero for some $x < x_0$, a contradiction. Now, we have that $x_0 \geq \left(\frac{\lambda(p+1)}{2\mu}\right)^{\frac{1}{p-1}} > \left(\frac{\lambda}{\mu}\right)^{\frac{1}{p-1}}$ (since if x_0 was smaller than this, it wouldn't be the supremum of A_0). Now, if $x_0 > \left(\frac{\lambda}{\mu}\right)^{\frac{1}{p-1}}$, we have

$$x_0^{p-1} > \frac{\lambda}{\mu} \implies x_0^p > x_0\frac{\lambda}{\mu} \implies -\lambda x_0 + \mu x_0^p > 0,$$

thus by (1.3.6) we can't have $u''(0) > 0$, otherwise for small $r > 0$, we will have three positive terms, which couldn't add to zero. If $u''(0) = 0$, then by taking the limit of (1.3.6) (using that u'' is continous by (1.3.6)) as $r \to 0$, we see that $u'(0) < 0$, a contradiction. So $u''(0) < 0$, which means that $u'(0^+) < 0$. Now, u' can't vanish, for if it did, we would have some $r_0 > 0$ such that $u(r_0) > 0$, $u'(r_0) = 0$ and $u''(r_0) \geq 0$. This implies that $u(r_0) \leq \left(\frac{\lambda}{\mu}\right)^{\frac{1}{p-1}}$, which implies that $E(u(r_0)) < 0$. By continuous dependence, it follows that for some $r_1 > x_0$, we have $E(u(r_1)) < 0$, which implies that $r_1 \in A_0$ by Lemma 1.3.3. This contradicts the property that $x_0 = \sup A_0$. Thus $u'(r) < 0 \ \forall\, r > 0$. Let

$$m = \inf_{r \geq 0} u(r) = \lim_{r \to \infty} u(r) \geq 0,$$

we claim that $m = 0$. Suppose instead that $m > 0$, we see from the equation that (since u' is bounded)

$$0 = \lim_{r \to \infty} u''(r) = \lambda m - \mu m^p.$$

Thus either $m = 0$, or $m = \left(\frac{\lambda}{\mu}\right)^{\frac{1}{p-1}}$. In the latter case, since there exists a sequence (r_n) such that $\lim_{n \to \infty} u'(r_n) = 0$, we have that $\lim\inf_{r \to \infty} E(u(r)) < 0$ (since m must be less than $\left(\frac{\lambda(p+1)}{2\mu}\right)^{\frac{1}{p-1}}$ since $u(0)$ is less than this, and u is decreasing), but this is impossible by Lemma 1.3.3, thus $m = 0$.

Now to prove the exponential decay, let $v(r) = \left(\frac{\mu}{\lambda}\right)^{\frac{1}{p-1}} u(\lambda^{-\frac{1}{2}} r)$, so that v is a solution of (1.3.18). Set

$$f(r) = v(r)^2 + v'(r)^2 - 2v(r)v'(r) = (v(r) - v'(r))^2. \quad (1.3.20)$$

Since $v''(r) \to 0$ as $r \to \infty$, we have $v(r),\ v'(r) \to 0$ as $r \to \infty$, so for all $\epsilon > 0$ there exists $R > 0$ such that for $r > R$, $v(r)^2,\ v'(r^2) < \frac{\epsilon}{2}$, thus for large r, we must have $v(r)v'(r) < 0$, otherwise from (1.3.20) we have

$$0 \leq f(r) < \epsilon - \epsilon = 0,$$

a contradiction. So for all r large enough, $v(r)v'(r) < 0$, hence for all r_0 large enough with $r > r_0$, we have

$$f(r) \geq v(r)^2 + v'(r)^2. \quad (1.3.21)$$

Now carrying out the calculation, we find that

$$f'(r) + 2f(r) = -2\frac{N-1}{r}(v'^2 - vv') + 2|v|^{p-1}(v^2 - vv') \leq 2|v|^{p-1}(v^2 - vv') \leq 2|v|^{p-1} f,$$

$$\implies \frac{f'(r)}{f(r)} + 2 - 2|v|^{p-1} \leq 0, \quad (1.3.22)$$

and so for r_0 large enough,

$$\frac{d}{dr}\left(\ln(f(r)) + 2r - 2\int_{r_0}^{r} |v(r)|^{p-1}\, dr \right) \leq 0. \quad (1.3.23)$$

