# Weak convergence methods for nonlinear partial differential equations.

Spencer Frei

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In this report, I have collected the proofs that Professor Gantumur Tsogtgerel, Dr. Brian Seguin, Benjamin Landon and I have developed in the summer of 2012 while studying various weak convergence methods for the purpose of the analysis of nonlinear partial differential equations. This research was supported by an Undergraduate Summer Scholarship from the Institut des sciences mathématiques, based in Montréal, Canada. We primarily followed L.C. Evans' textbook on the subject [1]; most of the theorems are taken almost directly from the text. Most of the intermediate comments in these sections are also taken directly from Evans' text; all proofs that Evans has supplied are included in the document for convenience of the reader, while many of the details that were left out of the text are fully fleshed out here.

## Contents

# 1 Convexity in scalar variational problems

## 1.1 Calculus of variations

This section corresponds to Chapter 2, §A of Evans. We are given a smooth function $F : \mathbb{R}^n \to \mathbb{R}$, and the goal is to find conditions under which we can guarantee existence of a minimizer to the variational problem

$$I[w] := \int_U F(Dw)dx \tag{1}$$

where $U$ is an open, bounded, smooth domain, and the minimizing problem (1) is taken over some family of admissable functions

$$\mathcal{A} = \{w \in W^{1,q}(U) : w = g \text{ on } \partial U\} \tag{2}$$

Here, $W^{k,q}(\Omega)$ is the Sobolev space whose weak derivatives $D^\alpha w$ are in $L^q$ for all multi-indices $|\alpha| \leq k$, and $g$ is some fixed function, with boundary values in $\mathcal{A}$ assumed in the trace sense. As mentioned, the problem is to find conditions on $F$ under which we can guarantee existence of a minimizer; i.e., some $u \in \mathcal{A}$ such that $I[u] = \inf_{w\in\mathcal{A}} I[w]$. To this end, let $(u_k)_{k=1}^{\infty}$ be a sequence of functions in $\mathcal{A}$ converging to the minimizer, i.e.,

$$\lim_{k\to\infty} I[u_k] = \inf_{w\in\mathcal{A}} I[w] := m \tag{3}$$

We suppose that the infimum is finite, and a coercivity condition on $F$:

$$F(p) \geq \alpha|p|^q - \beta \qquad (\forall\ p \in \mathbb{R}^n) \tag{4}$$

where $\alpha > 0$, $\beta \geq 0$ are some fixed constants. We want to deduce that the sequence $(u_k)$ is bounded in $W^{1,q}$. To see this, recall the Poincaré inequality: if $\Omega \subset \mathbb{R}^n$ is a bounded and open domain, and $1 \leq p \leq \infty$, then there is some constant $\gamma = \gamma(\Omega, p) > 0$ such that

$$\|u\|_{W^{1,p}(\Omega)} \leq \gamma \|Du\|_{L^p(\Omega)} \qquad (\forall\ u \in W_0^{1,2}(\Omega)) \tag{5}$$

Let $k$ be large enough so that we have $m + 1 \geq I[u_k]$. For such $k$, we have from the coercivity condition

$$\begin{aligned} m + 1 \geq I[u_k] &= \int_U F(Du_k)dx \geq \alpha \int_U |Du_k|^q - \beta \int_U dx \\ &= \alpha \|Du_k\|_{L^q(U)}^q - \beta\lambda(U) \end{aligned} \tag{$*$}$$

where $\lambda$ is the $n$-dimensional Lebesgue measure. Now, the Poincaré inequality does not directly apply to the functions $u_k$ since they are equal to $g$ on the boundary, which is not necessarily the zero function. We can, however, apply the inequality to the functions $u_k - g$, to give some constants $\gamma_1 = \gamma_1(U, q)$ such that

$$\begin{aligned} \|Du_k\|_{L^q(U)} &\geq \|Dg\|_{L^q(U)} + \gamma_1 \|u_k\|_{W^{1,q}} - \gamma_1 \|g\|_{W^{1,2}} \\ &= \gamma_1 \|u_k\|_{W^{1,q}} + c \end{aligned}$$

where $c$ is some fixed constant (since the given function $g$ is fixed) depending only on $U$, $q$, and $g$. This means there is some constant $\gamma_2$ depending on $U, q, g$ such that

$$\|Du_k\|_{L^q}^q \geq \gamma_2 \|u_k\|_{W^{1,q}}^q$$

We can now put the above inequality into (*) to get that

$$m + 1 \geq \gamma_3 \|u_k\|_{W^{1,q}}^q - \beta\lambda(U)$$

So for $k$ large enough, this means that $\|u_k\|_{W^{1,q}} \leq \gamma < \infty$, thus $(u_k)$ is bounded in $W^{1,q}$. Therefore, using Banach-Alaoglu theorem, we know that there is a weakly convergent subsequence: that is, a subsequence $(u_{k_j})_{j=1}^\infty \subset (u_k)_{k=1}^\infty$ and $u \in W^{1,q}(U)$ such that $u_{k_j} \rightharpoonup u$ in $W^{1,q}(U)$ (we use the notation $\rightharpoonup$ to denote weak convergence). In fact, we must have that $u \in \mathcal{A}$–that is, $u$ satisfies the boundary condition of $g$ on $\partial U$. If it were otherwise, then tails of the subsequence $(u_{k_j})$ would also not satisfy the boundary condition for $j$ large enough. Therefore we have that

$$u_{k_j} \rightharpoonup u \text{ in } W^{1,q}(U) \tag{6}$$

In particular, we have that $u_{k_j} \rightharpoonup u$ in $L^q(U)$ and that $Du_{k_j} \rightharpoonup Du$ in $L^q(U; \mathbb{R}^n)$. We recall that a sequence of $L^q(U)$ functions $(f_k)_{k=1}^\infty$ , $1 \leq q < \infty$, converges weakly to some $f \in L^q(U)$ if for every $g \in L^{q'}(U)$, we have that

$$\int_U g f_k d\lambda \to \int_U g f d\lambda$$

where $q$ is the conjugate exponent to $q$; i.e., the number $q'$ such that $\frac{1}{q} + \frac{1}{q'} = 1$. Now, since $u \in \mathcal{A}$, we have a candidate for our minimizer to the variational problem (1). Since $(u_k)$ was the sequence of functions converging to the minimizer, if we can show that (6) implies

$$I[u] \leq \liminf_{j\to\infty} I[u_{k_j}] \tag{7}$$

then we will have shown that $u$ is indeed the desired minimizer. We note that if $I[\cdot]$ satisfies (7) for weakly convergent sequences in a space $S$, then we say that $I$ is **lower semicontinuous with respect to weak convergence in** $S$.

So let us examine what type of structural conditions on $F$ will allow us to deduce lower semicontinuity of $I[\cdot]$. Say that the $u$ from above is a smooth minimizer. Then let us set, for $v$ a Lipschitz function with compact support in $U$,

$$i(t) := I[u + tv] = \int_U F(Du + tDv)dx \qquad\qquad (t \in \mathbb{R}) \tag{8}$$

Now, since $u$ is the minimizer to the variational problem, we know that $i$ has a minimum at $t = 0$, and hence we have that $i''(t) \geq 0$ by the second derivative test. Since $F$ is smooth, we can differentiate through the integral. For the first derivative of $F$, we have

$$\begin{aligned}\frac{\partial}{\partial t} F(Du + tDv)\Big|_{t=0} &= \frac{\partial F(Du)}{\partial p_1} D_1 v + \cdots + \frac{F(Du)}{\partial p_n} D_n v \\ &= \sum_{i=1}^n \frac{\partial F(Du)}{\partial p_i} v_{x_i}\end{aligned}$$

where again $p = (p_1, \ldots, p_n)$, and we use the notation $v_{x_i} := \frac{\partial v}{\partial x_i}$. Evidently this gives that the second derivative of $F$ is then

$$\frac{\partial^2}{\partial t^2} F(Du + tDv)\Big|_{t=0} = \sum_{i,j=1}^{n} \frac{\partial^2 F(Du)}{\partial p_i \partial p_j} v_{x_i} v_{x_j}$$

Differentiating through the integral, we then have that

$$0 \leq i''(0) = \int_U \sum_{i,j=1}^{n} \frac{\partial^2 F(Du)}{\partial p_i \partial p_j} v_{x_i} v_{x_j} \tag{9}$$

Relation (9) holds for all Lipschitz $v$ with support in $U$. So, for $\varepsilon > 0$ and $\xi \in \mathbb{R}^n$, let us consider

$$v(x) := \varepsilon \zeta(x) \rho\left(\frac{x \cdot \xi}{\varepsilon}\right) \tag{10}$$

Here, $\zeta(x) \in C_c^\infty(U)$, and $\rho(x)$ is the sawtooth periodic function (with period length of 2) that is equal to $x$ on $[0, 1]$ and $2 - x$ on $[1, 2]$. Note that $\rho'(z)^2 = 1$ for all $x$; this will be used in a moment. To find the derivatives $v_{x_j}$, let $z = \frac{x \cdot \xi}{\varepsilon}$. We then have that

$$\frac{\partial}{\partial x_i} \rho(z) = \frac{\partial \rho}{\partial z} \frac{\partial z}{\partial x_i} = \rho' \frac{\xi_i}{\varepsilon}$$

And so we have

$$\frac{\partial}{\partial x_i} v(x) = \varepsilon \zeta_{x_i} \rho\left(\frac{x \cdot \xi}{\varepsilon}\right) + \xi_i \zeta(x) \rho'$$

And therefore,

$$v_{x_i} v_{x_j} = \varepsilon^2 \zeta_{x_i} \zeta_{x_j} \rho^2 + \varepsilon \zeta_{x_i} \rho(z) \xi_j \zeta \rho' + \varepsilon \zeta_{x_j} \rho \xi_i \zeta \rho' + \xi_i \xi_j \zeta(x)^2 \rho'(z)^2$$

Now, as $\varepsilon \downarrow 0$, we have that $v_{x_i} v_{x_j} \to \xi_i \xi_j \zeta(x)^2$. Substituting this into (9), we have that

$$\int_U \sum_{i,j=1}^{n} \frac{\partial^2 F(Du)}{\partial p_i \partial p_j} \xi_i \xi_j \zeta(x)^2 dx \geq 0 \text{ for all } \zeta \in C_c^\infty(U)$$

Since the above holds for all test functions on $U$, we must have that

$$\sum_{i,j} \frac{\partial^2 F(Du)}{\partial p_i \partial p_j} \xi_i \xi_j \geq 0 \qquad\qquad (x \in U; \xi \in \mathbb{R}^n) \tag{11}$$

The above necessary inequality suggests that it is natural to assume that $F$ is convex:

$$\xi^T D^2 F(p) \xi \geq 0 \qquad\qquad (p, \xi \in \mathbb{R}^n) \tag{12}$$

## 1.2 Weak lower semicontinuity

This section corresponds to Evans, Chapter 2, §B. The analysis in the previous section suggests the following theorem, that convexity is indeed the proper structural hypothesis for our nonlinearity.

**Theorem 1.** *The functional $I[\cdot]$ is lower semicontinuous with respect to weak convergence in $W^{1,q}(U)$ if and only if $F$ is convex.*

*Proof.* ($\Rightarrow$) Let $p \in \mathbb{R}^n$ be fixed, and we suppose for simplicity that $U = Q$ where $Q$ is the open unit cube in $\mathbb{R}^n$. Fix any $v \in C_c^\infty(Q)$. For each $k \in \mathbb{N}$, subdivide $Q$ into $2^{kn}$ disjoint subcubes $\{Q_i\}_{i=1}^{2^{kn}}$, each with side length $\frac{1}{2^k}$. Then define functions $u_k$ by

$$u_k(x) := \frac{1}{2^k} v\left(2^k(x - x_l)\right) + p \cdot x \qquad (x \in Q_l)$$

where $x_l$ is the center of the cube $Q_l$: $u_k$ is defined on all of $Q$, but its value at a given $x$ is determined by which subcube $x$ lies in. Now define $u(x) := p \cdot x$ for $x \in Q$. Then we claim that $u_k \rightharpoonup u$ in $W^{1,q}(U)$. It suffices to show that

$$y_k(x) := \frac{1}{2^k} v\left(2^k(x - x_l)\right) \rightharpoonup 0.$$

To this end, we need to show that $y_k \rightharpoonup 0$ in $L^q(U)$ and that $Dy_k \rightharpoonup 0$ in $L^q(U; \mathbb{R}^n)$. Since the functions $y_k$ are bounded in $L^q$, it suffices to show that for every bounded, measurable $E \subset Q$ we have that $\int_E y_k \to 0$ strongly. This follows trivially since $y_k$ have compact support contained in $Q$:

$$\int_E y_k dx = \int_E \frac{1}{2^k} v\left(2^k(x - x_l)\right) \leq \frac{\sup v}{2^k} \lambda(E) \to 0 \text{ as } k \to \infty$$

Now, by assumption, $I[u] \leq \liminf_{k\to\infty} I[u_k]$, so we have that

$$\begin{aligned} I[u] &= \int_Q F(D(p \cdot x))dx = \int_Q F(p) = \lambda(Q)F(p) \\ &\leq \liminf_{k\to\infty} I[u_k] = \liminf_{k\to\infty} \int_Q F(p + \frac{1}{2^k} D(v(2^k(x - x_l)))dx \\ &= \int_Q F(p + Dv)dx \end{aligned}$$

But then this means that $u = p \cdot x$ is a minimizer of $I[\cdot]$ subject to its own boundary condition. Therefore inequality (11) gives that $F$ is convex.

($\Leftarrow$) Suppose that $u_k \rightharpoonup u$ in $W^{1,q}(U)$, and let us suppose first that $F$ is a maximum of finitely many affine functions:

$$F(p) = \max_{1\leq j\leq m} \left(b^j \cdot p + c^j\right) \qquad (p \in \mathbb{R}^n) \tag{13}$$

Let us define the sets $E_j$ as

$$E_j := \{x \in U : F(Du(x)) = b^j \cdot Du(x) + c^j\}$$

Then we have that $U = \cup_{j=1}^m E_j$. We can assume that the $E_j$ are disjoint, as otherwise we could define the sets $F_n$ as $F_1 = E_1$; $F_n = E_n \setminus (F_1 \cup \ldots \cup F_{n-1})$, then the sets $F_n$ would be disjoint and we would have that $\cup_1^m E_j = \cup_1^m F_j = U$, so we could proceed with the analysis using the sets $F_j$. (See the first chapter of Folland's *Real Analysis* for more details on this.) Now, since weak convergence is convergence of averages, $u_k \rightharpoonup u$ in $W^{1,q}$

implies that for every bounded measurable $E \subset U$ we have that $\int_E u_k \to \int_E u$ strongly. We thus have that

$$\begin{aligned}
I[u] = \int_U F(Du)dx &= \int_{\cup_{j=1}^m E_j} F(Du)dx \\
&= \sum_{j=1}^m \int_{E_j} F(Du)dx \\
&= \sum_{j=1}^m \int_{E_j} b^j \cdot Du(x) + c^j dx \\
&= \lim_{k\to\infty} \sum_{j=1}^m \int_{E_j} b^j \cdot Du_k(x) + c^j dx && \text{(weak convergence is convergence of averages)} \\
&= \liminf_{k\to\infty} \sum_{j=1}^m \int_{E_j} b^j \cdot Du_k(x) + c^j dx && \text{(limit exists means } \lim = \liminf\text{)} \\
&\le \liminf_{k\to\infty} \sum_{j=1}^m \int_{E_j} F(Du_k)dx = \liminf_{k\to\infty} I[u_k] && \text{(by supposition)}
\end{aligned}$$

Thus we are done if $F$ is a maximum of finitely many affine functions. In the general case, a convex function $F$ is the *supremum* of affine functions. So let us write

$$F^{(m)}(p) := \max_{1\le j\le m} \left(b^j \cdot p + c^j\right)$$

Then we have that $F^{(m)}$ is an increasing sequence of functions; $F^{(1)} \le \cdots \le F^{(m)} \le \cdots$, and further that $F^{(m)}$ converges pointwise to $F$:

$$F(p) = \lim_{m\to\infty} F^{(m)}(p)$$

Thus, by the monotone convergence theorem, we know that

$$\int F(Du) = \lim_{m\to\infty} \int F^{(m)}(Du)$$

And since we know that the inequality holds for each $m$ on the integral for the right, we are finished with the proof. $\square$

The proof of sufficiency clearly illustrates why convex nonlinearities are partially compatible with weak convergence: an affine function is weakly continuous and a convex function is the supremum of affine functions.

Now, recall $i(t)$ from (8): since $i$ has a minimum at $t = 0$, we have that

$$\begin{aligned}
0 = i'(0) &= \int_U \sum_{j=1}^n \frac{\partial F(Du)}{\partial p_j} v_{x_j} dx \\
&= -\int_U \sum_{j=1}^n \frac{\partial}{\partial p_j}\left(\frac{\partial F(Du)}{\partial p_j}\right) v dx,
\end{aligned}$$

when $F$ satisfies certain growth conditions; for instance, if $\nabla F(Du) \leq |Du|^q$. From this we see that the minimizer $u$ is a weak solution of the **Euler-Lagrange equation**:

$$\begin{cases} -\operatorname{div}(DF(Du)) = 0 & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

So we see that we can solve a nonlinear PDE using weak convergence methods, as per the title of Evans' text!

