The Einstein constraint equations on compact three-dimensional manifolds

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Contents

0.1 Abstract

The relevance of the Einstein constraint equations in physics is first presented. They arise in the initial-value formulation of general relativity, and must be satisfied by the metric and the extrinsic curvature of a Cauchy surface. Propagating the resulting tensor fields with the evolution equations is then equivalent to solving Einstein's field equation.

The constraint equations consist of a system of two coupled nonlinear second-order PDEs. Well-posedness of the system is addressed, following the work of Y.Choquet-Bruhat. Some brief comments about global solutions are made.

The conformal method is introduced. Using this approach, along with York splitting, the constraint equations now consist of a semilinear elliptic equation and a linear elliptic system that have to be solved for the conformal factor and a vector field.

The main part of the thesis addresses the questions of existence and uniqueness of solutions to the Einstein constraint equations on three-dimensional compact Cauchy surfaces without boundary. The Yamabe classification turns out to be a key tool, and is presented. Then follows a thorough literature review of the results in the cases where the mean curvature, which is part of the prescribed data, is constant or near-constant. Recent articles on the case where the mean curvature is far-from-constant are discussed qualitatively. We then turn to a specific toy-model investigated by D.Maxwell where a family of three-parameters is used to consider different regimes on a Yamabe-null manifold. A similar approach is then used to explicitly work out some existence and uniqueness results on a Yamabe-positive manifold.

0.2 Abrégé

Tout d'abord, le rôle des équations de contraintes d'Einstein en physique est présenté. Ces équations font partie de la formulation du problème de Cauchy de la relativité générale, et doivent être satisfaites par la métrique, ainsi que par la courbure extrinsèque moyenne d'une surface de Cauchy. Déterminer la propagation de ces champs tensoriels par les équations d'évolution équivaut à la résolution de l'équation de champ d'Einstein.

Les équations de contraintes d'Einstein consistent en un système de deux EDP couplées non-linéaires de deuxième ordre. Le travail d'Y.Choquet-Bruhat démontrant que le problème est bien posé, est résumé. S'ensuivent quelques brefs commentaires concernants les solutions globales.

La méthode conforme est exposée. L'utilisation de cette technique combinée à la décomposition de York tranforme les équations de contrainte en une équation semi-elliptique et une équation linéaire elliptique, ayant pour inconnues le facteur conforme et un champ vectoriel.

L'essentiel de la présente thèse se concentre sur les questions d'existence et d'unicité des solutions des équations de contraintes d'Einstein sur les variétés compactes tridimensionelles sans frontière. À cet effet, la classification de Yamabe est un outil important. Une revue de la littérature est alors détaillée dans les cas où la courbure moyenne, qui est une donnée prescrite, est constante, ou 'proche-de-constante'. Puis vient une présentation qualitative d'articles récents traitant du cas où la courbure moyenne est 'loin-de-constante'. On s'attarde ensuite sur un cas spécifique étudié par D.Maxwell, dans lequel une famille de trois paramètres est utilisée pour passer d'un régime à l'autre sur une variété Yamabe-nulle. Une approche similaire est ensuite utilisée pour obtenir des résultats d'existence et d'unicité explicites sur une variété Yamabe-positive.

0.3 Acknowledgements

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0.4 Conventions

Throughout the whole document, Einstein summation convention is used in expressions involving components of a tensor, i.e.: repetition of an index as an upper and as a lower index implies summation over all possible values of the index.

In the main body of the thesis, the following conventions are followed:

- M denotes a four-dimensional spacetime manifold, endowed with the Lorentzian metric g. Greek indices are used to label components of tensors living on M, and they are raised and lowered with the metric g.

- $\Sigma \subset M$ denotes the Cauchy surface on which the initial-value problem is formulated. The Riemannian metric induced by g on Σ is h. Latin indices are used to label components of tensors living on Σ, and they are raised and lowered with the metric h. Vectors normal to Σ are denoted by n, and the vectors in the canonical basis of $T_p\Sigma$ (where $p \in \Sigma$) are denoted by e.

- When referring to the geometrical object, a tensor is denoted by a single letter, whereas the expressions for its components in local coordinates involve (greek of latin) indices.

- Covariant derivatives are denoted by D, and connections by ∇.

- The notations ${}^{(g)}$ or ${}^{(h)}$ are used to make explicit which metric is used to perform an operation (such as $D^{(g)}$, for example), or to specify which tensor is meant (eg: $R^{(g)}$ for the Ricci scalar w.r.t. g).

- The Laplacian is denoted by Δ.

- If there is no ambiguity regarding the local coordinates being used, the notation $\partial_\mu := \frac{\partial}{\partial x^\mu}$ is used.

Chapter 1

Introduction

1.1 Motivation from physics: Einstein's equation of General Relativity

Einstein's work on gravity culminated in 1916, with his presentation of the theory of general relativity. Its main result is encompassed in what is now known as *Einstein's field equation*:

$$\mathrm{Ric} - \frac{1}{2}Rg = 8\pi G_N T \tag{1.1}$$

where

- Ric is the *Ricci tensor*,
- R is the *Ricci scalar*,
- g is the spacetime *metric*,
- T is the *energy-momentum tensor*,
- G_N is Newton's constant of gravitation.

(See Definitions A.27, A.45 & A.47) Equation (1.1) aims to describe how the geometry of spacetime reacts to the presence of energy-momentum. The energy-momentum tensor is usually known, and involves the metric. Since Ric and R are built from first and second derivatives of the metric and its inverse, Equation

(1.1) really is a set of ten non-linear equations to be solved for the ten independent components of g. [7]

1.2 The constraint equations

As such, Equation (1.1) is very hard to solve. To make it more manageable, it is recast into an *initial-value problem*: The underlying idea is that since Equation (1.1) is a second order tensorial PDE for the metric g, we should provide as initial conditions $g_{\mu\nu}$ and $\partial_t g_{\mu\nu}$ at an instant of time, in local coordinates. Using the terminology presented in Appendix B, this idea is now made precise.

The initial-value problem, also called the *Cauchy formulation* of general relativity consists in specifying two symmetric tensor fields, h and K on a Cauchy surface Σ, satisfying the *constraint equations*

$$R^{(h)} + (\mathrm{tr}_h K)^2 - K^{ab}K_{ab} = 16\pi\rho \qquad \text{(Hamiltonian constraint)} \tag{1.2}$$

$$D_b K^{ab} - D^a(\mathrm{tr}_h K) = 8\pi j^a \qquad \text{(Momentum constraint)} \tag{1.3}$$

obtained from Equation (1.1), Equations (B.10) & (B.11) and using

$$16\pi T_{\alpha\beta}n^\alpha n^\beta \equiv 16\pi\rho \qquad\qquad 8\pi T_{\alpha\beta}e^\alpha_a n^\beta \equiv 8\pi j_a \tag{1.4}$$

$R^{(h)}$ is the Ricci scalar on Σ, the covariant derivative is $D = D^{(h)}$, and $\mathrm{tr}_h K := K^a{}_a = h^{ab}K_{ba}$ is the trace of K. ρ and j are usually referred to as the *source terms*.