Now, since $v \to 0$ as $r \to \infty$, we can write

$$\ln(f(r)) \leq 2\int_{r_0}^{r} |v(r)|^{p-1}\, dr - 2r \leq 2\int_{r_0}^{r_1} |v(r)|^{p-1}\, dr + (r - r_0) - 2r = 2\int_{r_0}^{r_1} |v(r)|^{p-1}\, dr - r - r_0,$$

$$\implies \frac{d}{dr}\ln(f(r)) \leq \frac{d}{dr}\big(2\int_{r_0}^{r} |v(r)|^{p-1}\, dr - 2r\big) \leq \frac{d}{dr}\big(2\int_{r_0}^{r_1} |v(r)|^{p-1}\, dr - r - r_0\big)$$

$$\implies \frac{d}{dr}\ln(f(r)) \leq \frac{d}{dr}\big(2\int_{r_0}^{r_1} |v(r)|^{p-1}\, dr - r - r_0\big),$$

where r_1 is such that for $r > r_1$, $|v|^{p-1} < \frac{1}{2}$. Integrating, we see that for $r > r_1$,

$$\ln(f(r)) \leq K - r \implies f(r) \leq e^{K-r} = \alpha e^{-r}$$

for some constants K, α. For the interval $[r_0, r_1]$, since f is continuous on the compact interval, f achieves its maximum inside the interval and so there must exist a constant β such that $f(r) \leq \beta e^{-r}$ for all r in the interval. Let $C = \sup\{\alpha, \beta\}$, then $f(r) \leq Ce^{-r}$ for all $r \in [r_0, \infty)$. Applying this estimate in (1.3.21), we see that $|v(r)| \leq Ce^{-\frac{r}{2}}$. After plugging this estimate into (1.3.23), using the ML (maximum times length) inequality, and then integrating, we get

$$\ln(f(r)) \leq 2(r - r_0)Ce^{\frac{-(p-1)r}{2}} - 2r + K$$

for some constant K. The first term on the right of the inequality is bounded for nonnegative r, so we can bound it by a constant, and after absorbing constants together we have that

$$\ln(f(r)) \leq -2r \implies f(r) \leq Ce^{-2r}$$

where we have redefined C accordingly. Now using (1.3.21), and looking at the definition of $v(r)$, we have that $u(r)^2 + u'(r)^2 \leq Ce^{-2\sqrt{\lambda}r}$. $\square$

Theorem 1.3.5 : Assume $\lambda,\ \mu > 0$ and $(N-2)p < N+2$. There exists an increasing sequence (x_n) of positive numbers such that the solution un of (1.3.1) with the initial conditions $u_n(0) = x_n$ and $u_n(0) = 0$ is defined for all $r > 0$, has exactly n nodes, and satisfies for some constant C the estimate (1.3.7).

Lemma 1.3.6 : Let $n \in \mathbb{N}$, $x > 0$, and let u be the solution of (1.3.1) with the initial conditions $u(0) = x$ and $u'(0) = 0$. Assume that u has exactly n zeros on $(0, \infty)$ and that $u^2 + u'^2 \to 0$ as $r \to \infty$. There exists $\epsilon > 0$ such that if $|x - y| \leq \epsilon$, then the corresponding solution v of (1.3.1) has at most $n+1$ zeros on $(0, \infty)$.

Note: $E(u) = \frac{1}{2}u'(r)^2 - \frac{1}{2}\lambda u(r)^2 + \frac{\mu}{p+1}|u(r)|^{p+1}$

Recall Lemma 1.3.3 : For every $c > 0$, there exists an $\alpha(c) > 0$ with the following property. If u is a solution of (1.3.1), and if $E(u) = -c < 0$, and $u(R) > 0$ for some $R \geq 0$, then $u(r) \geq \alpha(c)$ for all $r \geq R$.