We will prove the theorem that informally means convergence of energies improves weak to strong convergence when $q = 2$, provided $F$ is uniformly strictly convex, and $F$ satisfies appropriate growth conditions. Evans' comments that the uniform convexity of $F$ somehow damps out wild oscillations in $\{Du_k\}_{k=1}^{\infty}$. For $q = 2$, if $F$ satisfies the growth condition

$$|F(p)| \leq C\left(1 + |p|^2\right) \qquad (p \in \mathbb{R}^n), \tag{14}$$

and $F$ is **uniformly strictly convex**,

$$\xi^T D^2 F(p)\xi \geq \gamma|\xi|^2 \qquad (\gamma > 0; p, \xi \in \mathbb{R}^n) \tag{15}$$

then the minimizing sequence converges *strongly* to $u$ in $W^{1,2}(U)$; and in fact, the following holds:

**Theorem 2.** *Assume that $F$ satisfies conditions* (14) *and* (15)*, and that $u_k \rightharpoonup u$ in $W^{1,2}(U)$. If we also have convergence of energies*

$$\int_U F(Du_k)dx = I[u_k] \to I[u] = \int_U F(Du)dx \tag{16}$$

*Then we have that*

$$u_k \to u \text{ strongly in } W^{1,2}(U)$$

*Proof.* Using the strict convexity (15) and a Taylor expansion, we see that for any $p, q \in \mathbb{R}^n$,

$$F(q) \geq F(p) + DF(p) \cdot (q - p) + \frac{\gamma}{2}|q - p|^2$$

Then for $p = Du$ and $q = Du_k$, we integrate over $U$ to deduce that

$$\underbrace{I[u_k] - I[u]}_{(*)} \geq + \underbrace{\int_U DF(du) \cdot (Du_k - Du)dx}_{(**)} + \frac{\gamma}{2} \underbrace{\int_U |Du_k - Du|^2 dx}_{(***)} \tag{17}$$

Now, the convergence of energies (16) then implies that the term $(*)$ tends to zero. For the second term, (14) and (15) imply that $|DF(p)| \leq c(1 + |p|)$. Since $U$ is a bounded domain, this means that $DF(Du) \in L^2(U; \mathbb{R}^n)$. Now, since $u_k \rightharpoonup u$ in $W^{1,2}(U)$, we have that $Du_k \rightharpoonup Du$ in $L^2(U; \mathbb{R}^n)$; thus $DF(Du) \in L^2(U; \mathbb{R}^n)$ implies that the term $(**)$ tends to zero. Together, these mean that the term $(***)$ tends to zero, which is precisely strong convergence of $Du_k \to Du$ in $L^2(U; \mathbb{R}^n)$. This completes the proof. $\square$

# 2 Convexity in vector variational problems

The preceding section was concerned with one dimensional variational problems. The next topic of interest is vector-valued problems in the calculus of variations.

## 2.1 Quasiconvexity

Here we follow Evans, Chapter 3, §A. We now consider the analysis of the functional

$$I[w] = \int_U F(Dw)dx \ , \tag{18}$$

where the functional is now take over the class of functions

$$\mathcal{A} := \{w \in W^{1,q}(U; \mathbb{R}^m) : w = g \text{ on } \partial U\}$$

for $1 < q < \infty$, and for some fixed function $g : \partial U \to \mathbb{R}^m$. We write $w = (w^1, \dots, w^m)$, and the gradient is thus the $m \times n$ matrix

$$Dw = \begin{pmatrix} w^1_{x_1} & \cdots & w^1_{x_n} \\ & \ddots & \\ w^m_{x_1} & \cdots & w^m_{x_n} \end{pmatrix} \ .$$

We assume that $F : M^{m\times n} \to \mathbb{R}$ is a given smooth function.

We want to use the same type of analysis we used in the scalar case to give a necessary condition on $F$ to deduce the existence of minimizers. So first, we assume the coercivity condition

$$F(P) \geq \alpha |P|^q - \beta \qquad\qquad (P \in M^{m\times n})$$

for some constants $\alpha > 0$ and $\beta \geq 0$. The same analysis as the previous section shows that the existence of a minimizer in $\mathcal{A}$ turns once more to the weak lower semicontinuity of $I[\cdot]$. So what type of nonlinearity is compatible with weak lower semicontinuity? Just as before, if $u$ is a smooth minimizer and $v = (v^1, \dots, v^m)$ is a Lipschitz function with compact support contained in $U$, then the second variation of the functional $i$,

$$i(t) := I[u + tv] = \int_U F(Du + tDv)dx$$

at $t = 0$ is

$$0 \leq i''(0) = \int_U \sum_{i,j,k,l} \frac{\partial^2 F(Du)}{\partial p^k_i \partial p^l_j} v^k_{x_i} v^l_{x_j} dx \tag{19}$$

For fixed $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$, let $\zeta \in \mathscr{D}(U)$ and let $\rho$ be the sawtooth function defined previously in Section 1.1. Again, we define, for $\varepsilon > 0$,

$$v(x) := \varepsilon \zeta(x) \rho\left(\frac{x \cdot \xi}{\varepsilon}\right)]\eta$$

and substitute $v(x)$ into the functional in (19), and send $\varepsilon > 0$ to get the requirement

$$\sum_{i,j,k,l} \frac{\partial^2 F(Du(x))}{\partial p^k_i \partial p^l_j}(Du(x))\eta_k \eta_l \xi_i \xi_j \geq 0 \tag{20}$$

for every $x \in U$, $\eta \in \mathbb{R}^m$, and $\xi \in \mathbb{R}^n$. This necessary inequality suggests that we should assume that $F$ is **rank-one convex**, i.e., that $F$ satisfies the **Hadamard-Legendre inequality**:

$$(\eta \otimes \xi)^T D^2 F(P)(\eta \otimes \xi) \geq 0 \qquad (P \in M^{m\times n},\ \eta \in \mathbb{R}^m,\ \xi \in \mathbb{R}^n) \tag{21}$$

In particular, this means that for each fixed $P$, $n$, $\xi$ as above, the scalar function $f$ defined by

$$f(t) := F(P + t\eta \otimes \xi) \qquad (t \in \mathbb{R})$$

It is important to note that rank-one convexity of $f$ does not imply that $F$ is itself convex.

Although the previous analysis may suggest that rank-one convexity is the proper structural hypothesis of nonlinearity for $F$, this is in fact not the case. We would like to demonstrate a necessary inequality on $F$ assuming weak lower semicontinuity.

To do so, let us fix $P \in M^{m\times n}$ and, as in the previous section, suppose for simplicity that $U = Q$. Take any $v \in \mathscr{D}(Q; \mathbb{R}^m)$, and as before, for each $k \in \mathbb{N}$ we subdivide $Q$ into subcubes $\{Q_l\}_{l=1}^{2^{kn}}$. Define functions $u_k(x)$ and $u$ by

$$u_k(x) = \frac{1}{2^k} v\left(2^k(x - x_l)\right) + Px \qquad (x \in Q_l)$$

$$u(x) = Px$$

Then, just as before, $u_k \rightharpoonup u$ in $W^{1,q}(U; \mathbb{R}^m)$. And supposing that $I[\cdot]$ is weakly lower semicontinuous, $I[u] \leq \liminf_{k\to\infty} I[u_k]$, then we have that

$$\lambda(Q)F(P) \leq \int_Q F(P + Dv)dx$$

From this necessary inequality, we introduce the following definition.

**Definition 1.** A function $F : M^{m\times n} \to \mathbb{R}$ is called **quasiconvex** if for every $P \in M^{m\times n}$ and $v \in \mathscr{D}(U; \mathbb{R}^m)$, we have that

$$\int_U F(P)dx \leq \int_U F(P + Dv)dx \tag{22}$$

Note that the above inequality (22) means that the plane $u(x) = Px$ is a minimizer on $U$ subject to its own boundary values. Therefore (20) tells us that **all quasiconvex functions are rank-one convex**. The converse to this statement was demonstrated to be false by a famous example by Sverak. (This example will be explored in later sections of this document.) By Jensen's inequality, we get that **every convex function is quasiconvex**.

## 2.2 Vector-valued weak lower semicontinuity

Here we follow Evans, Chapter 3, §B. Let us now suppose that $F$ satisfies the following growth condition.

$$0 \leq F(P) \leq C(1 + |P|^q) \qquad (P \in M^{m\times n}) \tag{23}$$

This growth condition will allow us to deduce a semicontinuity property of the functional $I[\cdot]$ in this section. However, in many applications of the calculus of variations in this context, such as nonlinear elasticity, the growth condition (23) is incompatible with physical requirements on $F$. This problem will be discussed in later sections when we analyze *polyconvex* functions $F$. More on this later, however; for now, it suffices to prove a lemma concerning the growth condition that follows for the gradient $DF(P)$.

**Lemma 1.** *Let $F$ be rank-one convex and assume the growth condition* (23)*. Then we have for some constant $C$,*

$$|DF(P)| \leq C(1 + |P|^{q-1}) \qquad (P \in M^{m\times n}) \tag{24}$$

*Proof.* Let $P \in M^{m\times n}$, and fix $1 \leq k \leq m$, $1 \leq i \leq n$. Choose $\eta_k = 1$, $\eta_l = 0$ $(l \neq k)$ and $\xi_i = 1$, $\xi_j = 0$ $(j \neq i)$. If $f$ is defined as in (**??**), then $f$ is convex, and we hence have the estimate

$$|f'(0)| \leq \frac{C}{r} \max_{B(0,r)} |f| \qquad (r > 0)$$

But then (23) implies that

$$\max_{B(0,r)} |f| \leq C(1 + |P|^q + r^q).$$

Setting $r = |P| + 1$, we get that

$$\max_{B(0,r)} |f| \leq C'(1 + |P|^q) \ ,$$

and taking the derivative we remove one power for $|P|$ to get the desired result. □

We will now prove that quasiconvexity is the proper assumption on $F$ that allows for the existence of minimizers when there are appropriate growth conditions on $F$. Additionally, we will prove that under the assumption of strict uniform quasiconvexity of $F$, then the convergence of energies $I[u_k] \to I[u]$ improves weak convergence to strong convergence in $W^{1,q}(U; \mathbb{R}^m)$. Both of these results will be packaged in the following theorem. The reference for this theorem is [2].

**Theorem 3.** *(a) Suppose that $F$ satisfies the growth condition* (23)*. Then the functional $I[\cdot]$ is lower semicontinuous with respect to weak convergence in $W^{1,q}(U; \mathbb{R}^m)$ if and only if $F$ is quasiconvex.*

*(b) Suppose additionally that $F$ is strictly uniformly quasiconvex; that is, there exists some $\gamma > 0$ such that*

$$\int_U F(P) + \gamma |Dv|^q dx \leq \int_U F(P + Dv) dx \qquad (\forall P \in M^{m\times n},\ v \in \mathscr{D}(U; \mathbb{R}^m)) \tag{25}$$

*Suppose that $u_k \rightharpoonup u$ in $W^{1,q}(U; \mathbb{R}^m)$ for a sequence $(u_k) \in W^{1,q}$. If additionally we have the convergence of energies,*

$$\lim_{k\to\infty} I[u_k] = I[u] \ ,$$

*then the weak convergence is improved to strong convergence $u_k \to u$ in $W^{1,q}_{loc}(U; \mathbb{R}^m)$.*

*Proof.* We begin with the proof, noting that the set-ups of the proof of part (a) and (b) are nearly identical for the first page.

The "only if" claim for (a) has already been demonstrated. As for sufficiency, suppose that $u_k \rightharpoonup u$ in $W^{1,q}(U;\mathbb{R}^m)$. Then we know that the norms $\|u_k\|_{W^{1,q}(U;\mathbb{R}^m)}$ are uniformly bounded, and by Rellich's theorem we know that $u_k \to u$ strongly in $L^q(U;\mathbb{R}^m)$. Pass to a subsequence, still denoted $u_k$, such that

$$\lim_{k\to\infty} I[u_k] = \liminf_{k\to\infty} I[u_k].$$

These are finite since the norms are bounded. Then let us define measures $\mu^k := 1 + |Du|^q + |Du_k|^q$. (See Evans, Chapter 1.) By the Banach-Alouglu theorem (or one of its variants), passing to a subsequence of measures, there is a measure $\mu$ such that $\mu^k \rightharpoonup \mu$ weakly. i.e.,

$$\int g d\mu_k \to \int g d\mu \qquad\qquad (\forall g \in C_0(U))$$

These measures form a Banach space under the total-variation norm. (Recall that the total variation $|\mu|(E)$ for a measurable (Borel) set $E$ is defined as $\sup \sum_{j=1}^{\infty} |\mu(E_j)|$, where the supremum is taken over all partitions $E = \cup_{j=1}^{\infty} E_j$, with the $E_j$ pairwise disjoint). As a general property on Banach spaces, norms are weakly lower semicontinuous. We thus have that $\mu$ is a finite measure, since

$$|\mu|(U) \leq \liminf_{k\to\infty} |\mu_k|(U) < \infty$$

Now, there are countably many $P$ such that $\mu(P \cap U) > 0$, where $P$ is a translate of a coordinate hyperplane. The idea is that each hyperplane $P$ is a dyadic translate of the coordinate plane. Now, split $U$ into cubes of side length $\frac{1}{2^i}$. Call $S_i$ the collection of cubes with side length $\frac{1}{2^i}$.

Now fix $\varepsilon > 0$. We want to show that $\liminf_{k\to\infty} I[u_k] \geq I[u] - \varepsilon$ for all such $\varepsilon$. Choose $V \subset\subset U$ (the closure of $V$ is compact and contained in $U$) such that

$$\int_{U\setminus V} F(Du)dx < \varepsilon$$

One can choose such a $V$ by the growth condition on $F$; since the integral is finite, take $V$ large enough so that this holds. Now denote by $(Du)_i$ the piecewise constant function such that for $x \in Q_i$, where $Q_i \in S_i$ is one of the cubes of side length $\frac{1}{2^i}$; i.e.,

$$(Du)_i(x) = \frac{1}{|Q_i|} \int_{Q_i} Dudx$$

Then we have that $(Du)_i \to Du$ strongly in $L^q(U; M^{m\times n})$. Now, by an alternative version of the dominated convergence theorem presented in Lieb and Loss' *Analysis*, that $0 \leq F((Du)_i) \leq 1 + |(Du)_i|^q$ gives that $1 + |(Du)_i|^q \to 1 + |Du|^q$ in $L^1(U;\mathbb{R}^m)$, and further that

$$\int_U F((Du)_i)dx \to \int_U F(Du)dx$$

and thus $\int |F((Du)_i) - F(Du)| \to 0$. So now take $i$ so large that

$$\begin{cases} \frac{1}{2^i} < \text{dist}(V, \partial U) \\ \|Du - (Du)_i\|_{L^q(U;M^{m\times n})} < \varepsilon \\ \|F(Du) - F((Du)_i)\|_{L^1(U)} < \varepsilon \end{cases}$$

Let $\{Q_l\}_{l=1}^m$ be cubes intersecting $V$, so that $V \subset \cup_{l=1}^m Q_l$. Let $0 < \sigma < 1$ and let $\hat{Q}_l$ be the concentric and parallel cube to $Q_l$, except with side length $\sigma \frac{1}{2^i}$. (Get $\hat{Q}_l$ by shrinking $Q_l$ by a factor of $\sigma$.) Choose smooth cutoff functions $\{f_l\}$ for each $\hat{Q}_l$ so that

$$\begin{cases} 0 \le f_l \le 1 \ , & \\ f_l \equiv 1 & \text{on } \hat{Q}_l \ , \\ f_l \equiv 0 & \text{on } (U \setminus Q_l) \cup \partial Q_l \ . \end{cases}$$

Also scale the $f_l$ so that $|Df_l| \le \frac{C2^i}{1-\sigma}$. And let $v_l^k := f_l(u_k - u)$. We know that $u_k \rightharpoonup u$ and that $u_k \to u$ strongly in $L^1$. Let $A_l$ be the value that $(Du)_i$ takes on $Q_l$; namely,

$$A_l := \frac{1}{|Q_l|} \int_{Q_l} Du dx \ .$$

We then have that

$$\begin{aligned} I[u_k] &= \int_U F(Du_k)dx \\ &\ge \sum_{l=1}^m \int_{Q_l} F(Du_k)dx && (F \text{ is positive; } \cup Q_l \subset U) \\ &= \sum_{l=1}^m \int_{Q_l} F\left(Du + D(u_k - u)\right) dx \\ &= \sum_{l=1}^m \int_{Q_l} \underbrace{F(A_l + Dv_l^k)}_{(**)} dx + E_1 + E_2 \ . \end{aligned}$$

Here $E_1$ and $E_2$ are defined as

$$\begin{cases} E_1 = \sum_{l=1}^m \int_{Q_l \setminus \hat{Q}_l} F(Du + D(u_k - u)) - F(Du + Dv_l^k)dx \\ E_2 = \sum_{l=1}^m \int_{Q_l} F(Du + Dv_l^k) - F(A_l + Dv_l^k)dx \end{cases}$$

Until now, the proofs are identical for part (a) and (b) of the theorem. For part (b), we will need to bound two more terms ($E_3'$ and $E_4'$, to be described.) Continuing with the proof of (a), we have that

$$\begin{aligned} I[u_k] &\ge \sum_{l=1}^m \int_{Q_l} F(A_l)dx + E_1 + E_2 && \text{(quasiconvexity applied to (**))} \\ &\ge \int_V F((Du)_i)dx + E_1 + E_2 \\ &:= E_1 + E_2 + E_3 + I[u] \\ &\ge I(u) - |E_1| - |E_2| - |E_3| \end{aligned}$$

Here we define $E_3 := \int_V F((Du)_i) - \int_U F(Du)dx$. We now go about bounding each of the $E_i$.