In this formulation:

- Σ is a Cauchy surface representing an 'instant of time'.
- Initial values for g are given by the six components of h.
- Initial values for the 'time derivative' of g are given by the specification of K, since as revealed by Claim B.7, they are closely related.

A few remarks are now in order:

Remark 1.1. Section 1.1 mentioned that the ten Einstein's field equations (1.1) were to be solved for the ten independent components of g. However, since the *Einstein tensor* defined as $G = \mathrm{Ric} - \frac{1}{2}Rg$ has to satisfy the Bianchi identity $g^{\mu\alpha}(D_\alpha G_{\mu\nu}) = 0$ for $\alpha = 0, 1, 2, 3$ (cf. Equation (A.33)), only six of the equations are truly independent. If a metric is a solution to Equation (1.1) in one coordinate system x^μ, it must also be a solution in any other coordinate system $x^{\mu'}$. This reveals that there are four unphysical degrees of freedom in g, and only six coordinate-independent degrees of freedom. [7] $\triangle$

Remark 1.2. The constraint equations only make up four of the ten Einstein's field equations. The others provide *evolution equations* for h and K. Those equations can be derived altogether from the Hamiltonian "ADM" formulation of general relativity (see [2] & [24] for references), which considers variations of the Einstein-Hilbert Lagrangian to derive the equations of motion of the system. Note that Appendix B does not follow this approach.

It can be shown that if the constraints are satisfied on the initial surface and the evolution equations are satisfied everywhere, then the constraints are satisfied everywhere. This is usually described by saying that "the constraints propagate". $\triangle$

1.3 Outline of the thesis

Before solving the Einstein constraint equations, Chapter 2 studies some qualitative features of the system, and its solutions. Y.Choquet-Bruhat's work on the well-posedness of the system is presented in Section 2.1. To this end, the meaning of having a unique solution is made precise. Her proof also demonstrates that the Einstein constraint equations are necessary and sufficient conditions for the existence of a solution to Einstein's field equation. We also highlight the issues that arise from the nonlinearity of the equations, and present an existence result for global solutions in Section 2.2. To this day, the most successful approach to tackle the Einstein constraint equations has been the conformal method, which is outlined in Section 2.3 in the vacuum case, when the Cauchy surface is a three-dimensional compact manifold. By considering that we only know the conformal class within which the metric h lies, and by splitting the extrinsic curvature K appropriately, the Einstein constraint equations are turned into a system which consists in two equations: the semilinear elliptic *Lichnerowicz equation*, and the *momentum constraint*, which is linear and elliptic. The new unknowns are the conformal factor ϕ and a symmetric and traceless $(2,0)$-tensor $\tilde{K}$. This tensor can be further decomposed using York splitting, which is presented in Section 2.4. The resulting conformal formulation of the Einstein constraint equations is given by Equations (2.18) & (2.19). They must be solved for the conformal factor ϕ and a vector field W, when the Cauchy surface Σ, the metric γ (s.t. $h \in [\gamma]$), the TT-tensor σ and the mean curvature τ are prescribed. h and K are reconstructed using Equations (2.20) & (2.21).

Chapter 3 presents existence and uniqueness results for solutions of the Einstein constraint equations. One key tool is Yamabe theorem presented in Section 3.1, which gives rise to the classification of manifolds according to their *Yamabe type*. Three situations are then considered, depending on the prescribed mean curvature τ which is either constant, near-constant, or far-from-constant ('CMC', 'near-CMC', and 'far-from-CMC' cases, resp.). In the CMC and near-CMC cases, explicit proofs for non-/existence and uniqueness are given, following [16] & [18].

When $\tau \equiv$ constant (Section 3.2), the momentum constraint is trivially satisfied by $W \equiv 0$, so that the only equation to consider is the Lichnerowicz equation. The strategy to prove non-existence is to work in a gauge where the scalar curvature is constant, and then make use of the maximum principle to prove that no positive solution ϕ exists. Existence results make use of the sub and super solution theorem. Uniqueness results are based on a technical lemma which relies on the monotonicity of $f(x, \phi)$ in ϕ, in the Lichnerowicz equation $\Delta\phi = f(x, \phi)$.

In the near-CMC case (Section 3.3), a theorem where sufficient conditions for existence and uniqueness of a solution is proved in detail. It allows for a non-constant prescribed mean curvature, but imposes some conditions on its gradient, the so-called 'near-CMC assumption'. The proof considers a sequence of semi-decoupled equations. Sub and super solutions are found for each system in the sequence, so that existence of a sequence of solutions is guaranteed by the sub and super solution theorem. Those sub and super solutions can be used to build uniform sub and super solutions. The sequence of solutions is shown to converge in C^0, and a bootstrapping argument shows that the limit is twice differentiable. The last step shows that the limit is a weak, and therefore a strong solution of the Einstein constraint equations. Uniqueness is shown by contradiction.

In the far-from-CMC case, we first present the results of [15] & [23] qualitatively in Section 3.4, along with some comments on the presence of matter in the equations. Those results are the first existence results for non-CMC data without the near-CMC assumption. We then turn to two toy-models in Section 3.5. The first one is investigated in [22] by D.Maxwell: using a family of three parameters that allows to go from the near-CMC to the far-from-CMC regime, as well as control the size of the TT-tensor, various non-/existence and uniqueness results are obtained on the conformally flat n-torus, which is a Yamabe-null manifold. A similar approach is used in the second toy-model that we investigated, which considers the constraint equations on $S^2 \times S^1$, a Yamabe-positive manifold.

To ease the reading of the main part, a lot of material has been relegated to the appendices. The reader familiar with pseudo-Riemannian geometry and elliptic PDEs should not need to refer to them extensively. Appendices A & B present the material required to understand the initial-value problem. Appendix C provides the tools used in the proofs of Chapter 3. Lastly, Appendix D is on conformal methods.

Chapter 2

The conformal method: Motivation & Presentation.

This chapter aims to present how the Einstein constraint equations have been tackled over the past 60 years. For simplicity, we will work with vacuum space-times (ie: $\rho = j_a \equiv 0,\ a = 1,2,3$) where Σ is a compact three-dimensional manifold, without boundary.

2.1 Well-posedness of the Cauchy problem

We first discuss the *well-posedness* of the Cauchy formulation presented in Section 1.2. That is, we wish to show

- *Existence:* For any choice of initial data, $\exists$ a consistent solution.

- *Uniqueness:* Moreover, that solution is unique.

- *Continuous dependence of the solution on the initial data:* The map from the space of initial data to the space of solutions is continuous.

[14] The Cauchy problem for the gravitational field differs in several important respects from that for other physical fields:

- The Einstein equations are non-linear. While other field theories are often non-linear because they involve several fields interacting with each other, the distinctive feature of the gravitational field is that it is self-interacting: it is non-linear even in the absence of other fields. This is because it defines the spacetime over which is propagates.