Proof of Lemma 1.3.6 : For simplicity, assume $\lambda,\ \mu = 1$. By lemma 1.3.3, $E(u(r)) > 0$ for all $r > 0$. Otherwise if $E(u(r)) < 0$ for some r, since E is decreasing and u has n zeros, there would be an R such that $u(R) > 0,\ E(u(R)) < 0$, which would imply that $u \geq k > 0$ by lemma 1.3.3, a contradiction. By looking at the definition of E, this implies that if $u'(r_0) = 0$, then $|u(r_0)|^{p-1} > \frac{p+1}{2} > 1$, so that by multiplying (1.3.1) by

$u(r_0)$, we have

$$u(r_0)u(r_0)'' = u(r_0)^2 - |u(r_0)|^{p-1}u(r_0)^2 < 0$$

and so $u(r_0)u''(r_0) < 0$. Now if $r_1 < r_2$ are two consecutive zeros of u', we see that

$$u(r_1)u(r_2)u(r_1)''u(r_2)'' > 0.$$

Since $r_1,\ r_2$ are consecutive zeros of u', it must be the case that $u(r_1)''u(r_2)'' < 0$ (since if $u(r_1)$ is a local minimum, $u(r_2)$ must be a local maximum), so that $u(r_1)u(r_2) < 0$, so that u has a zero in (r_1, r_2). Therefore since u has a finite number of zeros, u' has a finite number of zeros. Let $r_0 \geq 0$ be the largest zero of u' and assume, for example, that $u(r_0) > 0$. We have from the sixth line of this proof that $u(r_0) > 1$, and also u is decreasing on $[r_0, \infty)$ (since $u \to 0$ as $r \to \infty$). Therefore there exists an $r_1 \in [r_0, \infty)$ such that $u(r_1) = 1$, and we have that $u'(r_1) < 0$. By continuous dependance, there exists $\epsilon > 0$ such that if $|x - y| < \epsilon$, and if v is the solution of (1.3.1) with the initial condition $v(0) = y$, then the following properties hold:
(i) There exists $\gamma_0 \in [r_1 - 1, r_1 + 1]$ such that v has exactly n zeros on $[0, \gamma_0]$, (u has n zeros on $[0,r_1]$)
(ii) $v(\gamma_0) = 1$ and $v'(\gamma_0) < 0$.

So we only need to show that for a small enough $\epsilon > 0$, v has at most one zero on $[\gamma_0, \infty)$. Suppose v has a first zero $\gamma_1 > \gamma_0$. since $v(\gamma_1) = 0$, we must have $v'(\gamma_1) < 0$, and so $v'(r)$, $v(r) < 0$ for $r > \gamma_1$ and small. Also, it follows from (1.3.1) that v' can't vannish while $0 > v > -1$, since then we would have

$$v''(r) - v(r) + |v(r)|^{p-1}v(r) = 0,$$

with the first term positive, but the second two terms would also be positive, since suppose not

$$-v(r) + |v(r)|^{p-1} v(r) < 0 \implies |v(r)|^{p-1} > 1 \implies v(r) < -1,$$

a contradiction. Thus there exists $\gamma_3 > \gamma_2 > \gamma_1$ such that $v' < 0$ on $[\gamma_1, \gamma_3]$, and $v(\gamma_2) = -\frac{1}{4}$, $v(\gamma_3) = -\frac{1}{2}$. By lemma 1.3.3, we obtain the desired result if we show that $E(v(\gamma_3)) < 0$ for ϵ small enough. To see this, first observe that since $u > 0$ on $[r_0, \infty)$, for all $M > 0$, there exists an $\epsilon' \in (0, \epsilon)$ such that $\gamma_1 > M$ if $|x - y| \leq \epsilon'$. Let