To bound $E_1$, we use the growth estimate and the product rule to deduce that

$$\begin{aligned}|E_1| &\leq C\sum_{l=1}^{m}\int_{Q_l\setminus\hat{Q}_l}\left(\underbrace{1+|Du|^q+|Du_k|^q}_{\mu^k}+|Df_l|^q|u_k-u|^q\right)\\&\leq C\mu^k\left(\cup_{l=1}^m Q_l\setminus\hat{Q}_l\right)+C(1-\gamma)^{-q}\left\|u_k-u\right\|_{L^q}\cdot 2^i\end{aligned}$$

This implies that $\limsup_{k\to\infty}|E_1| \leq C\mu(\cup_{l=1}^m Q_l\setminus\hat{Q}_l)$. Then, as $\sigma\to 1$ in a countable manner, we get from continuity below the measure of the boundary of $Q_l$. Since the measure of the boundary of $Q_l$ is zero, this bounds the term $E_1$.

As for $E_2$, we have that

$$\begin{aligned}|E_2| &= \left|\sum_{l=1}^{m}\int_{Q_l}\underbrace{F(Du+Dv_l^k)-F(A_l+Dv_l^k)}_{(*)}dx\right|\\&\leq C\sum_{l=1}^{m}\int_{Q_l}\left(1+|Du|^{q-1}+|Du_k|^{q-1}+|Dv_l^k|^{q-1}\right)\underbrace{|Du-(Du)_i|}_{(**)}dx\end{aligned}$$

Using the multivariable fundamental theorem of calculus, we can write (*) as

$$\begin{aligned}&|F(Du+Dv_l^k)-F(A_l+Dv_l^k)| = \left|\int_0^1(-Du-A_l)\cdot DF(A_l+Dv_l^k+t(-Du-A_l))dt\right|\\&\leq c\int_0^1|Du-A_l|\left(1+|Du|^{q-1}+|Dv_l^k|^{q-1}+|Du-A_l|^{q-1}\right)dt\\&= c|Du-A_l|\left(1+|Du|^{q-1}+|Dv_l^k|^{q-1}+|Du-A_l|^{q-1}\right)dt\ .\end{aligned}$$

In the last line, we applied the triangle inequality with the power $q$ and using the $DF$ estimate. Now, we apply Hölder's inequality to the term (**) to get the bound

$$|E_2| \leq \|Du-(Du)_i\|_{L^q}\left(1+\|Du\|_{L^q}+\|Du\|_{L^q}\right) < (KC)\varepsilon$$

Now all that remains is a bound on $E_3$. We will use the $L^1$ estimate and account for the area of integration. Since $\int_{U\setminus V}F(Du)dx < \varepsilon$, we have that

$$|E_3| \leq \int_V|F(Du)_i - F(Du)|dx + \left|\int_{U\setminus V}F(Du)dx\right| \leq 2\varepsilon$$

Since we have appropriately bounded each of $E_1, E_2, E_3$, this completes the proof for part (a) of the theorem.

We can now finish off the proof for part (b) of the theorem. We have that $I[u_k] \to I[u]$ and that $\|u_k - u\|_{W^{1,q}} \to 0$. From before, we have that

$$
\begin{aligned}
I[u_k] &\geq \sum_{l=1}^{m} \int_{Q_l} F(A_l + Dv_l^k)dx + E_1 + E_2 \\
&\geq \sum_{l=1}^{m} \int_{Q_l} F(A_l) + \gamma |Dv_k^l| dx + E_1 + E_2 && \text{(strict uniform convexity)} \\
&\geq \sum_{l=1}^{m} \int_{Q_l} F((Du)_i)dx + \gamma \sum_{l=1}^{m} \int_{\hat{Q}_l} |Du_k - Du|^q dx + E_1 + E_2
\end{aligned}
$$

So we have

$$I(u_k) \geq I(u) + \gamma \int_V |Du_k - Du|^q + E_1 + E_2 + E_3 + E_4$$

Taking lim sup, we get that $E_i$, $i = 1, 2, 3$, are less than $\varepsilon$, whereas we are left with

$$E_4 = \gamma \sum_{l=1}^{m} \int_{Q_l \setminus \hat{Q}_l} |Du_k - Du|^q dx$$

Using the triangle inequality and the definition of our constructed measure, we get that

$$\limsup_{k\to\infty} |E_4| \leq \gamma C \mu \left( \cup_{l=1}^{m} Q_l \setminus \hat{Q}_l \right)$$

Taking $\sigma \to 1$, this quantity goes to zero. Putting this all together, we get that

$$\liminf_{k\to\infty} \underbrace{I(u_k)} \geq \underbrace{I(u)} - \limsup_{k\to\infty} \left( \int_V |Du - Du_k|^q + |E_1| + |E_2| + |E_3| + |E_4| \right)$$

By assumption, the difference of the quantities with the underbraces goes to zero as $k \to \infty$, and this proves the result. $\square$

It turns out that the growth condition (23) is actually incompatible with applications to nonlinear elasticity. In particular, in nonlinear elasticity we require an invertibility condition on solutions to variational problems, which is enforced in the model problem by requiring blowup of the function $F(A)$ as the determinant $\det A \to 0$.

We now turn our attention to another type of convexity which *is* in fact compatible with nonlinear elasticity applications—namely, polyconvexity. In the following we will restrict our focus to functions from $\mathbb{R}^3$ to itself. Recall that we denote $M^n$ as the space of all $n \times n$ matrices with real entries, and that $M^n_+$ is the set of all those matrices $A$ with $\det A > 0$.

**Definition 2.** A function $g : M^3 \to \mathbb{R}$ is said to be **polyconvex** if there is a function $G : M^3 \times M^3 \times (0, \infty) \to \mathbb{R}$ such that $g(A) = G(A, \operatorname{adj} A, \det A)$ for all $A \in M^3_+$, where $G$ is a convex function of each of its variables.

We now prove an existence theorem for minimizers of a polyconvex variational problem. This problem was mentioned in one of Ball's recent papers on open problems [3] and we will follow Ball & Murat's [4] in the presentation of the proof below.

We consider an elastic homogeneous body in reference configuration in a bounded domain $\Omega \subset \mathbb{R}^n$. We assume that the boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup N$ is strongly Lipschitz, with $\partial\Omega_1$ a measurable subset of $\partial\Omega$ with positive $(n-1)$-dimensional measure, and $\mathcal{H}^2(N) = 0$. We consider a mixed placement, zero traction boundary value problem, so that $u(\partial\Omega \setminus \partial\Omega_1)$ is traction-free. The deformation $u : \Omega \to \mathbb{R}^n$ is required to satisfy

$$u(x) = \bar{u}(x) \quad \text{a.e. } x \in \partial\Omega_1 \tag{26}$$

The total energy is given by

$$J(u) := \int_\Omega |g(\nabla u(x))| dx \tag{27}$$

where $g : M^n \to \bar{\mathbb{R}}$ is the stored energy function. The function space for which we consider possible minimizers is

$$\mathcal{A} := \{u \in W^{1,1}(\Omega; \mathbb{R}^n) : \text{ (1) holds and } J(u) < \infty\}$$

The existence theorem is the following:

**Theorem 4.** *Let $n = 3$, and suppose that the stored energy function $g$ satisfies*

*(H1) $g$ is continuous*

*(H2) $g(A) = \infty$ if and only if* $\det A \leq 0$

*(H3) $g$ is polyconvex, so that there is a convex function $G : M^3 \times M^3 \times (0, \infty) \to \mathbb{R}$ such that $g(A) = G(A, \operatorname{adj} A, \det A)$ for all $A \in M^3_+$.*

*(H4) $g(A) \geq K_1 + K\left(|A|^p + |\operatorname{adj} A|^q\right)$ for all $A \in M^3_+$, for some $K > 0$, $K_1$ constant, for $p \geq 2$ and $q \geq \frac{p}{p-1}$.*

*If $\mathcal{A}$ is non-empty, then $J$ attains its absolute minimum on $\mathcal{A}$, and the minimizer $u$ satisfies* $\det \nabla u(x) > 0$ *a.e. $x \in \Omega$.*

*Proof.* First, we would like to extend the corresponding convex function $G$ to be defined on all matrices, not just those with positive determinant. To this end, let us define

$$L := M^3 \times M^3 \times \mathbb{R} \equiv \mathbb{R}^{19} \ , \ L^+ := M^3 \times M^3 \times (0, \infty) \ ,$$

and $\Delta : M^3 \to L$ by $\Delta(A) = (A, \operatorname{adj} A, \det A)$. Let now $\tilde{G} : L \to \bar{\mathbb{R}}$ be the greatest convex function such that

$$g(A) = \tilde{G}(\Delta(A)) \quad \forall A \in M^3_+$$

Such a function exists since the function $\bar{G}$,

$$\bar{G}(H) := \begin{cases} G(H) & H \in L^+ \\ \infty & H \in L \setminus L^+ \end{cases} ,$$

is convex. From the above, we see that $\tilde{G}$ also satisfies $\tilde{G}(H) = \infty$ for all $H \in L \setminus L^+$. We now want to show that $\tilde{G}$ is continuous and agrees with $g$ on $L$. We recall Theorem 4.3 of Ball's paper [5]

**Theorem 5** (Ball [5], Theorem 4.3)**.** *Let $K \subset \mathbb{R}$ be convex and non-empty, and let $U = \{F \in M^3 : \det F \in K\}$. Then, denoting the convex hull of a set $A$ by* $\operatorname{Co} A$, *we have that*

$$\operatorname{Co} \Delta(U) = M^3 \times M^3 \times K \ ,$$

*where $\Delta : M^3 \to M^3 \times M^3 \times \mathbb{R}$ is defined by $\Delta(F) = (F, \operatorname{adj} F, \det F)$.*

Applying this theorem with $K = (0, \infty)$ we deduce that $\operatorname{Co} \Delta(M^3_+) = L^+$; i.e., the convex hull of matrices with positive determinant gives matrices with positive determinant. This means that any $H \in L^+$ can be written as $H = tA + (1-t)B$ with $A = \Delta(a)$ and $B = \Delta(b)$ for some $a, b \in M^3_+$. (We should note that this is not *exactly* true, as we should write $H = \sum_i \alpha_i A_i$ where $\sum_i \alpha_i = 1$, $A_i = \Delta(a_i)$ for $a_i \in M^3_+$, but the analysis following is the same in any case.) Then convexity of $\tilde{G}$ implies that

$$\begin{aligned}\tilde{G}(H) &\leq t\tilde{G}(\Delta(a)) + (1-t)\tilde{G}(\Delta(b)) \\ &= tg(a) + (1-t)g(b) \\ &< \infty & \text{(by (H2))}\end{aligned}$$

Thus we have that $0 \leq \tilde{G}(H) < \infty$ for all $H \in L^+$. Since $\tilde{G}$ is convex and finite on $L^+$, this means that it is continuous on $L^+$. Since it is infinite on the complement of $L^+$, this means that it is continuous on all of $L$.

Let now $M > 0$ be arbitrary, and define $\theta_M$ by

$$\theta_M(A) := \frac{g(A) - M}{\det A}$$

We can take $K_1 \geq 0$ without any loss of generality (otherwise we could just subtract the difference and proceed the proof with the corresponding positive $K_1'$). Since $K_1 \geq 0$, if $\det A \geq 1$ then $\theta_M(A) \geq -M$ since

$$\begin{aligned}\theta_M(A) = \frac{g(A) - M}{\det A} &\geq \frac{K_1 + K\left(|\operatorname{adj} A|^q + |A|^p\right) - M}{\det A} \\ &\geq \frac{-M}{\det A} \geq -M \ .\end{aligned}$$

Let $S := \{A \in M^3 : 0 < \det A \leq 1\}$, and $\xi := \inf_{A \in S} \theta_M(A)$, and let $(A_j) \in S$ be the minimizing sequence of $\theta_M(A)$, so that $\theta_M(A_j) \to \xi$ as $j \to \infty$. By hypothesis (H4), we have that

$$\begin{aligned}K|A_j|^p &\leq g(A_j) - K_1 - K|\operatorname{adj} A_j|^q \\ &= \tilde{G}(\Delta(A_j)) - K_1 - K|\operatorname{adj} A_j|^q < \infty \qquad \text{(since } 0 \leq \tilde{G}(H) < \infty \text{ for } H \in L^+\end{aligned}$$

Thus $A_j$ is bounded, and so a subsequence $A_\mu$ converges to some $A_0$. Since by (H1) $g$ is continuous and $\det A \leq 0$ if and only if $g(A) = \infty$ by (H4), we have that $g(A_\mu) \to g(A_0)$, and so we must have that $\det A_0 > 0$, since otherwise tails of the sequence $A_\mu$ would have non-positive determinant. Since $A_j$ is the minimizing sequence of $\theta_M(A)$ on $S$, we have that $\theta_M(A) \geq \theta_M(A_0)$ for all $A$ in $S$. But this means that

$$\frac{g(A) - M}{\det A} \geq \theta_M(A_0)$$

Thus there is a constant $m = m(M)$ such that $g(A) \geq M - m \det A$ for all $A \in M^3_+$. And hence the function $\theta(A, B, \delta) := M - m\delta$ is an affine function that satisfies $\theta(\Delta(A)) \leq g(A)$ for all $A \in M^3_+$. Since $\tilde{G}$ is the maximal convex function such that $g(A) = \tilde{G}(\Delta(A))$, we have

$$\tilde{G}(A, B, \delta) \geq M - m\delta \quad \forall (A, B, \delta) \in L$$

But $M > 0$ is arbitrary, so as $(A_k, B_k, \delta_k) \to (A, B, 0)$, we have that $\tilde{G}(A_k, B_k, \delta_k) \to \infty$. Thus $\tilde{G}$ is continuous on all of $L = M^3 \times M^3 \times \mathbb{R}$, and $g(A) = \tilde{G}(\Delta(A))$ for all $A \in M^3$.

For the second part of the proof, let now $\{u_j\} \subset \mathcal{A}$ be a minimizing sequence of $J$ over $\mathcal{A}$, and let $\xi := \inf_{w \in \mathcal{A}} J(w)$, so that $J(u_j) \to \xi$ as $j \to \infty$. Let $j$ be large enough so that $\xi + 1 \geq J(u_j)$. We then have that

$$\begin{aligned}
\xi + 1 \geq J(u_j) &= \int_\Omega |g(\nabla u_j)| dx \\
&\geq \int_\Omega |K_1 + K\left(|\nabla u_j|^p + |\operatorname{adj} \nabla u_j|^q\right)| dx && \text{(by (H4))} \\
&= K_1 \mathrm{meas}(\Omega) + K \int_\Omega |\nabla u_j|^p + K \int_\Omega |\operatorname{adj} \nabla u_j|^q \\
&= c + K \left\|Du_j\right\|^p_{L^p} + K \left\|\operatorname{adj} \nabla u_j\right\|^q_{L^q}
\end{aligned}$$

Recall now the Poincaré inequality: $\|Du\|_{L^p} \geq \|u\|_{W^{1,p}}$ for all $u \in W^{1,2}_0(\Omega)$ if $\Omega$ is an open and bounded set in $\mathbb{R}^n$. This inequality applies to $u_j - \bar{u} \in W^{1,p}_0$ so that

$$\begin{aligned}
\left\|Du_j\right\|_{L^p} &\geq \left\|D\bar{u}\right\|_{L^p} + \gamma_1 \left\|u_j\right\|_{W^{1,p}} - \gamma_1 \left\|\bar{u}\right\|_{W^{1,p}} \\
&= \gamma_1 \left\|u_j\right\|_{W^{1,p}} + c
\end{aligned}$$

So there is some $\gamma_2 = \gamma_2(\Omega, p, \bar{u})$ such that $\|Du_j\|^p_{L^p} \geq \|u_j\|^p_{W^{1,p}} + c$. This implies that

$$\xi + 1 \geq c + K\gamma_2 \left\|u_j\right\|^p_{W^{1,p}} + K \left\|\operatorname{adj} Du_j\right\|^q_{L^q}$$

Thus the tails of $u_j$ are bounded in $W^{1,p}$ and the tails of $\operatorname{adj} Du_j$ are bounded in $L^q$. So by Banach-Alaoglu theorem, there is a weakly convergent subsequence $\{u_\mu\}$ such that

$$\begin{cases} u_\mu \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^3) \\ u_\mu \to u \text{ a.e. in } \Omega \\ \operatorname{adj} \nabla u_\mu \rightharpoonup \chi \text{ in } L^q(\Omega; \mathbb{R}^9) \end{cases}$$

Now, Theorem 3.4(a) and (b) from Ball [6] give that

$$\begin{cases} \operatorname{adj} \nabla u_\mu \to \operatorname{adj} \nabla u \text{ in } \mathscr{D}'(\Omega) \\ \det \nabla u_\mu \to \det \nabla u \text{ in } \mathscr{D}'(\Omega) \end{cases},$$

so by uniqueness of limits for distributions, we get that $\chi = \operatorname{adj} \nabla u$. Recall now the elementary inequality $|\det \nabla u_\mu(x)| \leq C |\operatorname{adj} \nabla u_\mu(x)| |\nabla u_\mu(x)|$; using this and Hölder's inequality we get that $\{\det \nabla u_\mu\}$ is bounded in $L^1$:

$$\begin{aligned}
\int_\Omega |\det \nabla u_\mu(x)| dx &= \left\|\det \nabla u_\mu(x)\right\|_{L^1} \\
&\leq C \left\|\operatorname{adj} \nabla u_\mu \cdot \nabla u_\mu\right\|_{L^1} \\
&\leq \left\|\nabla u_\mu\right\|_{L^p} \left\|\operatorname{adj} \nabla u_\mu\right\|_{L^{p'}} < \infty\ .
\end{aligned}$$

That the last quantity is finite follows since $u_\mu \in W^{1,p}$ implies $\nabla u \in L^p$, and $\operatorname{adj} \nabla u_\mu \in L^{p'}$ by assumption. Since $\{\det \nabla u_\mu\}$ is bounded in $L^1$, it follows from Banach-Alaoglu that $\det \nabla u_\mu \to \eta$ for some $\eta$; but uniqueness of limits in distributions gives that $\eta = \det Du$.