- Two metrics g_1 and g_2 on a manifold M are physically equivalent if there is a diffeomorphism $\phi : M \to M$ which takes g_1 into g_2. Thus the solution of the field equations can be unique only up to a diffeomorphism.

That second point implies that the meaning of having a 'unique' solution has to be made more precise:

Definition 2.1. Let $\theta : \Sigma \hookrightarrow M$ be an embedding, and consider a solution (Σ, h, K) to the Cauchy problem. Then (M, θ, g) is called a *development* of (Σ, h, K). Another development (M', θ', g') of (Σ, h, K) is called an *extension* of M if there is a diffeomorphism $\alpha : M \to M'$ s.t. $\theta^{-1}\alpha^{-1}\theta = \text{id}$ on Σ, and $\alpha^* g' = g$.

Therefore, proving uniqueness amounts to proving *geometric uniqueness*. In order to obtain a definite member of the equivalence class of metrics which represents a spacetime, one introduces a fixed 'background' metric and has to impose four 'gauge conditions' on the covariant derivatives of the physical metric with respect to the background metric. These conditions remove the four degrees of freedom to make diffeomorphisms and lead to a unique solution for the metric components.

The well-posedness of the initial-value problem was first addressed by Y. Choquet-Bruhat in [8]. She showed that the Cauchy formulation of Einstein's (vacuum) theory of gravitation, for smooth initial data, is well-posed in the harmonic gauge. This proof is now sketched, following the presentations given in [4], [14] & [26].

Sketch of proof:

Step 1: The harmonic gauge. Looking at the principal part of the differential operator in equation (1.1), ie: the Ricci tensor:

$$R_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(\partial_\gamma \partial_\delta g_{\alpha\beta} + \partial_\alpha \partial_\beta g_{\gamma\delta} - \partial_\alpha \partial_\gamma g_{\beta\delta} - \partial_\beta \partial_\delta g_{\alpha\gamma}\right) + \dots \tag{2.1}$$

we see that it does not belong to any standard type such as elliptic, hyperbolic, etc. To freeze the diffeomorphism freedom, the harmonic gauge is imposed. Ie: in the given coordinates, $\Gamma^\mu = 0$, where

$$\Gamma^\mu = g^{\alpha\beta} \Gamma^\mu_{\beta\alpha} = g^{\mu\nu} g^{\alpha\beta} \left(\partial_\beta g_{\alpha\nu} - \frac{1}{2} \partial_\nu g_{\alpha\beta}\right) \tag{2.2}$$

Equivalently, the coordinates x^α should satisfy the wave equation. Indeed, in local coordinates [27]:

$$\Box_g x^a = \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} g^{\mu\nu}\right) \tag{2.3}$$

so that $\Gamma^\mu = 0 \iff \Box_g x^a = 0$.

Remark 2.2. Harmonic coordinates exist in (M, g): Indeed, the initial data can now be chosen so that $x^0 = 0$, $n^\alpha \partial_\alpha x^0 = 1$ and $n^\alpha \partial_\alpha x^a = 0$ there. Standard linear hyperbolic theory then gives the existence of functions x^α with these initial data satisfying the wave equation. The choice of initial data ensures that the derivatives of these functions are independent of the Cauchy surface and hence, by continuity, on a neighborhood of it. In these coordinates $g_{00} = -1$ and $g_{0a} = 0$ on the initial hypersurface. $\triangle$

In such coordinates, the last three terms in parenthesis in Equation (2.1) drop out and the principal part becomes identical to that of the wave equation. The equations so obtained are hyperbolic and are called the *reduced Einstein equations*.

Step 2: Equivalence of the equations. Sufficiency. It is however not yet clear that the spacetime obtained from solving the reduced equations satisfies the Einstein equations; for the two systems are only equivalent if the coordinates are harmonic. But whenever the reduced equations are satisfied, the Γ^α's satisfy a second order homogeneous linear hyperbolic equation. The construction of the initial data for the reduced equations ensures that $\Gamma^\alpha = 0$ on the initial surface; and the reduced equations and the constraints together imply that $\partial_t \Gamma^\alpha = 0$ there. Uniqueness in the Cauchy problem for that equation then ensures that $\Gamma^\alpha = 0$ everywhere.

Since the two systems of equations are equivalent, all we need to show is that the Cauchy problem for the reduced Einstein equations is well-posed. Good existence theorems for the Cauchy initial value problem for the reduced Einstein system are well-known, and the initial data $(g_{ab}(0), \partial_t g_{ab}(0))$ are freely prescribable, subject only to the condition that $g_{ab}(0)$ has Lorentz signature with M_0 spacelike.

Given a globally hyperbolic spacetime (see Appendix B), we have that the Einstein constraint equations must be satisfied on any Cauchy surface. A by-product of the well-posedness proof is that the constraint equations are not only necessary, but also sufficient conditions for the Einstein equations to have a solution, ie: local existence of a solution to the equations ensures existence of a development, which is a solution of Equation (1.1). ([4])

Step 3: Uniqueness. Suppose we have two Cauchy developments of the same initial data. Choose some coordinates on Σ. Based on these, we can uniquely construct harmonic coordinates as indicated above in each of the two spacetimes. Call them x^α and $\bar{x}^\alpha$. Define a mapping ψ from one spacetime to the other by the condition that $\bar{x}^\alpha = \psi \circ x^\alpha$. Then the first metric g and the metric obtained by pulling back the second metric g' with ψ both solve the reduced equations and induce the same data on the initial hypersurface. Thus, by uniqueness in the Cauchy problem for the reduced equations, they must be equal. This gives geometric uniqueness.

Or using the definition previously introduced, we have that any two developments of S are extensions of a common development. This common development represents a "neighborhood" of S in which the two developments must agree.

q.e.d. Sketch of proof

Since Einstein's equations are well-posed, the space of solutions of the constraints *parametrizes* the space of solutions Einstein's field equations.

2.2 Global solutions

[26] The distinction between local and global Cauchy problems is necessary for nonlinear equations. The question whether the Cauchy problem for the Einstein equations can be solved globally is related to the existence and nature of spacetime singularities.

There are two types of boundary conditions which are usually imposed on initial data sets:

- "Cosmological boundary conditions": where solutions of the Einstein equations should describe the universe as a whole.

- Asymptotically flat boundary conditions: where solutions of the Einstein equations should describe an isolated system, like the solar system.

Existence and uniqueness of global solutions is addressed in [10]:

Theorem 2.3. Given any set of initial data for Einstein's equations which satisfy the constraint conditions, there exists a development of that data which is maximal in the sense that it is an extension of every other development.