$$f(x) = \frac{|x|^{p+1}}{p+1} - \frac{x^2}{2}.$$

It follows from (1.3.6) that

$$\frac{d}{dr} E(v, r) + 2\frac{N-1}{r} E(v, r) = 2\frac{N-1}{r} f(v(r)),$$

and so

$$\frac{d}{dr} r^{2(N-1)} E(v, r) = 2(N-1) r^{2N-3} f(v(r)).$$

Integrating on (γ_0, γ_3), we obtain

$$\gamma_3^{2(N-1)} E(v, \gamma_3) = \gamma_0^{2(N-1)} E(v, \gamma_0) + 2(N-1) \int_{\gamma_0}^{\gamma_3} r^{2N-3} f(v(r))\, dr.$$

Note that by continuous dependence,

$$\gamma_0^{2(N-1)} E(v, \gamma_0)^{2(N-1)} \leq C,$$

with C independent of $y \in (x - \epsilon, x + \epsilon)$. Also, $f(v(r)) \leq 0$ on (γ_0, γ_3) since $-1 \leq v \leq 1$, and there exists $a > 0$ such that $f(t) \leq -a$ for $t \in (-\frac{1}{2}, -\frac{1}{4})$. It follows that

$$\gamma_3^{2(N-1)} E(v, \gamma_3) \leq C - 2(N-1)a \int_{\gamma_2}^{\gamma_3} r^{2N-3}\, dr$$

$$\le C - 2(N-1)a\gamma_2^{2N-3}(\gamma_3 - \gamma_2).$$

Since v' is bounded on (γ_2, γ_3) independently of y such that $|x-y| \le \epsilon'$, it follows that $\gamma_3 - \gamma_2$ is bounded from below. Therefore, we see that $E(v, \gamma_3) < 0$ if ϵ is small enough, which completes the proof. □

Proof of Theorem 1.3.5 : We have already showed in the proof of theorem 1.3.4 that

$$A_0 = \{x > 0 : u > 0 \text{ on } (0, \infty)\} \neq \emptyset,$$

with u(0)=x, u'(0)=0, and that $x_0 = \sup A_0$ has the desired properties (so that the solution u with $u(0) = x_0$ has $n = 0$ zeroes). Now we show by induction that

$$A_n = \{x > 0 : u \text{ has exactly } n \text{ zeroes on } (0, \infty)\} \neq \emptyset$$

and that $x_n = \sup A_n$ has the desired properties, for all $n \in \mathbb{N}$. Suppose it is true for the k^{th} case. Since the solution u with $u(0) = x_k$ has k zeroes, if we take $x > x_k$ close enough, by lemma 1.3.6 the solution u_x with $u_x(0) = x$ has at most $k+1$ zeroes, and it actually must to have $k+1$ zeroes. Suppose not, since $x > x_k$, u_x has at most $k-1$ nodes, and by Corollary 1.3.2 A_{k+1} is bounded. Thus if we consider $x > x_{k+1}$ small enough, it follows that the corresponding solution u has at most k zeroes, but it can't have k or $k-1$ zeroes, so it must have at most $k-2$ zeroes. Proceeding in this manner, we see that the number of nodes of of u_n is monotone, either increasing or decreasing. Since it is obvious that the number of nodes of u_1 is greater than the number of nodes of u_0, the number of nodes is montone increasing with x_n.

Now we must show that $x_{k+1} \in A_{k+1}$. Let u_{k+1} be the corresponding solution, and suppose x_{k+1} isn't in A_{k+1}, then u has a $(k+2)^{th}$ zero (it can't have less) at some r_0. By uniqueness, $u'(r_0) \neq 0$, so that u crosses the x-axis at

r_0. By continuous dependence, this is also true for for solutions with initial values close to x_{k+1}, which means that for all $x < x_{k+1}$ close enough, u has $k+2$ zeroes, but this contradicts that x_{k+1} is the supremum. This completes the proof. $\square$

