Probabilistic Approach to Einstein's Equations

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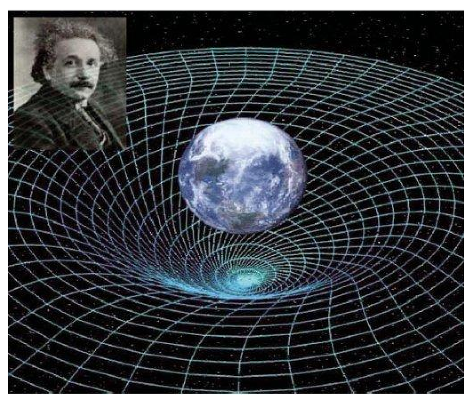
Introduction



Einstein's theory of General Relativity is one of the most profound achievements of science. It explains in detail how gravity works, the expansion of the Universe and the Big Bang. Despite its great achievements in predicting the behavior of the Universe, there are many questions left unanswered in the mathematics of the theory. Amongst those questions one that is of great importance is the initial value problem of General Relativity. In this problem Einstein's equations are a machine that evolves initial data given on a hypersurface inside the space-time. It turns out that any initial data must satisfy a system of nonlinear differential equations called Einstein's constraint equations. In this project, we studied the constraint equations using methods from probability theory, and as far as we know, it was the first time to apply probabilistic methods to Einstein's equations. Specifically, we have given probabilistic estimates for the almost constant mean curvature condition for the existence of solutions.

Conformal Formulation of Constraint Equation

The Einstein's Field Equations is $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}$, it is 10 nonlinear coupled PDEs. Such equations are very hard to study, thus it is reformulated as initial value problem with constraint and evolution part. The main interest for us is the constraint equation which can be written as:

$$R_{g_1} - |K|^2_{g_1} + \mathrm{tr}_{g_1}K^2 = 0, \quad \mathrm{div}_{g_1}K - d\,\mathrm{tr}_{g_1}K = 0.$$

Where g is a Riemannian metric on 3-manifold M; g_1 is another metric on M lying in a conformal class of g; and K a symmetric $(0,2)$-tensor. This equation is still very hard to study, thus it is further simplified using conformal methods. We let

$$g_1 = \psi^4 g, \quad K = \psi^{-2}(\sigma + \mathbf{L}W) + \frac{\tau}{3}g,$$

where $\mathbf{L}$ denotes the conformal Killing operator. With this in mind the Einstein's constraint equation becomes PDE of the form

$$-8\Delta\psi + R_g\psi = -\frac{2}{3}\tau^2\psi^5 + |\sigma + \mathbf{L}W|^2\psi^{-7}, \;\; \mathrm{div}\mathbf{L}W = \frac{2}{3}\psi^{-6}d\tau. \quad (1)$$

It has been shown by D.Maxwell and others the sufficient condition for existence of solution for (1) is

$$\frac{\max(|d\tau|)}{\min(|\tau|)} < \epsilon$$

We will study the likeliness of this condition.

Results Using Probabilistic Approach

We fix a metric g_0 denote the corresponding Laplacian Δ_0, and let $\{\lambda_j, \phi_j\}$ denote an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of $-\Delta_0$; we let $\lambda_0 = 0, \phi_0 = 1$. We assume that the mean curvature τ is of the form

$$\tau = (C + f(x)) = C - a \cdot \sum_{j=1}^{\infty} a_j c_j \phi_j(x) \quad (2)$$

where $a_j \sim \mathcal{N}(0,1)$ are i.i.d standard Gaussians, and c_j are positive real numbers and C is constant. We also let $c_j = F(\lambda_j)$, where $F(t)$ is an eventually monotone decreasing function of t, $F(t) \to 0$ as $t \to \infty$. For example, we may take $c_j = e^{-s\lambda_j}$ or $c_j = \lambda_j^{-s}$.

Theorem

Let τ is given by (2), and $c_j = O(\lambda_j^{-s})$ with $s > \frac{n+k}{2}$, then $f(x) \in C^k(M)$ a.s.

Theorem

Let τ is given by (2), with $c_j = \lambda_j^{-\beta}$ satisfying $\beta > \frac{1}{2}(n+1)$. Then

$$\mathbb{P}\{\frac{\max|d\tau|}{\min|\tau|} > \epsilon\} = \exp\left(C_1\epsilon - \frac{\epsilon^2}{a^2C_2}\right)$$

for some $C_1, C_2 > 0$

Conclusions

As an application of our result we see if τ is given by (2) it has nonzero probability of satisfying the condition, thus there exist τ such that it satisfies the sufficient condition. This sort of method can also be applied to similar PDEs and we are hoping to explore more about it.

Later on we would like to study (1) with less restrictions on the mean curvature τ.

Acknowledgement

I wish to thank both of my advisors Prof Dmitry Jakobson and Prof Gantumur Tsogtgerel for their tremendous contributions and support both morally and finantially without their help and guidence this project would not have been completed. I also show my gratitude to my friends and all who contributed in the course of the project.

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