Sketch of proof: Omit the embedding in (M, θ, g). The proof first considers the set of all possible developments of (Σ, h, K), and shows that it can be partially ordered by the relation $\leq$, writing $(M_2, g_2) \leq (M_1, g_1)$ if (M_1, g_1) is an extension of (M_2, g_2). If the collection (M_α, g_α) of all developments of (Σ, h, K) is totally ordered, $M' = \cup M_\alpha$ endowed with an appropriate metric is an upper bound for the set. Since every totally ordered set has an upper bound, by Zorn's lemma, there is a maximal development $(\tilde{M}, \tilde{g})$ of (Σ, h, K) whose only extension is itself.

Suppose (M', g') is another development of (Σ, h, K). By the local Cauchy theorem, there exist developments of (Σ, h, K) of which $(\tilde{M}, \tilde{g})$ are both extensions. The set of all such common developments is likewise partially ordered and so there will be a maximal development (M'', g''). Let $M^+ = \tilde{M} \cup M' \cup M''$. M^+ can be shown to be Hausdorff, and therefore (M^+, g^+) is a development, for an appropriate metric g^+. Hence, it is an extension of both $(\tilde{M}, \tilde{g})$ and (M', g'). But since the only extension of $(\tilde{M}, \tilde{g})$ is itself, we must have $(\tilde{M}, \tilde{g}) = (M^+, g^+)$ and $(M', g') \leq (M^+, g^+)$.

q.e.d. Sketch of proof.

2.3 The conformal method

A popular approach to the Einstein constraint equations is the conformal method, which manipulates Equations (1.2) & (1.3) as follows [9]:

- Consider that the metric h is given only *up to a conformal factor*: $h = e^{2\lambda}\gamma$ where γ is a given metric, but λ is a function to be determined, ie: $h \in [\gamma]$. We then have the following relation between the scalar curvatures of the metrics (cf. Equation (D.12) Appendix D):

$$R^{(h)} = e^{-2\lambda}\left(R^{(\gamma)} - 2\Delta^{(\gamma)}\lambda - 2\ h^{ab}\ \partial_a\lambda\ \partial_b\lambda\right) \tag{2.4}$$

Set $\phi^{2p} = e^{2\lambda}$, so that $h = \phi^{2p}\gamma$. Then choosing $p = 2$ cancels out the last term in Equation (2.4) to give:

$$R^{(h)} = \phi^{-5}\left(\phi\left(R^{(\gamma)}\right) - 8\Delta^{(\gamma)}\phi\right) \quad \text{where} \quad h = \phi^4\gamma \tag{2.5}$$

- Consider the following Lemma:

Lemma 2.4. If $h = \phi^4\gamma$, then for an arbitrary covariant $(2,0)$-tensor P, we have:

$$D_i^{(h)} P^{ij} \equiv \phi^{-10}\ D_i^{(\gamma)}\left(\phi^{10} P^{ij}\right) - 2\phi^{-1}\gamma^{ij}\partial_i\phi\ \mathrm{tr}_\gamma P \tag{2.6}$$

The lemma makes use of the definition: $\mathrm{tr}_\gamma P = \gamma_{ij}P^{ij} = P^i{}_i$, and suggests splitting the extrinsic curvature tensor K into a weighted traceless part and its trace as follows:

$$K^{ab} = \phi^{-10}\tilde{K}^{ab} + \frac{1}{3}h^{ab}\tau \tag{2.7}$$

where $\tau = \mathrm{tr}_h K = h^{ab}K_{ab} = K^b{}_b$ is the *mean extrinsic curvature* (see Appendix A.9), and $\tilde{K}$ is symmetric and traceless. Equation (2.7) gives:

$$|K|^2_{(h)} \ :=\ h_{ab}h_{cd}K^{ac}K^{bd} \ =\ \phi^{-12}\gamma_{ab}\gamma_{cd}\tilde{K}^{ac}\tilde{K}^{bd} + \frac{1}{3}\tau^2 \ =\ \phi^{-12}|\tilde{K}|^2_{(\gamma)} + \frac{1}{3}\tau^2 \tag{2.8}$$

The conformally formulated (CF) vacuum Einstein constraint equations then read as the following system:

$$\Delta^{(\gamma)}\phi - \frac{1}{8}\ R^{(\gamma)}\phi + \frac{1}{8}\ \phi^{-7}\ |\tilde{K}|^2_{(\gamma)} - \frac{1}{12}\ \phi^5\ \tau^2 \ =\ 0 \tag{2.9}$$

$$D_a^{(\gamma)}\tilde{K}^{ab} - \frac{2}{3}\ \phi^6\ \gamma^{ab}\ \partial_a\tau \ =\ 0 \tag{2.10}$$

to be solved for $(\tilde{K}, \phi)$. Equation (2.9) is known as the *Lichnerowicz equation*, and is a semilinear elliptic equation for ϕ when γ, τ, and $\tilde{K}$ are known. Equation (2.10), the *momentum constraint*, is a first-order linear system for $\tilde{K}$ when γ and ϕ are known.

The following theorem states that the solution to the system is independent of γ, the choice of representative of the equivalence class of conformally related metrics:

Theorem 2.5. Suppose that (K, ϕ) is a solution of the CF vacuum constraints for the metric γ and a given function τ. Then the CF vacuum constraints in the metric $\gamma' = \theta^4\gamma$, with $\tau' = \tau$, admit the solution $(\tilde{K}', \phi') = (\theta^{-10}\tilde{K}, \theta^{-1}\phi)$.

2.4 York splitting

We now focus on the CF momentum constraint, Equation (2.10):

$$D_a^{(\gamma)} \tilde{K}^{ab} = \frac{2}{3}\, \phi^6\, \gamma^{ab}\, \partial_a \tau \tag{2.11}$$

Since the system is linear, an arbitrary solution is the sum of a solution to the homogeneous system, and a particular solution.

For convenience, let $D := D^{(\gamma)}$ momentarily. By definition, symmetric $(2,0)$-tensors satisfying the equation $D_a \tilde{K}^{ab} = 0$ are known as *transverse*. Since $\tilde{K}$ is also traceless, it is a *TT-tensor*, for Transverse-Traceless.

The particular solution should be looked for in the formal L^2 dual of the kernel of the space of TT-tensors.