On the Dirichlet Principle

2.1 Weak Derivatives and Solutions :

Given $u \in L^p(\Omega)$, we say that that $v \in L^p(\Omega)$ is the weak derivative of u if

$$\int_\Omega u\varphi' = -\int_\Omega \varphi v$$

for all $\varphi \in C_0^\infty$. This definition is motivatved by integration by parts, since if u is differentiable, then using integration by parts on the left side gives the right side with $u' = v$ (in this case the weak derivative equals the strong derivative (the normal derivative)). We can generalize this to n dimensions by defining v to be the α^{th} weak derivative of u, and writing $D^\alpha u = v$ if

$$\int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega \varphi v$$

where $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1, ..., \alpha_n)$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, presuming that u, v are locally integrable on an open subset of $\mathbb{R}^n$, and φ is infinitely differentiable with compact support in the subset. The typical example of a function which has no derivative on $\mathbb{R}$ but has a weak derivative on $\mathbb{R}$ is the first continuous function with no derivative most students encounter: $|x|$. Intuitively, the weak derivative of $|x|$ should be

$$D|x| = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

It may seem a bit contrived that the weak derivative is 0 at $x = 0$, but it doesn't actually matter since the value of a function at one point does not affect the integral. In any case, if you consider $f(x) = |x|^r$ for $r > 1$ arbitrary, $f'(0) = 0$ always, so in this sense it does seem natural to define the weak derivative of $|x|$ to be 0 at $x = 0$. Now to check that we have the correct weak derivative: take $\varphi \in C_0^\infty$ arbitary, then we have that

$$\int_a^b |x|\varphi'\,dx = \int_a^0 |x|\varphi'\,dx + \int_0^b |x|\varphi'\,dx = -\int_a^0 x\varphi'\,dx + \int_0^b x\varphi'\,dx$$

and using integration by parts, this last part equals

$$\int_a^0 \varphi\,dx - \int_0^b \varphi\,dx = -\int_a^b \varphi D|x|\,dx\,,$$

which is was we aimed to show, hence we have chosen the right weak derivative (notice we only have to consider the case $a \leq 0 \leq b$, since if we integrate over any other region the weak derivative and the derivative coincide, so that the condition becomes trivial to show). Note two weak derivatives of the same function must be equal almost everywhere, and thus are identified together in L^p spaces.

We can now define weak solutions to Laplace's equation. Suppose $\Delta u = 0$, then for all $\varphi \in C_0^\infty$,

$$0 = \int \Delta u\varphi = -\int \nabla u \cdot \nabla\varphi$$

by integration by parts. Thus we say that u is a weak solution to $\Delta u = 0$ if $\int \nabla u \cdot \nabla\varphi = 0$ for all $\varphi \in C_0^\infty$.

2.2 Weak Convergence :

In a Hilbert space H, we say a sequence $x_n \in H$ converges weakly to $x \in H$ if for all $y \in H$, $\langle x_n|y\rangle \to \langle x|y\rangle$. We define the inner product on $W^{1,2}(\Omega)$ to be

$$\langle f|g\rangle = \int_\Omega f\overline{g} + \int_\Omega Df \cdot D\overline{g}\,,$$

and with this inner product $W^{1,2}$ is a Hilbert space. Note that strong convergence (the regular convergence) implies weak convergence.

Proof : Suppose $x_n \to x$ in some arbitrary Hilbert space. Let y be in this Hilbert space as well. Then by Hölder's inequality, $\|x_n y - xy\| \leq \|x_n - x\|_1 \|y\|_\infty \to 0$ as $n \to \infty$. $\square$

2.3 Sobolev Spaces ($\mathbf{W^{m,p}(\Omega)}$) :

In the proceeding section we will be covering the Dirichlet Principle, which is concerned with minimizing the functional $D(u) = \int_\Omega |\nabla u|^2$ with $u = g$ on $\partial\Omega$. Since $D(u) = \|\nabla u\|^2_{L^2}$, we will use this as the motivation for defining Sobolev spaces, which are convenient when discussing the Dirichlet problem. Define the Sobolev space as so:

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \ \forall \ |\alpha| \leq m\}.$$

When equipped with the norm

$$\|u\|_{W^{m,p}} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p},$$

$W^{m,p}$ becomes a Banach space. Notice that $\|u\|_{L^p} + \|\nabla u\|_{L^p} = \|u\|_{W^{1,p}}$. We will use this norm for the rest of this paper.