So we have that $u_\mu \rightharpoonup u$ in $W^{1,p}$, $Du_\mu \rightharpoonup Du$ in $L^p$, and since $C_c(\Omega)$ is dense in $L^p(\Omega)$, we have that $Du_\mu \rightharpoonup * \; Du$ in the sense of measures. In the same way, we have that $\operatorname{adj} Du_\mu \rightharpoonup * \; \operatorname{adj} Du$. We thus have that

$$(Du_\mu, \operatorname{adj} Du_\mu, \det Du_\mu) \rightharpoonup * \; (Du, \operatorname{adj} Du, \det Du) \text{ in the sense of measures.}$$

We now will use Proposition A.3 of [4], the proof of which is in the paper as well.

**Proposition 1** (Proposition A.3 [4])**.** *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set, and let $H : \mathbb{R}^s \to \bar{\mathbb{R}}$ be convex, lower semicontinuous and bounded below. Let $\theta, \theta_j \in L^1(\Omega; \mathbb{R}^s)$ with $\theta_j \rightharpoonup * \; \theta$ in the sense of measures, so that for each $\phi \in C_c(\Omega)$ we have*

$$\int_\Omega \theta_j \phi dx \to \int_\Omega \theta \phi dx \; .$$

*We then have that*

$$\int_\Omega H\left(\theta(x)\right) dx \leq \liminf_{j \to \infty} \int_\Omega H\left(\theta_j(x)\right) dx$$

We apply the above proposition to deduce that

$$\begin{aligned}
\int_\Omega g(Du(x))dx &= \int_\Omega \tilde{G}(Du, \operatorname{adj} Du, \det Du)dx \\
&\leq \liminf_{\mu \to \infty} \int_\Omega \tilde{G}(Du_\mu(x), \operatorname{adj} Du_\mu(x), \det Du_\mu(x))dx \\
&= \liminf_{\mu \to \infty} \int_\Omega g(Du_\mu(x))
\end{aligned}$$

This implies that

$$J(u) \leq \liminf_{\mu \to \infty} J(u_\mu) = \inf_{\mathcal{A}} J(\cdot)$$

Now, trace theory gives that $u \in \mathcal{A}$ just as before, and since $\int g(Du) < \infty$, we must have that $\det Du(x) > 0$ a.e. □

## 2.3 Some open problems

In the previous section we considered some results for the existence of minimizers to functionals under various structural hypotheses on the integrand. We now consider some of the open questions regarding convexity and existence of minimizers, such as the necessity of growth conditions or bounded domains, or the effect of the choice of function space $W^{1,p}$, or the need to control the determinant or adjugates of the gradient $Du$. The two main papers that we consulted for these open problems were by Ball [3], [7]. The following open problems are taken almost directly from these papers, and I have included some of the calculations that the working group has worked out that Ball left out when discussing these problems.

### 2.3.1 Open problems related to applications to nonlinear elasticity and the weak Euler-Lagrange equations.

Here we consider a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup N$ where $\partial\Omega_1, \partial\Omega_2$ are disjoint, relatively open subsets of $\Omega$ and $N$ has zero two-dimensional Hausdorff measure (area), $\mathcal{H}^2(N) = 0$; we can think of our domain as a cylinder and the corresponding decomposition of the boundary as surfaces where traction may or may not occur. We consider deformations $y : \Omega \to \mathbb{R}^3$ in the function space $W^{1,2}(\Omega; \mathbb{R}^3)$. The gradient is given by a 3-dimensional matrix $Dy(x) = \left(\frac{\partial y_i}{\partial x_j}\right)_{i,j=1}^3$, and we have some fixed boundary condition

$$y|_{\partial\Omega_1} = \bar{y} : \partial\Omega_1 \to \mathbb{R}^3 \tag{28}$$

The functional we consider is $I(\cdot)$ given by

$$I(y) = \int_\Omega W(Dy(x))dx \ , \tag{29}$$

where $W : M^3_+ \to \mathbb{R}$ is the stored-energy function of the material, and is assumed to be $C^1$ and bounded below. We turn our attention again to the existence of minimizers $y^*$ to the functional. A formal calculation (assuming sufficient conditions for differentiation through the integral for the moment) of $\frac{d}{d\tau} I(y^* + \tau\phi)\big|_{\tau=0}$ leads to the weak formulation of the Euler-Lagrange equilibrium equations:

$$\int_\Omega D_A W(Dy) \cdot D\phi dx = 0 \ . \tag{30}$$

which holds for all $\phi$ that are smooth and satisfy $\phi|_{\partial\Omega_1} = 0$. We will consider the assumptions necessary in leading to the passage of the limit momentarily. In nonlinear elastics, a common hypothesis is one of nondegeneracy of the determinant of the gradient; the idea is that only those deformations that may be "undone" are possible in real applications, and that fractures or other problems that may cause "uninvertible" deformations should require infinite energy. This is enforced through the condition that

$$W(A) \to \infty \ \text{ as } \ \det A \to 0+ \tag{31}$$

Now we notice that if a deformation $y$ has finite energy $I(y) < \infty$, then the nondegeneracy condition gives that

$$\det Dy(x) > 0 \,\text{a.e. in } \Omega \tag{32}$$

Now notice that (32) is not sufficient for the passage of the limit in (30). A stronger condition presented below is, however. Let us consider $y^* \in W^{1,\infty}$ as a $W^{1,\infty}$ local minimizer of $I$ in

$$\mathcal{A} = \{y \in W^{1,1}(\Omega; \mathbb{R}^3) : y|_{\partial\Omega_1} = \bar{y}\} \ .$$

In particular, this means that there is some $\varepsilon > 0$ such that for any $z \in \mathcal{A}$ satisfying $\|z - y^*\|_{W^{1,\infty}} \leq \varepsilon$, we have $I(y^*) \leq I(z)$. Suppose we have the nondegeneracy condition (31), and suppose further that we have strict positivity of the Jacobian, so that for some $\varepsilon > 0$,

$$\det Dy^*(x) \geq \varepsilon > 0 \text{ a.e. in } \Omega \tag{33}$$

Then since $Dy^*(x) + \tau D\phi \to Dy^*(x)$, since $\det(\cdot)$ is continuous, we have for $\tau$ small enough,

$$\det(Dy^*(x) + \tau D\phi(x)) \geq \frac{\varepsilon}{2} \text{ a.e. in } \Omega \tag{*}$$

Since $y^* \in W^{1,\infty}$, we have that $\|Dy^*\|_{L^\infty}$ is bounded. Since $\phi$ has compact support, for small $\tau$, $\{Dy^*(x) + \tau D\phi(x) : x \in \Omega\} \subset \mathbb{R}^3$ is in a bounded, compact set. So $W$ is uniformly continuous over its argument, and thus has a maximum. Since the range of $W$ is restricted to $\mathbb{R}$ (rather than $\bar{\mathbb{R}}$), from (*) we can take the Dominated Convergence Theorem to deduce

$$\begin{aligned}
&\lim_{\tau\to 0} \int_\Omega \frac{1}{\tau} \left[W(Dy^* + \tau D\phi) - W(Dy^*)\right] dx \\
&= \int_\Omega DW(Dy^*) \cdot D\phi dx \\
&= 0
\end{aligned}$$

Essential in this passage to the limit was the strict positivity of the Jacobian (33) and *a priori* knowledge that $y^* \in W^{1,\infty}$. What would be helpful in determining the satisfaction of the weak Euler-Lagrange equations (30) is *when* exactly the strict positivity of the Jacobian is satisfied, so that we could use this information as in the above proof. Ball listed the following two open problems after detailing this issue:

**Problem 1.** Prove or disprove, under reasonable growth conditions on $W$, global or suitably defined local minimizers of $I$ that satisfy (30).

**Problem 2.** Prove or disprove, under reasonable growth conditions on $W$, global or suitably defined local minimizers of $I$ that satisfy (33)

We should note that there are functions which satisfy (32) but fail to satisfy the strict positivity (33).

**Example.** An example of a *smooth* deformation $y$ satisfying $I(y) < \infty$, $\det Dy(x) > 0$ a.e., but failure of (33) would be

$$\begin{cases} & \vec{y}(\vec{x}) = |x|^2\vec{x} \\ W(A) = -\log\det A + g(A) & \end{cases},$$

where $g : M^3 \to \mathbb{R}$ is smooth. For such a deformation, its gradient is given by $\frac{\partial y_i}{\partial x_j} = |x|^2\delta_{i,j} + 2x_i x_j$,

$$Dy(x) = \begin{pmatrix} 3x_1^2 + x_2^2 + x_3^2 & 2x_1x_2 & 2x_1x_3 \\ 2x_1x_2 & x_1^2 + 3x_2^2 + x_3^2 & 2x_2x_3 \\ 2x_1x_3 & 2x_2x_3 & x_1^2 + x_2^2 + 3x_3^2 \end{pmatrix}$$

Its determinant is given by $\det Dy(x) = 3\left(x_1^2 + x_2^2 + x_3^2\right)^3 = 3|x|^6$ and we easily see that this satisfies all the mentioned properties.

### 2.3.2 Regularity and classification of singularities

We consider the problem of minimizing the functional

$$I(y) = \int_\Omega W(Dy(x))dx \ , \tag{*}$$

over a bounded domain $\Omega$ with boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup N$, where again $\mathcal{H}^2(N) = 0$. The permissible functions are in some given space $\mathcal{A}$, and $W : M^3_+ \to \mathbb{R}$ is $C^1$ and defined for all matrices with positive determinant. We suppose a traction condition on the boundary $\partial\Omega_1$, whereby $y|_{\partial\Omega_1} = \bar{y}$. A current open problem is the following:

**Problem 3.** When is the local or global minimizer of $I(\cdot)$ over $\mathcal{A}$ smooth, either on all of $\Omega$ or except at points where $x \in \partial\Omega_1 \cap \partial\Omega_2$?

Not much work has been done in this area. Some possible assumptions that may aid in making some work towards this problem include:

- $W$ is $C^\infty$
- $\partial\Omega$ is smooth, except possibly at $\partial\Omega_1 \cap \partial\Omega_2$
- $\bar{y}$ is $C^\infty$
- $W$ is strictly polyconvex, meaning that it is strictly convex in the matrix, adjugate, and determinant.

One known result along these lines is in the case of pure displacement boundary condition, meaning that $\partial\Omega_2 = \emptyset$.

A more ambitious project would be to characterize the various singularities of a possible solution. One result along these lines is the following. Suppose $W$ has a local minimizer. Then $W$ is strictly rank-one convex if and only if every locally weak solution of the corresponding Euler-Lagrange equation to (*) has a constant gradient.

An important physically relevant singularity is that of *cavitation*. An example of radial cavitation would be the deformation $y : B(0,1) \to \mathbb{R}^3$ defined by

$$y(x) = r(|x|)\frac{x}{|x|} \quad (x \in B(0,1)) \tag{**}$$

If $r(0) > 0$ in (**), then $y$ is discontinuous at zero, so that cavitation occurs.

In general, when does cavitation occur? It is known that if $W$ is polyconvex and $\mathcal{A} = W^{1,3}$ with specified boundary conditions, then the minimizer is of the form $\bar{y}(x) = \lambda x$, so that no cavitations occur. Howeer, if $W$ is polyconvex and satisfies

$$W(A) \geq c_0\left(|A|^p + |\operatorname{adj} A|^q\right) - c_1$$

for $2 \leq p < 3$ and $q < \frac{3}{2}$—so, the exact conditions of the existence theorem for polyconvexity, Theorem 4—then $I$ attains a minimizer over $\mathcal{A} = W^{1,1}$ among radial deformations of the form (**) satisfying boundary conditions, and when $\lambda$ is very large, $r(0) > 0$, and so we have cavitation.

Cavitation is itself an example of the *Laurentiev phenomenon*, i.e., for $\mathcal{A}_p = W^{1,p}$, that

$$\inf_{\mathcal{A}_1} I < \inf_{\mathcal{A}_3} I \ .$$

This phenomenon can happen in continuous function spaces as well, e.g. in one dimension. One open problem regarding the Laurentiev phenomenon is the following.

**Problem 4.** Can the Laurentiev phenomenon occur for elastostatics under growth conditions on $W$, ensuring that all finite-energy deformations are continuous?

We do know that there are minimizers of

$$I(y) := \int_\Omega f(Dy(x))dx \ , \qquad (\star)$$

where $f : M^{m\times n} \to \mathbb{R}$, that are not smooth. For example, if $m = n^2$, for large $n$ there are strictly convex $f$ such that the minimizer of $I(\cdot)$ subject to its own boundary condition is of the form

$$y^*(x) = \frac{1}{|x|} x \otimes x \ . \qquad (x \in B(0,1))$$

On the other hand, even if the minimizers are not smooth, one could ask if the set of singularities is "small". A recent result by Evans gives that the minimizer of $(\star)$, where $f$ is strictly convex and satisfies

$$c_1|A|^p - c_0 \le f(A) \le c_2\left(|A|^p + 1\right) \ , \qquad (p \ge 2)$$

is smooth almost everywhere in $\Omega$.

In elasticity, only certain singularities should be allowed: cavitation, fracture, and so on. Some possible reasons for not having other types of singularities may include:

- $W$ depends only on the gradient $Dy$, and not $y$ itself.
- We are working in low dimensions in general ($n = m = 3$).
- Frame indifference condition for the integrand $F$, although this seems to be rarely utilized in proofs that we have seen.
- Requirement of invertibility of $y$, i.e., that $\det Dy > 0$.

#### 2.3.3 Sverak example of a rank-one but not quasiconvex function

Although no longer an open problem, it was a long-standing question about the possible equivalency of quasiconvexity and rank-one convexity. Recall the definitions of these convexities. Let $F : M^{m\times n} \to \mathbb{R}$ where $m \ge 3$ and $n \ge 2$, and we consider $\Omega \subset \mathbb{R}^n$ as a bounded domain.

**Definition 3.** We say that $F$ is **rank-one convex** if for each $A, B \in M^{m\times n}$, where $B$ is rank-one (i.e., $B = \eta \otimes \xi$), the mapping $\mapsto F(A + tB)$ is convex.

**Definition 4.** We say that $F$ is **quasiconvex** if for every $A \in M^{m\times n}$ and for all $\phi \in C_c^\infty(\Omega; \mathbb{R}^m)$,

$$\int_\Omega F(A + D\phi)dx \ge \int_\Omega F(A)dx \ . \qquad \text{(QC)}$$

It has long been known that quasiconvexity implies rank-one convexity. It was also known that if $W$ has quadratic form, i.e. $F = c^{ij}_{kl} A^k_j A^l_i$, then rank-one convexity in fact does imply quasiconvexity. The statement for general $F$, however, was not known until the famous recent example by Sverak [8]. Here we will go through the proof of his example. We begin with a lemma.

**Lemma 2.** *Say that $f$ is $C^0$ on $M^{m\times n}$. Then $f$ is quasiconvex if and only if, for every smooth, periodic (with respect to $\mathbb{Z}^n$) functions $u$, we have that*

$$\int_{[0,1]^n} f(A+Du) \geq \int f(A) \ . \qquad (A \in M^{m\times n})$$

*Proof.* The "if" statement is obvious by extending compactly supported smooth functions to being periodic over $\mathbb{R}^n$.

For the other direction, fix such a smooth, periodic function $u$. For $\varepsilon > 0$, define a cut-off function $\eta_\varepsilon$ satisfying

$$\begin{cases} \eta_\varepsilon \equiv 1 \text{ on } [\varepsilon, 1-\varepsilon]^n \\ 0 \text{ on a neighbourhood of } \partial([0,1]^n) \\ |D\eta_\varepsilon| \leq \frac{2}{\varepsilon} \end{cases}$$

Let us now define

$$u_\varepsilon(x) := \varepsilon^2 \eta_\varepsilon(x) u(x^2/\varepsilon) \ . \qquad (*)$$

Then $u_\varepsilon$ is smooth with support in $\Omega$. We will prove the lemma for $A = 0$ in the following, but it will work for any $A$. We then have that

$$f(0) \leq \int_{[0,1]^n} f(Du_\varepsilon(x))dx$$

The idea is to take $\varepsilon$ to zero and get out $Du$; the $\varepsilon$ will cancel in (*). Let us take $\varepsilon = \frac{1}{k}$, and break $[0,1]^n$ into little cubes $\{Q_k\}$ with sides $\frac{1}{k}$. This brings the above inequality into

$$\begin{aligned} f(0) &\leq \sum_{Q_k} \int_{Q_k} f\left(\varepsilon^2 u \otimes D\eta_\varepsilon(x) + \eta_\varepsilon Du(x/\varepsilon^2)\right) dx \\ &= \lim_{k\to\infty} \frac{1}{k^n} \sum_{x\in\mathbb{Z}^n} \int_{[0,1]^n} f\left(\varepsilon^2 u \otimes D\eta_\varepsilon(x+\hat{x}) + \eta_\varepsilon Du(x)\right) dx \\ &= \lim_{k\to\infty} \frac{1}{k^n} \sum \int_{[0,1]^n} f\left(\eta_\varepsilon Du(x)\right) = \lim_{k\to\infty} \frac{1}{k^n} \sum \int_{[0,1]^n} f(Du(x)) \ . \end{aligned}$$

This completes the proof of the lemma. □

We will now construct the desired counter example for the case $n = 2$ and $m = 3$, and after the completion of the proof we will explain how to extend the proof to case of general $n, m$.