Lemma 2.6. The formal L^2 dual of the space of TT-tensors in a metric γ is the space of the conformal Lie derivative of γ with respect to some vector field W, which consists in the space of symmetric traceless $(2,0)$-tensors of the form:

$$(\mathfrak{L}_{\mathrm{conf}} W)^{ab} \equiv D^a W^b + D^b W^a - \frac{2}{3} \gamma^{ab} D_c W^c \tag{2.12}$$

Therefore, as a particular solution, we should seek a vector field W satisfying:

$$(\Delta_{\mathrm{conf}} W)^b := D_a (\mathfrak{L}_{\mathrm{conf}} W)^{ab} = \frac{2}{3}\, \phi^6\, \gamma^{ab} \partial_a \tau \tag{2.13}$$

where we define the *conformal Laplacian*. If given an arbitrary traceless $(2,0)$-tensor U, a TT-tensor can be found by solving: $(\Delta_{\mathrm{conf}} Z)^b = -D^b U$ for Z, since then $\sigma := \mathfrak{L}_{\mathrm{conf}} Z + U$ is transverse. Hence, $X := W + Z$ is a solution of:

$$\begin{aligned}
(\Delta_{\mathrm{conf}} X)^b &= \frac{2}{3} \phi^6 \gamma^{ab} \partial_a \tau - D^b U && (2.14)\\
\iff \quad D^b \left(\mathfrak{L}_{\mathrm{conf}} (W + Z)\right) + D^b U &= \frac{2}{3} \phi^6 \gamma^{ab} \partial_a \tau && (2.15)\\
\iff \quad D^b \left(\sigma + \mathfrak{L}_{\mathrm{conf}} W\right) &= \frac{2}{3} \phi^6 \gamma^{ab} \partial_a \tau && (2.16)\\
\iff \quad D^b \left(\mathfrak{L}_{\mathrm{conf}} W\right) &= \frac{2}{3} \phi^6 \gamma^{ab} \partial_a \tau && (2.17)
\end{aligned}$$

This suggest decomposing $\tilde{K}$ as $\sigma + \mathfrak{L}_{\text{conf}} W$ with σ prescribed, and solving Equation (2.17) for W. This way, we have recast Equation (2.10) into the elliptic equation (2.17).

In the conformal formulation using York splitting, the Einstein equations now read (using $\mathfrak{L} := \mathfrak{L}_{\text{conf}}$)

$$\begin{aligned} 8\Delta\phi &= R\phi - \phi^{-7}\left(\sigma^{ab} + (\mathfrak{L}W)^{ab}\right)\left(\sigma_{ab} + (\mathfrak{L}W)_{ab}\right) + \frac{2}{3}\,\phi^5\,\tau^2 && (2.18)\\ D^b\left(\mathfrak{L}W\right) &= \frac{2}{3}\phi^6\gamma^{ab}\partial_a\tau && (2.19)\end{aligned}$$

with prescribed data (γ, σ, τ), to be solved for (h, K), which are reconstructed using:

$$\begin{aligned} h &= \phi^4\gamma && (2.20)\\ K &= \phi^{-10}(\sigma + \mathfrak{L}_{\text{conf}} W) + \frac{1}{3}\phi^{-4}\gamma\tau && (2.21)\end{aligned}$$

Chapter 3

Existence and uniqueness results

We now turn to the question of existence and uniqueness of solutions to the Einstein constraint equations, given a set of initial data. The mean curvature τ turns out to be the most important factor in separating those sets of free data for which we know whether or not a solution exists from those sets for which we do not. Accordingly, results are divided as follows:

- Constant Mean Curvature (CMC) data, where $\tau \equiv$ constant,
- Near-CMC data, where $\tau \not\equiv$ constant, but some strict restrictions on τ apply,
- Far-from-CMC data, where τ meets neither of the above conditions.

Of additional interest is the question of roughness: How does the regularity of the initial data set relate to the existence/uniqueness/regularity of the solution? If understood, one can then construct solutions from data sets with minimal regularity, and get some insight on the features of the resulting solution. In turn, such information on local solutions could then be used to construct global solutions to the problem of general relativity. Moreover, some of the convergence results established by numerical relativists rest on the assumptions about the solution theory. In the following, we will be mostly concerned with statements on existence and uniqueness. Minimality of the requirements on regularity will not be addressed (see [15] and [21] for references).

But before turning to explicit results, we introduce the work of Yamabe, which turns out to be crucial in determining which prescribed set of data yield solutions to the Einstein constraint equations.

3.1 The Yamabe classification

The Yamabe classification characterizes manifolds according to the class of conformally related metrics they can admit. Theorem 3.9 was conjectured by Yamabe, and proved by Trudinger, Aubin and Schoen. We follow the treatment of [9], in the case $n = 3$. Let us work on (M, γ), where $\gamma \in W_2^p$ is a properly Riemannian metric, ie: $\inf_{\cup M_I} \det \gamma_{ij} > 0$ in a finite number of charts M_I covering M. e is a smooth metric on M.

Definition 3.1. Define the *q-Yamabe functional*

$$J_{\gamma,q}(\phi) := \frac{\int_M \left(8|\nabla\phi|^2_{(\gamma)} + R^{(\gamma)}\phi^2\right)\mu_\gamma}{\left(\int_M \phi^{2q}\mu_\gamma\right)^{1/q}} \tag{3.1}$$

Lemma 3.2. If $p > \frac{3}{2}$, the q-Yamabe functional is defined for every $\phi \in H_1$, $\phi \not\equiv 0$, $1 \leq q \leq 3$.

Proof. $\gamma \in W_2^p$, $p > \frac{3}{2} \implies \gamma \in C^0$ by Sobolev embedding theorem (see C.16) and γ is uniformly equivalent to e. Therefore, the pointwise norms of a tensor in the metrics γ and e are uniformly equivalent, and their L^p norms in the metrics γ and e are equivalent. The Sobolev embedding and multiplication theorems (see C.15) show that $R^{(\gamma)} \in L^p$. On the other hand, if $\phi \in H_1$, then $\phi \in L^{2q}$, $1 \leq q \leq 3$, hence $R^{(\gamma)}\phi^2 \in L^1$. □

Remark 3.3. By 'uniformly equivalent', it is meant that γ is uniformly bounded w.r.t. e. $\triangle$

Definition 3.4. $J_\gamma(\phi) := J_{\gamma,3}(\phi)$ is simply called the *Yamabe functional* on M.

Lemma 3.5. The Yamabe functional is a conformal invariant in the sense that if $\gamma' = \theta^4\gamma$, $\phi' = \theta^{-1}\phi$, then $J_{\gamma'}(\phi') = J_\gamma(\phi)$.

Proof. Since $\mu_\gamma = \theta^{-6}\mu_{\gamma'}$, we have $\int_M \phi^6\mu_\gamma = \int_M \phi'^6\mu_{\gamma'}$. It can be shown that

$$\Delta_{(\gamma)}\phi - \frac{1}{8}R^{(\gamma)}\phi = \theta^5\left(\Delta_{(\gamma')}\phi' - \frac{1}{8}R^{(\gamma')}\phi'\right) \tag{3.2}$$

Integrating by parts gives

$$\int_M \left(8|\nabla\phi|^2_\gamma + R^{(\gamma)}\phi^2\right)\mu_\gamma = \int_M \left(8|\nabla\phi'|^2_{\gamma'} + R^{(\gamma')}\phi'^2\right)\mu_{\gamma'} \tag{3.3}$$

$\square$

Lemma 3.6. The Yamabe functional admits an infimum for $\phi \in W^p_2$ and $\phi \not\equiv 0$, which depends only on the conformal class of γ.