Proof that a Sobolev space is a Banach space :

Let $\{u_j\}$ be a Cauchy sequence with respect to the $\|\cdot\|_{W^{m,p}}$ norm. This is equivalent to saying $\{D^\alpha u_j\}$ is a Cauchy sequence with respect to the $\|\cdot\|_{L^p}$ norm. Since L^p is complete, $D^\alpha u_j \to u^{(\alpha)}$, and $u^{(\alpha)} \in L^p$. Now we must show that $D^\alpha u \to u^{(\alpha)}$. Since $D^\alpha u_j$ is strongly convegent and this implies it is weakly convergent, we must show that

$$\int u^{(\alpha)}\varphi = (-1)^{|\alpha|}\int \varphi^{(\alpha)}u$$

for all $\varphi \in C_0^\infty$. We have that

$$\int u^{(\alpha)}\varphi = \lim_{j\to\infty}\int D^\alpha u_j\varphi = \lim_{j\to\infty}(-1)^{|\alpha|}\int \varphi^{(\alpha)}u_j = (-1)^{|\alpha|}\int \varphi^{(\alpha)}u$$

for all $\varphi \in C_0^\infty$. This completes the proof. $\square$.

There is an important inequality called the Poincaré inequality (whose proof is a bit beyond the scope of this paper) that will prove very useful to us in proving the existence of a Dirchlet energy minimizer: If $u \in W_0^{1,p}$, then

$$\|u\|_{L^p(\Omega)} \leq C(\Omega)\|u\|_{W^{1,p}(\Omega)},$$

for some constant $C(\Omega) > 0$.

We are now ready to get into the Dirichlet principle.

2.4 The Dirichlet Principle :

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \quad \text{weakly} \qquad (2.4.1) \\ u = g & \text{on } \partial\Omega, \end{cases}$$

if and only if u is a minimizer of the Dirichlet energy

$$D(u) = \int_{\Omega} |\nabla u|^2. \quad (2.4.2)$$

Proof : First suppose that u minimizes (2.4.2), then for any $v \in C_0^{\infty}$, and $\epsilon > 0$, we have that

$$\int |\nabla(u + \epsilon v)|^2 \geq \int |\nabla u|^2 \quad (2.4.3)$$

$$\Longrightarrow \int |\nabla u|^2 + 2\epsilon \int \nabla u \cdot \nabla v + \epsilon^2 \int |\nabla v|^2 \geq \int |\nabla u|^2$$

$$\Longrightarrow 2\epsilon \int \nabla u \cdot \nabla v + \epsilon^2 \int |\nabla v|^2 \geq 0.$$

Taking $\epsilon \to -\epsilon$ in (2.4.3), we get

$$-2\epsilon \int \nabla u \cdot \nabla v + \epsilon^2 \int |\nabla v|^2 \geq 0,$$

by dividing by ϵ and taking $\epsilon \to 0$, these last two inequalities together imply that $\int \nabla u \cdot \nabla v = 0$, and by section 2.1, this implies (2.4.1)

Now for the other direction. Suppose that $\Delta u = 0$ in Ω weakly, $u = g$ on $\partial\Omega$, and $v \in C_0^{\infty}$. Write

$$\int_{\Omega} |\nabla(u+v)|^2 = \int_{\Omega} |\nabla u|^2 + 2\int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} |\nabla v|^2 = \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\nabla v|^2 \geq \int_{\Omega} |\nabla u|^2,$$

since $v \in C_0^\infty$ is arbitrary, u is a minimizer of $D(u) = \int_\Omega |\nabla u|^2$. $\square$

Note that the Dirichlet energy which is usually written as $E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$, and as this integral has connections with physics (liquid crystals, superconductors, etc.) it is this integral that we are really interested in minimizing, but clearly u minimizes D if and only if u minimizes E anyway.