Let $W : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$W(x) = \frac{1}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_2 \\ \sin 2\pi(x_1+x_2) \end{pmatrix} \ .$$

Then the gradient of $W$ is given by

$$DW(x) = \begin{pmatrix} \cos 2\pi x_1 & 0 \\ 0 & \cos 2\pi x_2 \\ \cos 2\pi(x_1+x_2) & \cos 2\pi(x_1+x_2) \end{pmatrix} .$$

Let now $L = \left\{ \begin{pmatrix} r & 0 \\ 0 & s \\ t & t \end{pmatrix} \ : \ r,s,t \in \mathbb{R} \right\}$, and let $f : L \to \mathbb{R}$ be defined by $f(x) = -rst$. We will extend $f$ to a function on all matrices, and will show that the functional satisfies rank-one but not quasi- convexity.

In $L$, there are only three rank-one directions:

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \ , \ B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ B_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} .$$

i.e., the coordinates must satisfy

$$\begin{cases} x_1y_1 = a & x_1y_2 = 0 \\ x_2y_1 = 0 & x_2y_2 = b \\ x_3y_1 = c & x_3y_2 = c \end{cases} .$$

Then $f(A+tB_i)$ is linear in $t$ for rank-one $B_i$: $f(A+tB_i) = -(a_{11}+t)a_{22}a_{33}$. Thus $f$ is rank-one on $L$. We then have that

$$\begin{aligned} \int_{[0,1]^2} f(Du)dx &= -\int_{[0,1]^2} \cos(2\pi x_1)\cos(2\pi x_2)\cos\left(2\pi(x_1+x_2)\right)dx_1dx_2 \\ &= -\int_{[0,1]^2} \cos^2(2\pi x_1)\cos^2(2\pi x_2)dx_1dx_2 \\ &+ \int_{[0,1]^2} \sin(2\pi x_1)\cos(2\pi x_1)\underbrace{\sin(2\pi x_2)}_{=0}\cos(2\pi x_2)dx_1dx_2 \\ &< 0 = f(0) \end{aligned}$$

The last inequality follows since the first term in the second line is negative due to the positive integrand. Thus $f$ is *not quasi-convex on* $L$. We now want to extend the function to all of $M^2$. Before doing so, we will need the following lemma.

**Lemma 3.** *Let $L$ and $f$ be as before, and define*

$$F(x) = f(Px) + \varepsilon|x|^2 + \varepsilon|x|^4 + k|x-Px|^2$$

*where $|\cdot|^2$ represents the $\ell^2$ norm, i.e., the sums of squares, and $P$ is the orthogonal projection of $M^{3,2} \to L$, so that*

$$P\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \\ e & f \end{pmatrix} .$$

*Then for every $\varepsilon > 0$ there is $k$ such that $F$ is rank-one convex.*

*Proof.* Fix $A, Y \in M^{3\times 2}$. Then, using the product rule and the fact that $|A+tY|^2 = \sum(a_i + ty_i)^2$,

$$\begin{aligned}
&\frac{d^2}{dt^2} F(A+tY)\Big|_{t=0} \\
&= \frac{d^2}{dt^2} f(A+tPY)\Big|_{t=0} + 2\varepsilon|Y|^2 + 4\varepsilon|A|^2 + 7\varepsilon\left(\frac{d}{dt}\frac{|A+tY|^2}{2}\right)^2 + 2k|Y-PY|^2
\end{aligned}$$

Now, all of these terms are strictly positive except for $\frac{d}{dt^2}f(\cdot)$, for which we want a lower bound. We have

$$f(PA+tPY) = (a_{11}+ty_{11})(a_{22}+ty_{22})(a_{33}+ty_{33}) \ .$$

Taking the second derivative, we get a constant $c$, independent of $A, Y \in M^{3\times 2}$, such that

$$\frac{d}{dt^2} f(PA+tPY)\Big|_{t=0} \geq -c|A|\cdot|Y|^2 \ .$$

This means that

$$\frac{d^2}{dt^2} F(A+tY)\Big|_{t=0} \geq \left(-c|A| + 4\varepsilon|A|^2\right)|Y|^2 \geq 0 \ ,$$

where the last inequality follows provided that $|A| \geq \frac{c}{4\varepsilon}$. In this case, $F$ is rank-one convex. To prove the lemma, we need to show that the second derivative is non-negative everywhere. We have that

$$\frac{d^2}{dt^2} F(A+tY) \geq \frac{d^2}{dt^2} f(PA+tPY)\Big|_{t=0} + 2\varepsilon|Y|^2 + 2K|Y-PY|^2 =: g(A,Y,k) \ .$$

Then $g$ is $C^0$.

Let us define $\mathcal{K} := \{(A,Y) \in M^{3\times 2}\times M^{3\times 2} : |A| \leq \frac{c}{4\varepsilon}, |Y| = 1, Y \text{ is rank-one convex}\}$. Then $\mathcal{K}$ is a closed and bounded subset of $\mathbb{R}^s$, and hence compact.

**Claim.** *There is $k_0$ such that $g(A,Y,k_0) \geq \varepsilon$ for all $(A,Y) \in \mathcal{K}$.*

*Proof of claim.* Suppose not. Then for $k = 1, \ldots,$ there is $(A_k, Y_k) \in \mathcal{K}$ such that $g(A_k, Y_k) \leq \varepsilon$. Since $\mathcal{K}$ is compact, passing to a subsequence we get that $(A_k, Y_k) \to (\hat{A}, \hat{Y})$. This implies that

$$\lim_{k\to\infty} 2\varepsilon|Y_k|^2 + 2k|Y_k - PY_k|^2 + \frac{d^2}{dt^2} f(PA_k + tPY_k)\Big|_{t=0} \leq -\varepsilon \ .$$

Therefore $\hat{Y} = P\hat{Y}$, and so $\frac{d^2}{dt^2} f(P\hat{A} + tP\hat{Y})\Big|_{t=0} \leq -\varepsilon$. But this is a contradiction to the fact that $f$ is rank-one convex. This proves the claim. ☐

So there is $k_0$ such that $\frac{d^2}{dt^2}F(A+tY)|_{t_0} = \frac{d^2}{dt^2}F((A+t_0Y)+tY)|_{t=0} \geq 0$ for all $A \in M^{3\times 2}$, with $Y$ rank-one and $|Y| = 1$. So we have shown that for all $\varepsilon > 0$ there is $k_\varepsilon$ such that $F$ is rank-one convex, and this proves the lemma. ☐

We can now prove the desired theorem.

**Theorem 6.** *There is $\varepsilon$, $k_\varepsilon$ such that $F$ is rank-one convex but not quasiconvex.*

*Proof.* We have that $\int_{[0,1]^2} f(Dw)dx < 0$. We see that $|Dw|$ is bounded (see the previous calculation at the beginning of the section), so that there is $\varepsilon > 0$ such that

$$\int_{[0,1]^2} \left( f(Du) + \varepsilon|Dw|^2 + \varepsilon|Dw|^4 \right) dx < 0 \ .$$

Now $\varepsilon > 0$ is fixed. Take $k_\varepsilon$ as in the previous lemma. Then $F$ is rank-one convex, but now

$$F(Dw) = f(Dw) + \varepsilon|Dw|^2 + \varepsilon|Dw|^4 + k_\varepsilon \underbrace{|PDw - Dw|^2}_{=0 \text{ since already in subspace}} \ .$$

This implies that

$$\int_{[0,1]^2} F(Dw) = \int_{[0,1]^2} f(Dw) + \varepsilon|Dw|^2 + \varepsilon|Dw|^4 < 0 = F(0) \ .$$

Thus $F$ is *not* quasiconvex in the case $n = 2$ and $m = 3$. To extend to $n > 2, m > 3$, take $T : M^{m\times n} \to M^{3\times 2}$ defined by

$$T \begin{pmatrix} x_{11} & \cdots \\ \ddots & \cdots \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} \ .$$

Then $T$ preserves rank-one lines, and so $\tilde{F} : M^{m\times n} \to \mathbb{R}$ defined by $\tilde{F}(X) = F(TX)$ is rank-one convex, and we can take $\hat{W} : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$\tilde{W}(x) = \frac{1}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_2 \\ \sin 2\pi(x_1 + x_2) \\ 0 \\ 0 \\ \vdots \end{pmatrix} \ .$$

□

# 3 Compactness methods for nonlinear PDEs

## 3.1 Concentrated compactness

The idea for this section is to analyze the sets where compactness may fail, in the hopes of somehow showing these sets are negligible in some sense, or ideally that these sets are empty. The reference is Evans, Chapter 4.

### 3.1.1 Variational problems

To do so, we will study variational problems involving (Sobolev) critical growth nonlinearities. Such problems barely fail to satisfy the usual compactness (Sobolev embedding) criteria.

To that end, we will consider the following model problem. For $n \geq 3$, we want to study the existence of minimizers for the functional,

$$I(w) = \int_{\mathbb{R}^n} |Dw|^2 dx \ , \tag{34}$$

over the set of admissable functions $\mathcal{A} = \{w \in L^{2^*}(\mathbb{R}^n) : \|w\|_{L^{2^*}} = 1, Dw \in L^2\}$. Here, $p^* = \frac{np}{n-p} > p$ is the Sobolev conjugate of the exponent $p$.

Recall the Gagiardo-Nirenberg-Sobolev inequality: for $1 \leq q < n$, we have the inequality

$$\|f\|_{L^{q^*}(\mathbb{R}^n)} \leq C_q \|Df\|_{L^q(\mathbb{R}^n;\mathbb{R}^n)} \ , \tag{GNS}$$

which holds for all $f \in C_0^1(\mathbb{R}^n)$; by the usual density arguments, the inequality is also valid when $f \in L^{q^*}$ and $Df \in L^q$. In this inequality, $C_q = C_q(q,n)$ is an optimal constant. Note that the infimum of the functional (34) can be expressed in terms of this constant. We have that

$$\left(\int |Df|^q\right)^{\frac{1}{q}} \geq C_q^{-1} \left(\int |f|^{q^*}\right)^{\frac{1}{q^*}} \ ,$$

which gives that $\int |Df|^q \geq C_q^{-q} \|f\|_{L^{q^*}}^q$; thus we have

$$I(f) = \int |Df|^2 \geq C_2^{-2} \|f\|_{L^{2^*}}^2 \ .$$

Since $C_2$ is an optimal constant, we get that

$$I := \inf_{w \in \mathcal{A}} I(w) = C_2^{-2} \ .$$

Now we turn to the question at hand: is the infimum $I$ attained by a function in $\mathcal{A}$? As usual, consider a minimizing sequence $\{u_k\} \subset \mathcal{A}$, so that

$$I(u_k) = \int |Du_k|^2 \to I \ .$$

Since $I = C_2^{-2} < \infty$, this means that $\sup_k \|Du_k\|_{L^2} < \infty$, and since $\|u_k\|_{L^{2^*}} = 1$ for each $k$, we can extract a weakly convergent subsequence, not relabeled, such that

$$\begin{cases} Du_k \rightharpoonup Du \text{ in } L^2 \\ u_k \rightharpoonup u \text{ in } L^{2^*} \end{cases} .$$

Since $I(\cdot)$ is convex, we know that it is lower semicontinuous with respect to weak convergence from the preceding sections, so that $\liminf_{k\to\infty} I(u_k) = I$. So, *if* $u \in \mathcal{A}$, then we indeed have a minimizer to $I$ over $\mathcal{A}$. Clearly $Du \in L^2$, so we need only concern ourselves with $\|u\|_{L^{2^*}}$. As a general property of weak convergence in Banach spaces, we know that

$$\|u\|_{L^{2^*}} \leq \liminf_{k\to\infty} \|u_k\|_{L^{2^*}} = 1 \ . \tag{*}$$

We need to prove that the above is a legitimate equality. We will cast the above problem in terms of probability measures, given by integration of functions along $\mathbb{R}^n$. We first need a definition.

**Definition 5.** Let $(X, \tau)$ be a topological space, and let $\Sigma$ be a $\sigma$-algebra containing $\tau$ (namely, containing the Borel $\sigma$-algebra). A collection of measures $M$ on $\Sigma$ is called **tight** if for every $\varepsilon > 0$ there is compact set $K_\varepsilon \subset X$ such that for each measure $\mu \in M$, we have that $|\mu(X \setminus K_\varepsilon) < \varepsilon$.

Notice that in the specific case of probability measures, the above reduces to the existence of compact sets $K_\varepsilon$ such that $\mu(K_\varepsilon) > 1 - \varepsilon$. We now recall a result about probability measures on separable metric spaces, known as Prokhorov's theorem.

**Theorem 7.** *Let $(X, d)$ be a separable metric space, and let $\mathcal{P}(X)$ denote the collection of all probability measures on $(X, \mathcal{B}_X)$. Then a collection of measures $K \subset \mathcal{P}(X)$ is tight if and only if the closure of $K$ is weakly sequentially compact in $\mathcal{P}(X)$.*

To return to the problem at hand, we want to show that $\|u\|_{L^{2^*}} = 1$. The inequality in (*) may be strict if either of these two possibilities occur:

(1) The measures $\{\nu_k\} := \{|u_k|^{2^*}\}$ (defined by integration over $\mathbb{R}^n$) are not tight. In this case, $\{\nu_k\}$ does *not* have a weakly convergent subsequence in $\mathcal{P}(X)$; i.e., there is no probability measure $\bar{\nu} = |\bar{u}|^{2^*}$ such that $\nu(\mathbb{R}^n) = \int_{\mathbb{R}^n} |\bar{u}|^{2^*} dx = 1$.

(2) The measures $\{\nu_k\}$ are actually tight, but the limit $\nu_k \to \nu$ does not correspond to the limit $u$; i.e.,
$$1 = \nu(\mathbb{R}^n) \neq \int_{\mathbb{R}^n} |u|^{2^*} dx \ .$$
Such a possibility may occur if $\nu$ has a singular part. Recall that by the Lebesgue-Radon-Nikodym theorem, we can decompose the measure $\nu$ into the sum of an absolutely continuous measure and a singular measure. That is, there are measures $\nu_0$ and $\nu_1$, where $\nu_0$ is absolutely continuous with respect to the Lebesgue measure $m$ on $\mathbb{R}^n$, and $\nu_1$ is mutually singular with respect to $m$. (This means that there is some $g \in L^1(\mathbb{R}^n)$ such that $\nu_0(A) = \int_A g dm$ for all measurable sets $A \subset \mathbb{R}^n$.) We then have that
$$1 = \nu(\mathbb{R}^n) = \nu_0(\mathbb{R}^n) + \nu_1(\mathbb{R}^n) \ .$$
So if $\nu_1$ is not identically zero—namely, if $\nu$ has a non-trivial singular component—then $\nu$ will not directly correspond to integration over $\mathbb{R}^n$ to the Radon-Nikodym derivative $g$. If it were zero, however, then since the Radon-Nikodym derivative is unique, we would hope that $u = g$.

Singular components of measures generally arise when there are highly oscillatory functions that are being dealt with. What is particularly unfortunate is that our problem at hand is not affected by changes in translations or oscillations: our choice of minimizing sequence could be quite unfortunate in this respect. Let $v \in \mathcal{A}, y \in \mathbb{R}^n, s > 0$ be arbitrary. Then define
$$v^{y,s}(x) := s^{-(n-2)/2} v\left(\frac{x-y}{s}\right) \ . \qquad (x \in \mathbb{R}^n)$$

Then we have that $1 = \int_{\mathbb{R}^n} |v|^{2^*} dx = \int_{\mathbb{R}^n} |v^{y,s}|^{2^*} dx$, as well as $\int_{\mathbb{R}^n} |Dv|^2 = \int_{\mathbb{R}^n} |Dv^{y,s}|^2$. To see this, write $u = (x-y)/s$, so that $x = us + y$, and note that $2^* = \frac{2n}{n-2}$, so that $-2^*/2 = -n/(n-2)$, and so

$$|v^{y,s}(u)|^{2^*} = |s^{-(n-2)/2}|^{2^*} |v(u)|^{2^*} = s^{-n} |v(u)|^{2^*} \ .$$

The Jacobian cancels out the $s^{-n}$ so that the integral of $|v^{y,s}|^{2^*}$ is also one. For the gradient, we have that

$$|Dv^{y,s}(u)|^2 = |\frac{1}{s} s^{-(n-2)/2} Dv(u)|^2 = |s^{-2} s^{-(n-2)}||Dv(u)|^2 = s^{-n} |Dv(u)|^2 \ ,$$

so that the Jacobian again cancels out the $s^{-n}$. So we see that by taking $s > 0$ very small, we can construct highly oscillatory functions which satisfy the requirements of the minimizing sequence, but whose measures may have singular parts and hence preclude the existence of a minimizer.