Proof. If $\gamma \in W^p_2$, $p > \frac{3}{2}$, then $R^{(\gamma)} \in L^r\ \forall\ 1 \leq r \leq p$, hence $||R^{(\gamma)}||_{L^{q/(q-1)}}$ is bounded if $q \leq 3$. By Hölder inequality:

$$\int_M \left(8|\nabla\phi|^2_\gamma + R^{(\gamma)}\phi^2\right)\mu_\gamma \quad \geq \quad -\left|\int_M R^{(\gamma)}\phi^2\mu_\gamma\right| - ||R^{(\gamma)}||_{L^{q/(q-1)}}||\phi^2||_{L^q} \tag{3.4}$$

$\square$

Definition 3.7. The *Yamabe number* is defined as:

$$\mu := \inf_{\phi\in W^p_2,\ \phi\not\equiv 0} J_\gamma(\phi) \tag{3.5}$$

By Lemma 3.5, it only depends on the conformal class of γ.

Definition 3.8. (M,γ) is said to be in the *negative Yamabe class* if $\mu < 0$, in the *zero Yamabe class* if $\mu = 0$, and in the *positive Yamabe class* if $\mu > 0$.

Theorem 3.9. *(Yamabe Theorem)* Let M be compact, and $p > \frac{3}{2}$. Then:

- If γ is in the negative Yamabe class it is conformal to a metric with scalar curvature -1. We write $\gamma \in \mathcal{Y}^-(M)$.
- If γ is in the zero Yamabe class it is conformal to a metric with scalar curvature 0. We write $\gamma \in \mathcal{Y}^0(M)$.
- If γ is in the positive Yamabe class it is conformal to a metric with continuous and positive scalar curvature. We write $\gamma \in \mathcal{Y}^+(M)$.

See [20] or [21] for a proof. Note that the last statement in Theorem 3.9 is not as strong as the previous two. However, for metrics with higher regularity, we have the following result, which we state without proof [20]:

Theorem 3.10. [20] If γ is a smooth metric in the positive Yamabe class, then γ is conformally equivalent to a metric with scalar curvature $+1$.

Yamabe types link the Yamabe properties to the topology of M:

Definition 3.11. Let M be a compact manifold of dimension $n \geq 3$. It is said to be:

- *Yamabe-positive* if it admits a metric of positive Yamabe class.
- *Yamabe-null* if it admits a metric in the zero Yamabe class, but no metric in the positive Yamabe class.
- *Yamabe-negative* if it admits no metric in the zero or positive Yamabe class.

Note that a manifold is of exactly one Yamabe type. It is a fact proved by Aubin that every compact manifold admits a metric in the negative constant scalar curvature. However, Schoen and Yau used minimal surfaces to find topological obstructions to the existence of metrics with scalar curvature ≥ 0. For example, Gromov and Lawson proved that for all n, the n-torus has no metric with strictly positive scalar curvature [19]. However, since it can be equipped with a flat product metric, it is a Yamabe-null manifold. The sphere is an obvious example of a Yamabe-positive manifold. The torus with n handles, $n \geq 2$ is an example of a Yamabe-negative manifold.

3.2 CMC-results

[16] The assumption $\tau \equiv$ constant implies that $\mathfrak{L}W$ must vanish in Equation (2.19), even if γ admits a conformal Killing vector field (see Appendix C.3.1). Equation (2.18) now takes the form

$$\Delta\phi = \frac{1}{8}R\phi - \frac{1}{8}\left(\sigma^{ab}\sigma_{ab}\right)\phi^{-7} + \frac{1}{12}\tau^2\phi^5 \tag{3.6}$$

and has to be solved for $\phi > 0$. Consider the map

$$\mathscr{L} : \hat{\mathscr{C}}(\Sigma) \to \mathscr{E}(\Sigma) \qquad (\gamma, \sigma, \tau) \mapsto (h, K) \tag{3.7}$$

where

- $\hat{\mathscr{C}}(\Sigma) = \{(\gamma, \sigma, \tau) : (3.6) \ \text{ can be solved}\}$
- $\mathscr{E}(\Sigma) = \{(h, K) : \gamma$ is given by (2.20), K is given by (2.21), (ϕ, W) satisfy (3.6) $\}$.

$\mathscr{L}$ is surjective: if given $(h, K) \in \mathscr{E}(\Sigma)$, Equation (3.6) can be solved for ϕ (we get $\phi = 1$) if we take $(\gamma, \sigma, \tau) = (h, K - \frac{1}{3}h(\mathrm{tr}K), \mathrm{tr}K)$

$\mathscr{L}$ is not injective: Indeed, $\mathscr{L}$ is invariant under the group of conformal transform maps

$$\Theta_\theta : \hat{\mathscr{C}}(\Sigma^3) \to \hat{\mathscr{C}}(\Sigma^3) \qquad (\gamma, \sigma, \tau) \mapsto (\theta^4\gamma, \theta^{-10}\sigma, \tau) \tag{3.8}$$

with θ any positive definite scalar function on Σ. Hence, in testing a given set (γ, σ, τ), we may choose a conformal factor θ^4 which casts Equation (3.6) into a more easily studied form. This is a direct consequence of the Yamabe theorem.

Before stating the main result of this section, we prove a useful proposition:

Proposition 3.12. Let Σ be a closed three-dimensional manifold with a C^2 Riemannian metric γ of arbitrary Yamabe class. Let $\hat{\Sigma} \subset \Sigma$ be open with regular boundary $\partial\hat{\Sigma}$ and with $\Sigma - (\hat{\Sigma} \cup \partial\hat{\Sigma}) \neq \varnothing$ open. Then $\exists$ a $C^3(\Sigma)$ function $\theta > 0$ s.t. on $\hat{\Sigma}$, $R^{(\theta^4\gamma)} < -\xi < 0$ where $\xi > 0$ is some constant.

Proof. Since $\gamma \in C^2$, $R^{(\gamma)}$ is continuous on Σ. Let $R^{(\gamma)} < \mu$, for some constant μ, and $\alpha^4 = 2\mu$. Then $R^{(\tilde{\gamma})} < \frac{1}{2}$ on Σ, where $\tilde{\gamma} = \alpha^4\gamma$. The following

$$\begin{cases} 8\Delta u = u & \text{on } \hat{\Sigma} \\ u = 1 & \text{on } \partial\hat{\Sigma} \end{cases} \tag{3.9}$$

has a unique C^2 solution, which is positive definite by the maximum principle. Let $u < m$ for some positive constant m. Let θ be a C^3 function on Σ such that $\theta = u$ on $\hat{\Sigma} \cup \partial\hat{\Sigma}$, then on $\hat{\Sigma}$, $R^{(\theta^4\tilde{\gamma})} = u^{-4}\left(R^{(\tilde{\gamma})} - 1\right) < -\frac{1}{2}u^{-4} < -\frac{1}{2}m^{-4}$. Let $\xi := \frac{1}{2}m^{-4}$. ☐

This following theorem provides a complete function space parametrization of the set of CMC solutions of the Einstein constraints on a given closed manifold.