The existence and uniqueness of solutions to Laplace's equation is very important in areas of physics such as electrostatics, where if V is the electric potential, and $\Delta V = 0$ in Ω then the charge density in Ω is zero, and solving Laplace's equation gives us the potential, hence the electric field in Ω. The uniqueness of the solution is very important for it lets us use methods of solving the equation that would be very difficult otherwise, namely the method of images, which on first glance seems unlikely to give the correct solution, but it must as solutions to Laplace's equation are unique.

2.5 On the existence of a minimum :

We have showed necessary and suficient conditions for a function to be a minimizer, but it is still unknown if a minimizer does exist. Since $D(u)$ is bounded below by zero, it has a finite infimum. Let $A \equiv \inf\{\|u\|_{L^2} : u = g \text{ on } \partial\Omega\}$. Let $(v_j)_{j=1}^\infty$ be a sequence of functions such that

$$\|\nabla v_j\|_{L^2} \leq A + \frac{1}{j} \ . \quad (2.5.1)$$

Note that $D(u) = \|\nabla u\|_{L^2}^2$. By the triangle inequality, we have

$$\|v_j\|_{L^2} \leq \|v_j - g\|_{L^2} + \|g\|_{L^2} \leq \alpha\|\nabla v_j - \nabla g\|_{L^2} + \|g\|_{L^2} \leq \alpha\|\nabla v_j\|_{L^2} + \alpha\|\nabla g\|_{L^2} + \|g\|_{L^2}$$

$$\leq \alpha(A+1)^{\frac{1}{2}} + (\alpha+1)\|g\|_{W^{1,2}} \quad (2.5.2)$$

for some $\alpha > 0$, where we have used the Poincaré inequality, and in (2.5.2) that $\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|\nabla u\|_{L^p}$. By this inequality $\|v_j\|_{L^2} < \infty$, thus together with (2.5.1) we conclude that $(v_j)_{j=1}^{\infty}$ exists in and is bounded in $W^{1,p}(\Omega)$. Thus there exists a function $u \in W^{1,p}(\Omega)$ such that $\lim_{j\to\infty} v_j = u$ weakly in $W^{1,p}(\Omega)$. Now $W_g^{1,p}(\Omega) \neq \emptyset$ and is weakly closed, so $u \in W_g^{1,p}(\Omega)$. Now u is guaranteed to be a minimzer of D since D is convex (which we show in the next section). This completes showing the existence. $\square$

2.6 On the uniqueness of a minimum :

Let u be a minimizer of the functional $D(u)$. Write $d(\epsilon) = D(u + \epsilon v)$, it must be true that $d''(\epsilon)\big|_{\epsilon=0} \geq 0$. We will show that it is so. Noting that $|\nabla(u + \epsilon v)|^2 = \nabla(u + \epsilon v) \cdot \nabla(u + \epsilon v) = |\nabla u|^2 + 2\epsilon \nabla u \cdot \nabla v + \epsilon^2 |\nabla v|^2$, we have that

$$d''(\epsilon) = \frac{d^2}{d\epsilon^2} D(u + \epsilon v) = \int_\Omega \frac{\partial^2}{\partial \epsilon^2} |\nabla(u + \epsilon v)|^2 = \int_\Omega 2|\nabla v|^2 \geq 0$$

$$\implies d''(\epsilon)\big|_{\epsilon=0} \geq 0.$$

Notice that $d''(\epsilon) \geq 0$ for all ϵ, and any functions u, v; this tells us that $d(\epsilon)$ is convex. This tells us that if there were another minimizer w with the same boundary value as u, that if we let $v = w - u$ and let $\epsilon = 1$ then $D(u) = d(0) \leq d(1) = D(w)$, with equality if and only if $\int_\Omega |w - u|^2 = 0$, which means that if $w \neq u$ almost everywhere, then w and u differ by a nonzero constant in Ω, thus by continuity they don't have the same boundary value. So the inequality is strict, a contradiction. Hence we conclude by the convexity of $d(\epsilon)$ that u is the unique minimizer of D. This completes the entire proof. $\square$

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