What is a more interesting possibility is that for a given minimizing sequence $\{u_k\}$, one can construct translations and dilations so that the new minimizing sequence $\{u_k^{y_k,s_k}\}$ actually is a minimizer.

**Theorem 8.** *Let $\{u_k\} \subset \mathcal{A}$ be a minimizing sequence to $I(\cdot)$, so that $I(u_k) \to I = \inf_{\mathcal{A}} I(w)$. As before, $Du_k \rightharpoonup Du$ in $L^2$, and $u_k \rightharpoonup u$ in $L^{2^*}$. Then there are translations $\{y_k\} \subset \mathbb{R}^n$ and dilations $\{s_k\} \subset (0,\infty)$ such that the rescaled family $\{u_k^{y_k,s_k} \subset \mathcal{A}$ is strongly precompact in $L^{2^*}$. In particular, there is a minimizer $u \in \mathcal{A}$.*

*Proof.* First we define the Lévy concentration functions

$$Q_k(t) := \sup_{y \in \mathbb{R}^n} \int_{B(y,t)} |u_k|^{2^*} dx \ . \qquad (t > 0, k = 1, \ldots)$$

Then $Q_k^{y,s}(t) = Q_k^{y,1}(t/s)$, where $Q_k^{y,s}$ is the concentration of function of $u_k^{y,s}$. This identity follows since $|s^{-(n-2)/2}|^{2^*} = s^{-n}$, so

$$Q_k^{y,s}(t) = \sup_{z \in \mathbb{R}^n} \int_{B(z,t)} \left| s^{-(n-2)/2} u_k \left( \frac{x-y}{s} \right) \right|^{2^*} dx = \sup_{z \in \mathbb{R}^n} \int_{B(z,t/s)} |u_k(x-y)|^{2^*} dx \ .$$

Note that $Q_k(0) = 0$ and that $\lim_{t\to\infty} Q_k(t) = \lim_{t\to\infty} \sup_{y\in\mathbb{R}^n} \int_{B(y,t)} |u_k|^{2^*} \to 1$ since $u_k \in \mathcal{A}$ implies that $\|u_k\|_{L^{2^*}} = 1$. For fixed $k$, $Q_k$ is also a continuous, increasing function of $t$, as the integral is taken over a larger area for larger $t$ and the integrand is positive. So $Q_k(t)$ takes values in $[0,1]$. Hence we can choose dilations $\{s_k\}$ such that $Q_k^{y,s_k}(1) = \frac{1}{2}$ for all $y \in \mathbb{R}^n$ $(k = 1, \ldots)$. This is possible since

$$Q_k^{y,s_k}(1) = Q_k^{y,1}(\frac{1}{s_k}) = \sup_{z \in \mathbb{R}^n} \int_{B(z,\frac{1}{s_k})} |u_k(x-y)|^{2^*} dx \ ,$$

and $Q_k(t) = \sup_{z\in\mathbb{R}^n} \int_{B(z,t)} |u_k(x)|^{2^*} dx$.

This allows us to choose translations $\{y_k\}$ such that $\nu_k^{y_k,s_k} := |u_k^{y_k,s_k}|^{2^*}$ are tight in $\mathcal{M}(\mathbb{R}^n)$. (Recall Definition 5). See Lions [9], Lemma I.1 for more information on this.

Now for notational simplicity, let us assume that the translations and dilations above were unnecessary, so that the measures $\{\nu_k\}$ are tight. (In the following, we could instead write $\nu_k^{y_k,s_k}$ instead). By Prokhorov's theorem (Theorem 7), there is a subsequence, still denoted $\nu_k$, such that

$$\nu_k \rightharpoonup \nu \text{ in } \mathcal{M}(\mathbb{R}^n),\ \nu(\mathbb{R}^n) = 1 \ .$$

Since $Du_k \in L^2$ for all $k$, and $\int |Du_k|^2 \to C_2^{-2} < \infty$, we also have that

$$\mu_k \rightharpoonup \mu \text{ in } \mathcal{M}(\mathbb{R}^n) \ ,$$

for $\mu_k := |Du_k|^2$.

For the third step of the proof, we would like to show that $u \neq 0$. Let us recall Theorem 9 of Chapter 1 of Evans.

**Theorem 9.** *For $n \geq 3$, suppose that $f_k \to f$ in $L^2_{loc}$, $Df_k \rightharpoonup Df$ in $L^2$, $|Df_k|^2 \rightharpoonup \mu$ in $\mathcal{M}$, and that $|f_k|^{2^*} \rightharpoonup \nu$ in $\mathcal{M}$. Then we have the following.*

*(i) There is a countable index set $J$, and countable set of distinct points $\{x_j\} \subset \mathbb{R}^n$ and non-negative weights $\{\mu_j, \nu_j\}_{j\in J}$ such that*

$$\nu = |f|^{2^*} + \sum_J \nu_j \delta_{x_j} \ ,$$

*and*

$$\mu \geq |Df|^2 + \sum_J \mu_j \delta_{x_j} \ .$$

*(ii) $\nu_j \leq C_2^{2^*} \mu_j^{2^*/2}$ for each $j$.*

*(iii) If $f \equiv 0$ and also $\nu(\mathbb{R}^n)^{1/2^*} \geq C_2 \mu(\mathbb{R}^n)^{1/2}$, then $\nu$ is concentrated at a single point.*

Now, notice that since $u_k \rightharpoonup u$ in $L^{2^*}$, we have that $u_k \to u$ strongly in $L^2_{\text{loc}}$, since $2^* > 2$. We also have that $\mu_k(\mathbb{R}^n) = \int_{\mathbb{R}^n} |Du_k|^2 \to I$, and by weak convergence we have that

$$\mu(\mathbb{R}^n) \leq \liminf_{k\to\infty} \mu_k(\mathbb{R}^n) = \liminf_{k\to\infty} \int_{\mathbb{R}^n} |Du_k|^2 = I = C_2^{-2} \ .$$

Therefore $\mu(\mathbb{R}^n)^{1/2} \leq C_2^{-1}$, and so $C_2\mu(\mathbb{R}^n)^{1/2} = \nu(\mathbb{R}^n) = \nu(\mathbb{R}^n)^{1/2^*}$. Thus we are in the setup of Theorem 9. Suppose for the contradiction that $u \equiv 0$. Then by (iii), $\nu$ is concentrated at a single point $x_0$. Thus

$$\frac{1}{2} = Q_k(1) \geq \int_{B(x_0,1)} |u_k|^{2^*} dx \ .$$

But for large $k$, the quantity on the right hand side goes to one, which is a contradiction. Thus $u \neq 0$.

The final step of the proof is to show that $u \in \mathcal{A}$; namely, that the $L^{2^*}$ norm is actually equal to one. Suppose not. Then for some $0 < \alpha < 1$, we have $\|u\|_{L^{2^*}}^{2^*} = \alpha$. ($\alpha \neq 0$ since $u \neq 0$.) Let us define

$$I_\alpha := \inf_{w\in\mathcal{A}_\alpha} I(w) \ ,$$

where

$$A_\alpha := \{w \in L^{2^*} : \|w\|_{L^{2^*}}^{2^*} = \alpha, Dw \in L^2\} \ .$$

Then as before, the Sobolev inequality states that $I(f) = \int |Df|^2 \geq C_2^{-2} \|f\|_{L^{2^*}}^2$, and since $C_2$ is an optimal constant, we get that

$$\begin{aligned} I_\alpha &= \inf_{w \in \mathcal{A}_\alpha} I(w) = \inf_{\|w\|_{L^{2^*}} = \alpha^{-2^*}} I(w) \\ &= C_2^{-2} \alpha^{2/2^*} \\ &= I \alpha^{2/2^*} \ . \end{aligned}$$

We now can conclude the proof. By (i) and (ii) of Theorem 9, we have that

$$\nu = |u|^{2^*} + \sum_J \nu_j \delta_{x_j} \ , \mu \geq |Du|^2 + \sum_J \mu_j \delta_{x_j}$$

for some countable points $\{x_j\}_{j \in J}$ and positive weights $\{\nu_j, \mu_j\}$ such that $\nu_j \leq C_2^{2^*} \mu_j^{2^*/2}$. This implies that for each $j \in J$,

$$\mu_j \geq C_2^{-2} \nu_j^{2/2^*} = I \nu_j^{2/2^*} \ .$$

Integrating the first quantity over $\mathbb{R}^n$, we get that $1 = \alpha + \sum_J \nu_j$. This gives the contradiction, as

$$\begin{aligned} I &\geq \mu(\mathbb{R}^n) \\ &\geq \int_{\mathbb{R}^n} |Du|^2 + \sum_J \mu_j \\ &\geq I_\alpha + \sum_J \mu_j = I \alpha^{2/2^*} + \sum_J \mu_j \\ &\geq I \left( \alpha^{2/2^*} + \sum_J \nu_j^{2/2^*} \right) \\ &> I \ . \end{aligned}$$

The final strict inequality follows since $2^* > 2$ and $\alpha < 1$ implies that $\alpha^{2/2^*} > \alpha$; and if any of the $\nu_j \geq 1$, then we needn't worry about the satisfaction of the inequality, and if any $\nu_j < 1$, then $\nu_j^{2/2^*} > \nu_j$ so that $(\alpha^{2/2^*} + \sum_j \nu_j^{2/2^*}) > 1$. This completes the proof. $\square$

We now consider a variation on the preceding problem, where we deal with a similar functional but with a lower order perturbation. The problem we consider is that of minimizers for

$$I^\lambda(w) = \int_U |Dw|^2 - \lambda w^2 dx \ ,$$

defined over the set of admissable functions

$$\mathcal{A} := \{w \in W_0^{1,2}(U) : \|w\|_{L^{2^*}} = 1\} \ .$$

Here, $U \subset \mathbb{R}^n$ is a bounded domain. As usual, let $\{u_k\} \subset \mathcal{A}$ be the minimizing sequence to $I(\cdot)$ over $\mathcal{A}$, so that

$$I^\lambda(u_k) \to \inf_{w\in\mathcal{A}} I^\lambda(w) =: I_\lambda \ .$$

Write $I_0 = \inf_{v\in\mathcal{A}} I^0(v)$ and $I^0(w) = \int_U |Dw|^2$. As to prove compactness, we begin with the following lemma, which shows that the minimum energy $I_\lambda$ is strictly smaller than $I_0$ when $\lambda > 0$. This should intuitively make sense since we are *subtracting* a positive quantity in the functional.

**Lemma 4.** *If $\lambda > 0$ and $n \geq 4$, then $I_\lambda < I_0$.*

*Proof.* From Section 1 of this chapter in Evans, we know that if $U = \mathbb{R}^n$, then the infimum is attained by functions of the form

$$u_{y,\varepsilon} = \frac{c_\varepsilon}{(\varepsilon + |x-y|^2)^{\frac{n-2}{2}}} \ ,$$

for some $\varepsilon > 0$, and $c_\varepsilon$ is a normalization constant so that $u_{y,\varepsilon}$ has the appropriate norm to belong to the desired function space. But $U \subsetneq \mathbb{R}^n$ is bounded. So let us assume for simplicity that $0 \in U$. One would guess that $u_{0,\varepsilon}$ would be a good candidate for a minimizer to the functional for small $\varepsilon$, except that $u_{0,\varepsilon} \neq 0$ on $\partial U$. So let us fix this defect by writing

$$v^\varepsilon(x) := \zeta(x) u_{0,\varepsilon}(x) \ ,$$

where $\zeta \in C_c^\infty(U)$ is a cutoff function such that $\zeta \equiv 1$ near the origin. Choose $c_\varepsilon$ so that $\|v^\varepsilon\|_{L^{2^*}(U)} = 1$. An analysis performed by Brezis and Nirenberg [10] (see Lemma 1.1) makes the estimates

$$\begin{cases} \int_U |Dv^\varepsilon|^2 = \frac{K_1}{\varepsilon^{(n-2)/2}} + O(1) \\ \left(\int_U |v^\varepsilon|^{2^*} dx\right)^{2/2^*} = \|v^\varepsilon\|_{2^*}^2 = \frac{K_2}{\varepsilon^{(n-2)/2}} + O(\varepsilon) \\ \int_U |v^\varepsilon|^2 dx = \begin{cases} \frac{K_3}{\varepsilon^{(n-4)/2}} + O(1) & \text{if } n \geq 5 \\ K_3|\log\varepsilon| + O(1) & \text{if } n = 4 \end{cases} \end{cases} .$$

The paper goes through all of the necessary calculations and merely involve basic calculus. These estimates imply that

$$I^\lambda(v^\varepsilon) = \begin{cases} I_0 + O(\varepsilon^{(n-2)/2}) - \lambda K\varepsilon & \text{if } n \geq 5 \\ I_0 + O(\varepsilon) - \lambda K|\log\varepsilon| & \text{if } n = 4 \end{cases} .$$

So choose $\varepsilon > 0$ small enough so that $I_\lambda \leq I^\lambda(v^\varepsilon) < I_0$, as desired. The reason for a different bound for the $n = 4$ case in the final estimate is because if we just assume a general $n \geq 4$, we have the following. Recall that $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to 0$ means that there are $\delta, M > 0$ such that for $|\varepsilon| < \delta$, $|f(\varepsilon)| \leq M|g(\varepsilon)$. This means that $I^\lambda(v^\varepsilon) \leq I_0 - \lambda K\varepsilon + M\varepsilon^{(n-2)/2}$; if we want this quantity to be strictly smaller than $I_0$, we need that $\varepsilon^{(n-4)/2} < \frac{\lambda K}{M}$, and so we cannot take $\varepsilon$ small enough for the desired strict inequality to hold unless $n \geq 5$. $\square$

We now begin with the actual theorem on compactness.

**Theorem 10.** *Let $0 < \lambda < \lambda_1$, where $\lambda_1$ is the principal eigenvalue for $-\Delta$ on $W_0^{1,2}(U)$. Let $\{u_k\} \subset \mathcal{A}$ be a minimizing sequence for $I$ and $n \geq 4$. Then there is a subsequence $\{u_{k_j}\} \subset \{u_k\}$ and a function $u \in \mathcal{A}$ such that $u_{k_j} \to u$ strongly in $W_0^{1,2}(U)$. In particular, $u \in \mathcal{A}$ is a minimizer.*

*Proof.* Step 1. Since $\{u_k\}$ is a minimizing sequence, we have that

$$I^\lambda(u_k) = I_\lambda + o(1) \ . \tag{*}$$

Since $\{u_k\}$ is bounded in $W_0^{1,2}(U)$, there is a subsequence $u_{k_j}$ such that

$$\begin{cases} u_{k_j} \rightharpoonup u \text{ in } W_0^{1,2} \\ u_{k_j} \to u \text{ a.e. and strongly in } L^2(U) \end{cases} . \tag{**}$$

The reason $u_{k_j} \to u$ strongly in $L^2$ is, of course, due to Rellich compactness.

Step 2. Define $v_{k_j} := u_{k_j} - u$. Then $v_{k_j} \rightharpoonup 0$ in $W_0^{1,2}$, $v_{k_j} \to 0$ a.e., and so (*) gives that

$$I^\lambda(u) + I^0(v_{k_j}) = \int_U |Du|^2 - \lambda|u|^2 dx + \int |Dv_{k_j}|^2 = I_\lambda + o(1) \ . \tag{***}$$

This $o(1)$ comes from writing $u = v_{k_j} + u_{k_j}$.

We now need a refinement of Fatou's lemma (from Evans, Chapter 1) to go on with the proof.

**Lemma 5.** *Suppose that $U$ is a bounded, open, and smooth domain, and that $1 \leq q < \infty$, with a sequence $(f_k) \in L^q$ such that $f_k \rightharpoonup f$ in $L^q(U)$, and $f_k \to f$ a.e. in $U$. We then have that the norms decouple in the limit,*

$$\lim_{k\to\infty} (\|f_k\|_{L^q}^q - \|f_k - f\|_{L^q}^q) = \|f\|_{L^q}^q \ .$$

The proof of this lemma is fairly elementary. Recall first the inequality

$$||a+b|^q - |a|^q| \leq \varepsilon |a|^q + C(\varepsilon)|b|^q \ , \tag{\#}$$

which holds for $a, b \in \mathbb{R}$ and $C(\varepsilon)$ depends only on $\varepsilon$ and $q$. Define the function

$$g_k^\varepsilon := (||f_k|^q - |f_k - f|^q - |f|^q| - \varepsilon|f_k - f|^q)^+ \ ,$$

where $(\cdot)^+$ denotes the positive part. Then $g_k^\varepsilon(x) \to 0$ a.e. as $k \to \infty$. Applying the inequality (#) with $a = f_k - f$, $b = f$ with the triangle inequality gives

$$\begin{aligned} g_k^\varepsilon &\leq (||f_k|^q - |f_k - f|^q - |f|^q| + |f|^q - \varepsilon|f_k - f|^q)^+ \\ &\leq (\varepsilon|f_k - f|^q + C(\varepsilon)|f|^q + |f|^q - \varepsilon|f_k - f|^q)^+ \\ &= (1 + C(\varepsilon))\,|f|^q \ . \end{aligned}$$

Dominated convergence then gives that

$$\lim_{k\to\infty} \int_U g_k^\varepsilon dx = 0 \ .$$

Since $||f_k|^q - |f-f_k|^q - |f|| \leq g_k^\varepsilon + \varepsilon|f_k - f|^q$, we get that

$$\limsup_{k\to\infty} \int_U ||f_k|^q - |f-f_k|^q - |f||\, dx \leq \varepsilon \limsup_{k\to\infty} \int_U |f_k - f|^q dx = O(\varepsilon)$$

since weakly convergent sequences are bounded. This completes the desired refinement of Fatou's lemma.