Theorem 3.13. Let $\gamma \in C^3(\Sigma)$ and $\sigma \in W^p_2(\Sigma)$, for $p > 3$. The Lichnerowicz equation (3.6) admits or does not admit a positive definite solution $\phi \in C^{2,\alpha}(\Sigma)$ as indicated in the following table:

Table 1: Existence / Non-Existence results in the CMC-case.

	$\sigma^2 \equiv 0$, $\tau = 0$	$\sigma^2 \equiv 0$, $\tau \neq 0$	$\sigma^2 \not\equiv 0$, $\tau = 0$	$\sigma^2 \not\equiv 0$, $\tau \neq 0$
$\gamma \in \mathscr{Y}^+(\Sigma)$	No	No	Yes	Yes
$\gamma \in \mathscr{Y}^0(\Sigma)$	Yes	No	No	Yes
$\gamma \in \mathscr{Y}^-(\Sigma)$	No	Yes	No	Yes

For data (γ, σ, τ) in the class $(\mathscr{Y}^0, \sigma^2 \equiv 0, \tau = 0)$, any constant is a solution. For data in all other classes for which solutions exist, the solution is unique.

Proof. Existence: The proofs for non-existence all follow the same scheme: We first choose a conformal gauge so that the scalar curvature is constant, and conclude from the maximum principle that no positive solution exists. The choice of gauge, and the form of the Lichnerowicz equation are presented in Table 2.

Table 2: Data used in the proof of non-existence results in the CMC-case.

$(\mathscr{Y}^+, \sigma^2 \equiv 0, \tau = 0)$	$R = +1$	$\Delta\phi = \frac{1}{8}\phi$
$(\mathscr{Y}^-, \sigma^2 \equiv 0, \tau = 0)$	$R = -1$	$\Delta\phi = -\frac{1}{8}\phi$
$(\mathscr{Y}^+, \sigma^2 \equiv 0, \tau \neq 0)$	$R = +1$	$\Delta\phi = \frac{1}{8}\phi + \frac{1}{12}\tau^2\phi^5$
$(\mathscr{Y}^0, \sigma^2 \equiv 0, \tau \neq 0)$	$R = 0$	$\Delta\phi = \frac{1}{12}\tau^2\phi^5$
$(\mathscr{Y}^0, \sigma^2 \not\equiv 0, \tau = 0)$	$R = 0$	$\Delta\phi = -\frac{1}{8}\sigma^2\phi^{-7}$
$(\mathscr{Y}^-, \sigma^2 \not\equiv 0, \tau = 0)$	$R = -1$	$\Delta\phi = -\frac{1}{8}\phi - \frac{1}{8}\sigma^2\phi^{-7}$

- $(\mathscr{Y}^0, \sigma^2 \equiv 0, \tau = 0)$: Choose the conformal gauge so that $R = 0$, then any constant function satisfies $\Delta\phi = 0$.

The following proofs of existence make use of the sub and super solution theorem, presented in Appendix C.3.3: finding positive definite sub and super solutions ensures existence of a solution.

- $(\mathscr{Y}^-, \sigma^2 \equiv 0, \tau \neq 0)$: Choose the conformal gauge so that $R = -1$, then $\phi = \left(\frac{3}{2\tau^2}\right)^{1/4}$ is a solution of $\Delta\phi = -\frac{1}{8}\phi + \frac{1}{12}\tau^2\phi^5$.
- $(\mathscr{Y}^-, \sigma^2 \not\equiv 0, \tau \neq 0)$: Choose the conformal gauge so that $R = -1$, then

$$\phi_- = \left(\frac{3}{2\tau^2}\right)^{1/4} \qquad \phi_+ = \max\left\{1, \left(\frac{3}{2\tau^2}\left(1 + \max_\Sigma \sigma^2\right)\right)^{1/4}\right\} \tag{3.10}$$

are sub and super solutions of $\Delta\phi = -\frac{1}{8}\phi - \frac{1}{8}\sigma^2\phi^{-7} + \frac{1}{12}\tau^2\phi^5$.

- $(\mathscr{Y}^0, \sigma^2 \not\equiv 0, \tau \neq 0)$, *Case 1:* $\sigma > 0$: Choose the conformal gauge so that $R = 0$, then

$$\phi_- = \left(\frac{3}{2\tau^2}\min_\Sigma \sigma^2\right)^{1/12} \qquad \phi_+ = \left(\frac{3}{2\tau^2}\max_\Sigma \sigma^2\right)^{1/12} \tag{3.11}$$

are sub and super solutions of $\Delta\phi = -\frac{1}{8}\sigma^2\phi^{-7} + \frac{1}{12}\tau^2\phi^5$.

- $(\mathscr{Y}^0, \sigma^2 \not\equiv 0, \tau \neq 0)$, *Case 2:* $\sigma \not> 0$: We choose the conformal gauge as follows: Let $p \in \Sigma$ be a point at which σ^2 is not zero. Then there exists an open ball B containing p such that $\sigma^2 \geq \zeta > 0$ everywhere in B, $\zeta \in \mathbb{R}^+$. Then $\hat{\Sigma} = \Sigma - B$ contains all the zeros of σ^2. Applying Proposition 3.12, we can choose a conformal gauge so that the conformal data satisfy $\tilde{R} \leq -\kappa < 0$ on $\hat{\Sigma}$ and $\tilde{\sigma} \geq \tilde{\zeta} > 0$, for $\kappa, \tilde{\zeta} \in \mathbb{R}^+$. Then

$$\phi_- = \min\left\{1, \left(\frac{\tilde{\zeta}}{\max_\Sigma |\tilde{R}| + \frac{2}{3}\tau^2}\right)^{1/8}, \left(\frac{3\kappa}{2\tau^2}\right)^{1/4}\right\} \tag{3.12}$$

$$\phi_+ = \max\left\{1, \left(\frac{3}{2\tau^2}\left(\max_\Sigma |\tilde{R}| + \max_\Sigma \tilde{\sigma}^2\right)\right)^{1/4}\right\} \tag{3.13}$$

are sub and super solutions of $\Delta\phi = \frac{1}{8}\tilde{R}\phi - \frac{1}{8}\sigma^2\phi^{-7} + \frac{1}{12}\tau^2\phi^5$.

- $(\mathscr{Y}^+, \sigma^2 \not\equiv 0, \tau \neq 0)$, *Case 1:* $\sigma \not> 0$: The proof is identical to the case just treated.

- $(\mathscr{Y}^+, \sigma^2 \not\equiv 0, \tau \neq 0)$, *Case 2:* $\sigma > 0$: Choose the conformal gauge so that $R = +1$. Then

$$\phi_- = \min\left\{1, \left(\min_\Sigma \sigma^2\left(\frac{1}{1 + \frac{2}{3}\tau^2}\right)\right)\right\} \qquad \phi_+ = \left(\max_\Sigma \sigma^2\right)^{1/8} \tag{3.14}$$

are sub and super solutions of $\Delta\phi = \frac{1}{8}\phi - \frac{1}{8}\sigma^2\phi^{-7} + \frac{1}{12}\tau^2\phi^5$.