We can now apply the above to deduce that

$$1 = \left\|u_{k_j}\right\|_{2^*}^{2^*} = \|u\|_{2^*}^{2^*} + \left\|v_{k_j}\right\|_{2^*}^{2^*} + o(1)\ .$$

Then the elementary inequality $(a+b)^\eta \leq a^\eta + b^\eta$ (which holds for $a, b > 0$ and $0 \leq \eta \leq 1$) gives us that

$$1 + o(1) \leq \left(\int_U |u|^{2^*} + |v_{k_j}|^{2^*}\right)^{\frac{2}{2^*}} \leq \|u\|_{L^{2^*}}^2 + \left\|v_{k_j}\right\|_{L^{2^*}}^2\ . \tag{$\triangle$}$$

Now (***) and ($\triangle$) give that

$$I^\lambda(u) + I^0(v_{k_j}) \leq I_\lambda\left(\|u\|_{L^{2^*}}^2 + \left\|v_{k_j}\right\|_{L^{2^*}}^2\right) + o(1)\ . \tag{$\triangle\triangle$}$$

Step 3. By definition, we know that $I_\lambda \leq I^\lambda(u)$. By weak convergence, $\|u\|_{L^{2^*}} \leq 1$, and so

$$I_\lambda \|u\|_{L^{2^*}}^2 \leq I^\lambda(u)\ ,$$

so that ($\triangle\triangle$) is now

$$I^0(v_{k_j}) \leq I_\lambda \left\|v_{k_j}\right\|_{L^{2^*}}^2 + o(1)\ .$$

For large $j$, since $v_{k_j} \to 0$ a.e., $\left\|v_{k_j}\right\|_{L^{2^*}} \leq 1$, and by definition of infimum, we have that

$$I_0 \left\|v_{k_j}\right\|_{L^{2^*}} \leq I^0(v_{k_j})\ .$$

This implies

$$(I_0 - I_\lambda) I^0(v_{k_j}) \leq o(1)\ .$$

By Lemma 4, $I_0 - I_\lambda > 0$, and so $I^0(v_{k_j}) \to 0$ *strongly*, i.e., $\int_U |Du_{k_j} - Du|^2 \to 0$. This means that $u_{k_j} \to u$ strongly in $W_0^{1,2}(U)$, so that $u \in \mathcal{A}$ is a minimizer to $I^\lambda(\cdot)$. $\square$

### 3.1.2 Nonvariational problems

In the last section we focused on concentration methods concerning the lack of compactness of the injection $L^{2^*} \hookrightarrow W^{1,2}$. We now consider some nonvariational problems where compactness is again in question due to critical growth nonlinearities.

We consider a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, where $\Omega$ is a bounded domain. The non-variational problem of interest is

$$\begin{cases} -\Delta u = b(Du) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}. \tag{*}$$

We subject the function $b$ to the growth condition

$$b(P) \le C\left(1 + |P|^2\right) \ . \tag{GC}$$

Suppose that we have a sequence of approximate solutions

$$-\Delta u_k = b_k(Du_k), \ \ u_k|_{\partial\Omega} = 0 \ ,$$

and suppose that $\{u_k\}$ is bounded in $W_0^{1,2} \cap L^\infty$, and suppose that each $b_k$ satisfies (GC), and that $b_k \to b$ locally uniformly. Now we ask the obvious: if $u_k \rightharpoonup u$ in $W^{1,2}$, does $u$ solve (*)? We first consider a scalar problem to gain some insight into the problem.

**Theorem 11.** *Suppose we have a sequence of approximating scalar problems $-\Delta u_k = f_k$. Suppose that $\{u_k\}$ is bounded in $W^{1,2}(\Omega)$, and that $\{f_k\}$ is bounded in $L^1(\Omega)$. Then $\{u_k\}$ is precompact in $W^{1,p}(\Omega)$ for $1 \le p < 2$.*

*Proof.* Passing to a subsequence, we get that $u_k \rightharpoonup u$ in $W^{1,2}$ and $u_k \to u$ a.e. and strongly in $L^2$.

**Claim.** *$Du_k \to Du$ a.e.*

Assuming the claim is true, let $v_k = |Du_k - Du|^p$; then $v_k \to 0$ a.e. By Egorov's theorem, for each $\delta > 0$, there is a measurable set $E_\delta$ such that $v_k \to 0$ uniformly on $E_\delta$ with $m(\Omega \setminus E_\delta) < \delta$. So we have

$$\int v_k \le \int_{E_\delta} |Du_k - Du|^p + \int_{E_\delta^c} |Du_k|^p + \int_{E_\delta^c} |Du|^p \ ,$$

and we want to show that these last two terms go to zero as $\delta \downarrow 0$. We know that $\{Du_k\}$ is uniformly bounded in $L^2$. Thus by Hölder's inequality,

$$\int_{E_\delta^c} |Du_k|^p \le \left( \int_{E_\delta^c} |Du_k|^2 \right)^{p/2} \delta^{\frac{2-p}{2}} = o\left(\delta^{\frac{2-p}{2}}\right) \ .$$

Thus this term goes to zero as $\delta \to 0$ so long as $p < 2$.

We now prove the claim. Let $\sigma, \delta > 0$ and introduce the function $\beta$ defined by

$$\beta = \begin{cases} \sigma & x \ge \sigma \\ -\sigma & x \le -\sigma \\ \text{linear function connecting function} & x \in (-\sigma, \sigma) \end{cases} \ .$$

For any function $u$, the composition $\beta \circ u$ truncates the function to be between $\pm\sigma$. By Egorov's theorem, $u_k \to u$ uniformly on $E_\delta$. So let $\zeta$ be a cut-off function such that $\zeta \equiv 1$ on $E_\delta$, and we may suppose that $|u_k - u| \le \frac{\sigma}{2}$ on $E_\delta$. (We can do this by just considering the tails of $\{u_k\}$. Then, taking into consideration the weak formulation of the problem at hand,

$$\begin{aligned} \int_{E_\delta} |Du_k - Du|^2 &\le \int_\Omega \zeta D(u_k - u) \cdot D\beta(u_k - u) \\ &= \int f_k \zeta \beta(u_k - u) - \int D\zeta \cdot Du_k \cdot \beta(u_k - u) - \int \zeta Du - D\beta(u_k - u) \ . \end{aligned}$$

This implies that

$$\limsup_{k\to\infty} \int_{E_\delta} |Du_k - Du|^2 \le \sigma \sup_k \|f_k\|_{L^1} = O(\sigma \ .$$

Taking $\sigma \downarrow 0$ finishes the proof. □

We now consider a new type of method for analysing possible failures of compactness, known as the *capacity method*. We define the $p$-capacity of a set as follows.

**Definition 6.** The **$p$-capacity** of a set is defined by

$$C_p(A) := \inf \left\{ \|Df\|_{L^p}^p : f \in L^{p^*}, Df \in L^p, A \subset \operatorname{int}\{f \ge 1\} \right\} .$$

Let us suppose that $Du_k \to Du$ in $L^p$ for some $p < 2$. We define the **reduced defect measure** $\theta$ by

$$\theta(E) := \limsup_{k\to\infty} \int_E |Du_k - Du|^2 dx \ ,$$

where $E \subset \Omega$ is a Borel set. We say that the $p$-capacity of a reduced defect measure $\theta$ is zero, denoted $C_p[\theta] = 0$, if there is a sequence of open sets $\{V_i\}$ with $\theta(\Omega \setminus V_i) = 0$ with $C_p(V_i) \to 0$ as $i \to \infty$. We now utilize these tools in proving compactness for vector non-variational problems.

**Theorem 12.** *Let $m > 1$ and suppose that $u_k \rightharpoonup u$ in $W^{1,2}$. Then $C_p[\theta] = 0$ for $1 \le p < 2$. Furthermore, if we also know that $C_2[\theta] = 0$, then $u$ is a solution to the PDE (*) introduced at the beginning of this section.*

*Proof.* Let $1 \le p < 2$. By Theorem 7, Chapter 1 of Evans, for any $\delta > 0$ there is a relatively closed set $E_\delta \subset \Omega$ such that $u_k \to u$ uniformly on $E_\delta$ with $C_p(\Omega \setminus E) \le \delta$. (Note that this $E_\delta$ is not the same $E_\delta$ from Egorov's theorem.) By the same argument as before, we get that

$$\int_{E_\delta} |Du_k - Du|^2 \le \sigma \sup \|b_k(Du_k)\|_{L^1} \le C\left(\sigma + \sigma \sup \|Du_k\|_{L^2}^2\right) = O(\sigma) \ .$$

Since $\sigma > 0$ is arbitrary, $\theta(E_\delta) = 0$. So we can choose $V_i = E_{\delta_i}^c$ for $\delta_i \to 0$ to get that $C_p[\theta] = 0$.

Now suppose that $C_2[\theta] = 0$. Then there are open sets $\{V_i\}$ such that $\theta(\Omega \setminus V_i) = 0$ with $C_2(V_i) \to 0$. By definition of capacity, there are $\phi_i$, $0 \le \phi_i \le 1$, such that $\phi_i \equiv 0$ on $V_i$, $\phi_i \to 1$ a.e., and $\|D\phi_i\|_{L^2} \to 0$.

Now pick $v \in W^{1,2} \cap L^\infty$, and test $\phi_i v$. It is identically zero on $V_i$, which are the sets where we do *not* have strong convergence. We have that

$$\int_\Omega \phi_i Du_k : Dv + D\phi_i^T Du_k \cdot v = \int_\Omega \phi_i b_k(Du_k) \cdot v \ .$$

We have that $Du_k \to Du$ strongly in $L^2(\Omega \setminus V_i)$ by defect measure property, and so

$$\int_\Omega \phi_i Du : Dv + D\phi_i^T Du \cdot v = \int_\Omega \phi_i b(Du) \cdot v \ .$$

Using dominated convergence, let now $i \to \infty$. Then $\phi_i \to 1$, and Cauchy-Schwarz on $D\phi_i^T \in L^2$, $v \in L^\infty$ allows us to deduce

$$\int_\Omega Du : Dv = \int_\Omega b(Du) \cdot v \ .$$

This is precisely that $u$ is a weak solution to $-\Delta u = b(Du)$. ☐

## 3.2 Compensated compactness

The idea of this section is to consider some nonvariational PDEs for which oscillations in minimizing sequences may cause some types of issues for compactness. We first consider some direct methods that can be used for specific PDEs to address some passages to limits. The reference is Evans, Chapter 5.

We consider first harmonic maps into spheres. That is, functions $u : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is a bounded domain, that satisfy

$$\begin{cases} -\Delta u = |Du|^2 u & \text{in } U \\ |u| = 1 & \text{a.e. in } U \end{cases} . \tag{*}$$

These equations arise from the (weak) Euler-Lagrange equations for

$$I(w) = \int_U |Dw|^2 dx \ ,$$

taken over the admissible set of functions

$$\mathcal{A} = \{w \in W^{1,2}(U; \mathbb{R}^m) : |w| = 1 \text{ a.e.}, \ w = g \text{ on } \partial U\} \ .$$

A **harmonic map** is a weak solution $u \in \mathcal{A}$ of (*). We will first show that the harmonic map structure of functions is preserved under weak convergence in $W^{1,2}$.

**Theorem 13.** *Suppose $\{u_k\} \subset \mathcal{A}$ are harmonic maps with $u_k \rightharpoonup u$ in $W^{1,2}(U; \mathbb{R}^m)$. Then $u$ is also a harmonic map.*

*Proof.* Step 1. Fix indices $1 \leq i, j \leq m$. For $v^i, v^j \in W_0^{1,2} \cap L^\infty$, we have

$$\int_U Du_k^i \cdot Dv^i dx = \int |Du_k|^2 u_k^i v^i dx, \ \ \int_U Du_k^j \cdot Dv^j dx = \int |Du_k|^2 u_k^j v^j dx \ .$$

So let now $v^i = u_k^j w$ and $v^j = u_k^i w$ for $w \in C_c^\infty(U)$. Subtracting the expressions, we obtain

$$\begin{aligned} 0 &= \int_U Du_k^i \cdot D(u_k^j w) - Du_k^j \cdot D(u_k^j w) dx \\ &= \int_U \left(Du_k^i \cdot Dw\right) u_k^j - \int_U \left(Du_k^j \cdot Dw\right) u_k^i dx \ . \end{aligned}$$

Since $u_k \rightharpoonup u$ in $W^{1,2}$, by Rellich compactness we have that $u_k \to u$ strongly in $L^2$, and so as $k \to \infty$ we have that

$$0 = \int_U u^j Du^i \cdot Dwdx - \int_U u^i Du^j \cdot Dwdx \ . \tag{**}$$

By density, (**) holds for $w \in W_0^{1,2}(U) \cap L^\infty(U)$.

*Step 2*. Given $v \in W_0^{1,2}(U) \cap L^\infty(U)$, set $w = u^j v^i$ in (**) to get that

$$0 = \int u^j Du^i \cdot D(u^j v^i) - u^i Du^j \cdot D(u^j v^i) dx \ .$$

The above holds in summation $\sum_{i,j}$ as well. Expanding this out and using the product rule, we get that

$$\begin{aligned} 0 = \sum_{i,j} \int & |uv|^2 \left(Du^i \cdot Du^i\right) + u^j \left(Du^9 \cdot Du^j\right) v^i \\ & - u^i \left(Du^j \cdot Du^j\right) v^i - u^i \left(Du^j \cdot Dv^i\right) u^j \ . \end{aligned}$$

Summing over $j$ first, and then $i$, we get that

$$0 = \int Du : Dv + 0 - |Du|^2 u \cdot v - 0 \ .$$

This is exactly what it means for $u$ to be a weak solution. The reason the second and fourth terms are zero is because $|u|^2 = \sum_i (u^i)^2 = 1$, which implies $(Du)^T u = 0$, so differentiating gives us that

$$\frac{1}{2} \frac{\partial}{\partial x^j} \left( \sum_i u^{i^2} \right) = 0 = \sum_j \left( \sum_i u^i \frac{\partial u^i}{\partial x_j} \right) \ .$$

The second term is $u^j \left( \frac{\partial u^i}{\partial x_k} \frac{\partial u^i}{\partial x^k} \right) v^i$ and thus goes to zero. $\square$

We now consider the homogenization of divergence structure PDEs. The model problem we consider is

$$\begin{cases} \left(a_{ij}(x/\varepsilon) u^\varepsilon_{x_i}\right)_{x_j} = f \text{ in } U \\ u^\varepsilon = 0 \text{ on } \partial U \ . \end{cases} \tag{*}$$

Here $A = (a_{ij})_{ij}$ is a matrix, $\varepsilon > 0$ is small and we employ summation notation. We suppose a uniform ellipticity condition

$$\xi^T A(u) \xi \geq \nu |\xi|^2 \ , \tag{UE}$$

for $\xi \in \mathbb{R}^n$, $\nu > 0$, and $y \in Y$ where $Y$ is the unit cube in $\mathbb{R}^n$; we will restrict our focus to the unit cube in $\mathbb{R}^n$ as the result for general bounded domains follows easily from this. We also suppose that $A$ is uniformly bounded and $Y$ periodic, so that

$$\begin{cases} |A(y)| \leq C \\ y \mapsto A(y) \text{ is } Y\text{-periodic} \ . \end{cases}$$

Let now $u^\varepsilon \in W_0^{1,2}$ be a weak solution to the PDE (*), so that for each $v \in W_0^{1,2}(U)$,

$$\int_U a_{ij}(x\varepsilon) u^\varepsilon_{x_i} v_{x_j} dx = \int_U f v dx \ . \tag{**}$$

Set now $v = u^\varepsilon$ and use (UE) with Hölder's inequality to deduce that

$$\nu \int_U |Du^\varepsilon|^2 dx \leq \int_U f u^\varepsilon dx \leq \|f\|_{L^2} \|u^\varepsilon\|_{L^2} \ .$$

We can now use Young's inequality, Poincaré's inequality and divide by the constant in Poincaré's inequality to get a uniform bound $\sup_{\varepsilon>0} \|u^\varepsilon\|_{W_0^{1,2}(U)} < \infty$. We can therefore pass to a subsequence $u_k \rightharpoonup u \in W_0^{1,2}(U)$.

There is no obvious way to pass to the limit in (**). We instead consider a sequence of "adjoint corrector problems" for $1 \leq \ell \leq n$:

$$\begin{cases} -\left(a_{ij}(y) w^\ell_{y_j}\right)_{y_i} = \left(a_{i\ell}(y)\right)_{y_i} \\ w^\ell \text{ is } Y\text{-periodic} \ . \end{cases} \tag{***}$$

Since $a_{i\ell}$ is $Y$-periodic, the right hand side integrates to zero, so by the Fredholm alternative there is a weak solution $w^\ell$ to (***). Let us now define the **averaged coefficients**

$$\tilde{a}_{i\ell} := \int_Y a_{ij}(y) \left(\delta_{j\ell} + w^\ell_{yj}(y)\right) dy \ .$$

We will show that $u$ satisfies the PDE with these *averaged* coefficients $\tilde{a}$ rather than $a$.

**Theorem 14.** *The weak limit $u$ is a weak solution to the homogenized problem*

$$\begin{cases} -\left(\tilde{a}_{i\ell} u_{x_i}\right)_{x_\ell} = f \ \text{ in } U \\ u = 0 \ \text{ on } \partial U \ . \end{cases} \tag{HP}$$

*Proof.* Step 1. We know that $a_{ij}(x/eps_k) u^{\varepsilon_k}_{x_i} \rightharpoonup h^j$ in $L^2(U)$ as $k \to \infty$. Since $u_{x_i}$ are bounded, and the coefficients $a_{ij}$ are uniformly bounded, we have, for each $v \in W_0^{1,2}$,

$$\int_U h \cdot Dv dx = \int_U f v dx = \int_U a_{ij}(x\varepsilon) u^\varepsilon_{x_i} v_{x_j} dx \ .$$

Step 2. Fix $1 \leq \ell \leq m$ and define the **corrector**

$$v^\varepsilon(x) := x_\ell + \varepsilon w^\ell(x/\varepsilon) \ , \qquad (x \in U)$$

where $x_\ell$ denotes the $\ell$th component of $x$. We know that $w^\ell$ satisfies $-(a_{ij}(y) w^\ell_{y_j})_{y_i} = (a_{i\ell}(y))_{y_i}$. This means, by taking derivatives, that $v^\varepsilon$ is a weak solution of

$$-\left(a_{ij}(x/\varepsilon) v^\varepsilon_{x_j}\right)_{x_i} = 0 \ .$$

Let now $\zeta in C_c^\infty(U)$ and substitute $v = \zeta v^\varepsilon$ into (**). We then have

$$\begin{aligned} \int_U f v \zeta dx &= \int_U a_{ij}(x/\varepsilon) u^\varepsilon_{x_i} (v^\varepsilon \zeta)_{x_j} dx \\ &= \int_U a_{ij}(x/\varepsilon) u^\varepsilon_{x_i} \left(\zeta_{x_j} v^\varepsilon + v^\varepsilon_{x_j} \zeta\right) dx && \text{(product rule and int. by parts)} \\ &= \int_U a_{ij}(x/\varepsilon) u_{x_i} \zeta_{x_j} v^\varepsilon - a_{ij}(x/\varepsilon) v^\varepsilon_{x_j} u^\varepsilon \zeta_{x_i} dx && \text{(IV)} \end{aligned}$$

Step 3. We know that $v^{\varepsilon_k} \to x_\ell$ in $L^2(U)$. Differentiating and recalling the definitions of $v^\varepsilon = v^{\varepsilon,\ell}$, $\tilde{a}_{i\ell}$ we get

$$a_{ij}(x/\varepsilon_k)v^{\varepsilon_k}_{x_j} = a_{ij}(x/\varepsilon_k)\left(\delta_{j_k} + w^\ell_{x_j}(x/\varepsilon_k)\right) \rightharpoonup \tilde{a}_{i\ell} \text{ in } L^2(U) \ .$$

Now send $\varepsilon \to 0$ in (IV). Since $v^{\varepsilon_k} \to x_\ell$, and by weak convergence for $h$, $\tilde{a}_{i\ell}$, we have

$$\int f\zeta x_\ell dx = \int hv_{x_\ell}\zeta_{x_j} - \tilde{a}_{i\ell}u\zeta_{x_i}dx \ .$$

Since $v^\varepsilon\zeta \in W^{1,2}_0$, we know that $\int fv^\varepsilon\zeta = \int h \cdot D(v^\varepsilon\zeta)$, and so differentiating again gives

$$\int fx_\ell\zeta dx = \int_U h_\ell\zeta + h^j x_\ell\zeta_{x_j}dx \ ,$$

so that

$$\int (\tilde{a}_{i\ell}u)_{x_i}\zeta dx = \int h_\ell\zeta dx \ .$$

This means that for each component $\ell$, we have the identity (since $\tilde{a}_{i\ell}$ is constant)

$$\sum (\tilde{a}_{i\ell}u)\, x_i = \sum \tilde{a}_{i\ell}u_{x_i} = h_\ell \ .$$

Plugging this into $\int h \cdot Dv = \int fv$, we get that $\int \tilde{a}_{i\ell}u_{x_i} \cdot Dv = \int fv$, so that (HP) is indeed satisfied,

$$-\left(\tilde{a}_{i\ell}u_{x_i}\right)_{x_\ell} = f \text{ in } U \ .$$

□

We now consider a new type of structural constraint on the behaviour of gradients in divergence-free fully nonlinear PDEs. The problem of interest is

$$\begin{cases} -\operatorname{div}(E(Du)) = f \text{ in } U \\ u = 0 \text{ on } \partial U \ . \end{cases} \tag{*}$$

Here, $E : \mathbb{R}^n \to \mathbb{R}^n$ is given. If $E = DF$, $F : \mathbb{R}^n \to \mathbb{R}$, then we have a variational problem (for which (*) will be the associated Euler-Lagrange equation) and we know from Chapter 2 of Evans that convexity is the proper structural hypothesis. In this case, we know from convexity that

$$(E(p) - E(q)) \cdot (p - q) = (DF(p) - DF(q)) \cdot (p - q) \geq 0 \ .$$

In the case that $E$ is not given by $DF$, the above inequality suggests that we suppose that $E$ is **monotone**, meaning that for each $p, q \in \mathbb{R}^n$,

$$(E(p) - E(q)) \cdot (p - q) \geq 0 \ . \tag{M}$$

Now suppose also that we have the growth condition

$$E(p) \leq C(1 + |p|) \tag{GC}$$

for $E$. And further suppose that we have a sequence of approximating solutions $\{u_k, f_k\}$ where $f_k \to f$ in $L^2$ and $u_k \in W_0^{1,2}$ are weak solutions to $- \div (E(Du_k)) = f_k$ in $U$, so that

$$\int E(Du_k) \cdot Dvdx = \int f_k v dx \ , \tag{**}$$

with $u_k \rightharpoonup u$ in $W_0^{1,2}$. Then we have the following result.

**Theorem 15.** *The weak limit $u$ is a solution to the PDE (*).*

*Proof.* By (M), we know that for each $v \in W_0^{1,2}$,

$$0 \leq \int_U (E(Dv) - E(Du_k)) \cdot (Dv - Du_k) dx \ .$$

Putting $v - u_k$ into ($**$), we get that

$$0 \leq \int_U E(Dv) \cdot (Dv - Du_k) - f_k(v - u_k) dx \ .$$

Using the growth condition, we take $k \to \infty$ to get

$$0 \leq \int_U E(Dv) \cdot (Dv - Du) - f(v - u) dx \ .$$

For $\lambda > 0$ and fixed $w \in W_0^{1,2}(U)$, let $v = u + \lambda w$. We then have that

$$0 \leq \int_U E(Du + \lambda Dw) \cdot Dw - fwdx \ .$$

Take now $\lambda \to 0$ to get the inequality $0 \leq \int_U E(Du) \cdot Dw - fwdx$; but we can apply this same analysis to $-w$ to get the opposite inequality, so that $u$ solves the PDE (*). $\square$

# 4 Maximum principle methods

The maximum principle is often used for nonlinear PDE to provide estimates, but our interest here is to use maximum principles for justifying weak convergence techniques. The reference for this section is Evans, Chapter 6. To this end, let us consider the model problem of a fully nonlinear PDE

$$\begin{cases} F(D^2 u) = f \text{ in } U \\ u = 0 \text{ on } \partial U \ . \end{cases} \tag{*}$$

Assume $F : S^n \to \mathbb{R}$ is given, where $S^n$ denotes the space of $n \times n$ symmetric matrices. We assume that the problem (*) is *elliptic*, i.e., that $F$ is monotone decreasing with respect to matrix ordering in $S^n$ (so that $A \geq B$ means $x^T Ax \geq x^T Bx$ for all column vectors $x$). In other words, $F$ satisfies

$$F(P) \leq F(P) \text{ if } P \geq R \ \ (S, R \in S^n) \tag{E}$$

Suppose that $f_k \to f$ uniformly, and that additionally $u_k$ is a smooth solution of the approximating problem $F(D^2 u_k) = f_k$ in $U$, $u_k = 0$ on $\partial U$. Suppose that $\{u_k\}$ is bounded in $W^{2,\infty}(U)$, so that we can extract a subsequence $\{u_{k_j}$ with

$$u_{k_j} \to u \text{ uniformly}$$

$$D^2 u_{k_j} \rightharpoonup \star D^2 u \text{ in } L^\infty(U; S^n)$$

Then does $u$ satisfy (*)? If $F$ is convex and uniformly elliptic, then there are strong estimates that are available (see Evans' text for references on this), so we instead focus on the non-convex case. This setup is similar to that of the last section where we considered divergence-structure quasilinear PDEs. Let us recast the above into a more abstract form to gain an insight into the fundamental structures that allow the desired satisfaction of PDEs under weak convergence.

Let us suppose that we are given a sequecne of approximation problems in a Hilbert space $\mathcal{H}$, so that $A(u_k) = f_k$, where $A(\cdot)$ is a given operator mapping its domain $D(A) \subset \mathcal{H}$ into $\mathcal{H}$. We suppose that $A$ satisfies a *monotonicity condition*

$$0 \leq \langle A(u) - A(v), u - v \rangle \quad (u, v \in D(A)) \ . \tag{MC}$$

For $u = u_k$, we then have that for each $k$,

$$0 \leq \langle Av - f, v - u_k \rangle \quad (k = 1, \ldots) \ .$$

So if $u_k \rightharpoonup u$ in $\mathcal{H}$ and $f_k \to f$ in $\mathcal{H}$, we deduce that in the limit

$$0 \leq \langle Av - f, v - u \rangle \ .$$

For $w \in D(A)$, let $v = u + \lambda w$. If we assume $A$ to be continuous on finite dimensional subspaces of $\mathcal{H}$—which is an appropriate assumption if $A$ represents a differential operator—then we know $A(u + \lambda w) = A(u) + o(1)$, and so dividing by $\lambda > 0$ and sending $\lambda \to 0$ gives us

$$0 \leq \langle A(u) - f, w \rangle \ ,$$

for each $w \in D(A)$. Thus this is also true for $-w \in D(A)$, and so we have equality in the above. If we assume $D(A)$ to be dense in $\mathcal{H}$, since we have shown that $0 = \langle A(u) - f, w \rangle$ for a dense subspace of $\mathcal{H}$, this means that

$$A(u) = f \ .$$

Unfortunately, the operator $A(u) := F(D^2 u)$ does *not* satisfy (MC) for even smooth functions $u, v$ where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H} = L^2(U)$. On the other hand, solutions of elliptic PDEs should satisfy some maximum and/or comparison principles, which are pointwise in nature. This suggests that we should shift our focus from $L^2$ to $L^\infty$ or even $C(\bar{U})$. The question, then, is to ask whether or not suitably defined operators $A$ on such spaces satisfy (MC). However, the lack of inner product structure on $L^\infty$ and $C(\bar{U})$ occlude this immediate possibility.

Instead let us consider another possible type of monotonicity condition. Consider a real Banach space $X$. The Sato bracket $[\cdot, \cdot]_+$ is defined as

$$[f, g]_+ = \lim_{\lambda \to 0+} \frac{\|g + \lambda f\|^2 - \|g\|^2}{2\lambda} := \lim_{\lambda \to 0+} \Delta(f, g, \lambda)$$

This limit exists for all $f, g \in X$. First notice that $t \mapsto \|g + tf\|$ is convex, so that the mapping $t \mapsto \Delta(f, g, t)$ is monotonic increasing for all $t > 0$. Also notice that $\Delta$ is bounded below by $-\|g\|^2$ for all $t > 0$, and since $\Delta$ is monotonic and bounded above and below, its limit exists for all $f, g$. Notice that if $X = \mathcal{H}$, a Hilbert space, then $[f, g]_+ = \langle f, g \rangle$, since

$$\langle g + \lambda f, g + \lambda f \rangle - \langle g, g \rangle = 2\lambda \langle g, f \rangle + |\lambda|^2 \langle f, f \rangle \ .$$

Now consider an operator $A : D(A) \subset X \to X$ as "monotone in X" if

$$0 \leq [A(u) - A(v), u - v]_+ \quad (u, v \in D(A)) \tag{MC'}$$

For the case $X = C(\bar{U})$ under supremum norm (we need $\bar{U}$ so that we are dealing with a compact set), it turns out that the Sato bracket can be characterized by

$$[f, g]_+ = \max\{f(x_0)g(x_0) : x_0 \in \bar{U}, g(x_0) = \|g\|_{C(\bar{U})} \ .$$

For the proof of this identity, see Sato's paper [11]. The proof is elementary, and Sato considers the pairing

$$\tau(f, g) = \lim_{\varepsilon \to 0+} \frac{\|f + \varepsilon g\| - \|f\|}{\varepsilon} \ ,$$

and proves that for $X = C(\bar{U})$, the pairing is equal to

$$\tau(f, g) = \max_{x \in \chi(f)} ((\operatorname{sgn} f(x))g(x)) \ .$$

Thus we have $[f, g]_+ = \|g\| \, \tau(g, f)$.

We first show that our $F$ satisfies the (MC').

**Theorem 16.** *The operator $A(u) := F(D^2 u)$, defined for $C^2$ functions vanishing on the boundary of $U$, satisfies (MC') in $C(\bar{U})$.*

*Proof.* Let $u, v \in C^2(U)$ be functions vanishing on the boundary. We want to show that $0 \leq [F(D^2 u) - F(D^2 v), u - v]_+$. This occurs iff

$$\max\{(F(D^2 u(x_0)) - F(D^2 v(x_0))) \cdot ((u - v)(x_0)) : x_0 \in \bar{U}, (u - v)(x_0) = \|u - v\|_{C(\bar{U})}\} \ .$$

So let $x_0 \in U$ be such that $(u - v)(x_0) = \|u - v\|_{C(\bar{U})}$. Then $u - v$ has a maximum at $x_0$, so by the second derivative test,

$$D^2(u - v)(x_0) \leq 0 \Rightarrow \ D^2 u(x_0) \leq D^2 v(x_0) \ .$$

Therefore (E) gives that

$$F(D^2 u(x_0)) \geq F(D^2 v(x_0)) \ ,$$

and so $A(u) - A(v) \geq 0$ at $x_0$. Since $(u - v)(x_0) = \|\cdot\| \geq 0$, we have that

$$(A(u) - A(v))(u - v) \geq 0 \text{ at } x_0 \ .$$

On the other hand, if $x_0$ is such that $(v - u)(x_0) = \|u - v\|_{C(\bar{U})}$, then the same argument gives that $A(v) - A(u) \geq 0$ at $x_0$ and $(v - u)(x_0) \geq 0$, so that (MC') is satisfied. ☐

Now for the problem set above: $F(D^2u_k) = f_k$, $u_k = 0$ on $\partial U$, $W^{2,\infty} \ni u_{k_j} \to u$ uniformly, $D^2u_{k_j} \rightharpoonup \star\, D^2u$ in $L^\infty$. Condition (MC') gives

$$0 \le A(u_k) - A(v), u_k - v]_+ = [f_k - A(v), u_k - v]_+ \ .$$

It is easy to verify that $[\ ,\ ]$ is upper semicontinuous (just apply the exact definition), so that $\limsup_{k\to\infty}[\ ,\ ] \le [\limsup_{k\to\infty}, \limsup_{k\to\infty}]$, and hence we arrive at

$$0 \le [A(v) - f, v - u]_+ \ , \tag{35}$$

for each $C^2$ function $v$ such that $v = 0$ on $\partial U$. However, since we only know that $u \in W^{2,\infty}$—and not that $u$ is $C^2$—we cannot justify the method as before whereby we set $v = u + \lambda w$. We instead proceed as follows.

**Theorem 17.** *The weak limit $u$ solves the PDE (*) almost everywhere.*

*Proof.* Step 1. As above, we arrive at (35) for all $C^2$ functions $v$ vanishing on $\partial U$. We would like to design test functions $v$ to read off useful information.

Step 2. We know that $u \in W^{2,\infty}(U)$. Rademacher's theorem states that if $u \in W^{2,p}(U)$ for $p > n$, then $u$ is twice differentiable a.e. in the classical sense. So let $x_0 \in U$ be any point where $D^2u(x_0)$ exists. We handcraft a $C^2$ function $v_k$ having the form, for $\varepsilon > 0$,

$$v(x) = u(x_0) + Du(x_0)\cdot(x - x_0) + \frac{1}{2}(x - x_0)^T D^2u(x_0)(x - x_0) + \varepsilon|x - x_0|^2 \ ,$$

for $x$ near $x_0$, such that $|u - v|$ attains a maximum over $\bar{U}$ only at $x_0$. (Notice the similarity to second order Taylor expansion for $u$.) This means that $|u - v|(x_0) = \|u - v\|_{C(\bar{U})}$, and hence (35) and the identity for $[\ ,\ ]$ give us that

$$\begin{aligned}&[f - A(v), u - v]_+\\ &= \max\{(f(x_0) - A(v(x_0)))\cdot((u - v)(x_0)) : x_0 \in \bar{U}, (u - v)(x_0) = \|u - v\|_{C(\bar{U})}\}\\ &\ge 0 \ .\end{aligned}$$

Therefore $f - A(v) \ge 0$ at $x_0$, and so $A(v) = F(D^2v(x_0)) \le f(x_0)$. Taking the derivative of $v$ then gives

$$F(D^2u(x_0) + 2\varepsilon I) \le f(x_0) \ .$$

Assuming $F$ to be continuous, send $\varepsilon \to 0$ to get $F(D^2u(x_0)) \le f(x_0)$. We deduce the opposite inequality by performing the same analysis for $[A(v) - f, v - u]_+ \ge 0$, and this proves the theorem.

□

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