- $(\mathscr{Y}^+, \sigma^2 \not\equiv 0, \tau = 0)$, *Case 1:* $\sigma > 0$: Choose the conformal gauge so that $R = +1$. Then $\phi_- = \left(\frac{1}{8}\min_\Sigma \sigma^2\right)^{1/8}$ and $\phi_+ = \left(\frac{1}{8}\max_\Sigma \sigma^2\right)^{1/8}$ are sub and super solutions of $\Delta\phi = \phi - \frac{1}{8}\sigma^2\phi^{-7}$.

- $(\mathscr{Y}^+, \sigma^2 \not\equiv 0, \tau = 0)$, *Case 2:* $\sigma \not> 0$: Choose the conformal gauge so that $R = +1$. Let $A := \max\left\{1, \frac{1}{8}\max_\Sigma \sigma^2\right\}$ and consider the linear PDE

$$-\Delta\phi_- + \phi_- = \frac{\sigma^2}{8}A^{-7} \tag{3.15}$$

Since $-\Delta + 1$ has trivial kernel, $\lambda_{ab} \in C^3(\Sigma)$, and $\sigma^2 \in W_2^p(\Sigma)$, $\exists!$ solution $\phi_- \in W_4^p(\Sigma)$. We show that ϕ_- is a subsolution of $\Delta\phi = \phi - \frac{1}{8}\sigma^2\phi^{-7}$. Since

$\frac{\sigma^2}{8}A^{-7} \geq 0$, $\phi_- > 0$. Consider the function

$$G(x,s) := \frac{\sigma^2}{8}s^{-7} \quad \Longrightarrow \quad \frac{\partial G}{\partial s} = -\frac{7}{8}\sigma^2 s^{-8} \tag{3.16}$$

so that $G(x,s)$ is monotonically nonincreasing in s. Since $A \geq 1$:

$$G(x,A) \leq G(x,1) = \frac{\sigma^2}{8} \leq A \quad \Longrightarrow \quad G(x,A) \leq A \tag{3.17}$$

Hence

$$-\Delta\phi_- + \phi_- \leq A \quad \overset{\text{max.princ.ver.3}}{\widehat{\Longrightarrow}} \quad \phi_- \leq A \Longrightarrow G(x,A) \leq G(x,\phi_-) \tag{3.18}$$

$$\Longrightarrow \quad -\Delta\phi_- + \phi_- \leq G(x,\phi_-) = \frac{\sigma^2}{8}\phi_-^{-7} \tag{3.19}$$

Finally, $\phi_+ = A$ is a supersolution. $\square$

Proof. <u>Uniqueness:</u> We first prove a Lemma:

Lemma: Let $f : \Sigma \times \mathbb{R}^+ \to \mathbb{R}$ be C^1, with $\frac{\partial}{\partial s}f(x,s) \geq 0$, $\frac{\partial}{\partial s}f(x,s) \not\equiv 0$, for $s \in I$, I some interval of $\mathbb{R}^+$. If ψ_1 and ψ_2 are both solutions of the PDE $\Delta\psi = f(x,\psi(x))$ and if both $\psi_1(x)$ and $\psi_2(x)$ take values in I for all $x \in \Sigma$, then $\psi_1(x) = \psi_2(x)$ for all $x \in \Sigma$.

Proof of Lemma: Suppose $\psi_1(x)$ and $\psi_2(x)$ are both solutions of $\Delta\psi = f(x,\psi(x))$ and consider the quantity:

$$\mathscr{I}(\psi_1,\psi_2)(x) := \int_0^1 \left\{\frac{\mathrm{d}}{\mathrm{d}\nu}\left[\Delta(\nu\psi_1(x) + (1-\nu)\psi_2(x)) - f(x,\nu\psi_1(x) + (1-\nu)\psi_2(x)\right]\right\}\mathrm{d}\nu \tag{3.20}$$

We compute expressions for $\mathscr{I}$ in two ways: First using the Fund. Thm. of Calc.:

$$\begin{aligned}\mathscr{I}(\psi_1,\psi_2)(x) &= \left[\Delta(\nu\psi_1(x) + (1-\nu)\psi_2(x)) - f(x,\nu\psi_1(x) + (1-\nu)\psi_2(x)\right]\Big|_{\nu=0}^{\nu=1} \\ &= [\Delta\psi_1 - f(x,\psi_1)] - [\Delta\psi_2 - f(x,\psi_2)] = 0 \end{aligned} \tag{3.21}$$

then computing the derivative first:

$$\begin{aligned}\mathscr{I}(\psi_1,\psi_2)(x) &= \int_0^1 \{(\Delta\psi_1 - \Delta\psi_2) - D_2 f(x,\nu\psi_1 + (1-\nu)\psi_2)(\psi_1 - \psi_2)\}\,\mathrm{d}\nu \\ &= \Delta(\psi_1 - \psi_2) - \left[\int_0^1 D_2 f(x,\nu\psi_1 + (1-\nu)\psi_2)\right](\psi_1 - \psi_2) \end{aligned} \tag{3.22}$$

where D_2 takes the derivative w.r.t. the second argument of f. Hence

$$\begin{aligned}
& \Delta(\psi_1-\psi_2) - \left[\int_0^1 D_2 f(x,\nu\psi_1+(1-\nu)\psi_2)\right](\psi_1-\psi_2) = 0 \\
\Longleftrightarrow \quad & \Delta(\psi_1-\psi_2) = \mathscr{F}[x,\psi_1,\psi_2](\psi_1-\psi_2) \qquad (3.23) \\
\text{where} \quad & \mathscr{F}[x,\psi_1,\psi_2] := \int_0^1 D_2 f(x,\nu\psi_1+(1-\nu)\psi_2) \qquad (3.24)
\end{aligned}$$

Multiplying both sides by $(\psi_1-\psi_2)$, integrating over Σ, and integrating the first term by parts yields:

$$\int_\Sigma |\nabla(\psi_1-\psi_2)|^2 + \mathscr{F}[x,\psi_1,\psi_2]|\psi_1-\psi_2|^2 = 0 \qquad (3.25)$$

Since by the hypothesis of the lemma, $F[x,\psi_1,\psi_2] \geq 0$, we must have $\psi_1 = \psi_2$.

q.e.d. Lemma

We shall use the lemma to prove uniqueness, in five of the cases that admit solutions. Indeed, the case $(\mathscr{Y}^0, \sigma^2 \equiv 0, \tau = 0)$ has infinitely many solutions. This corresponds to a freedom of choice in the spatial scale of the spacetimes. Note that the choice of conformal gauge does not have to be the same for the uniqueness proof as for the existence proof.

- $(\mathscr{Y}^+, \sigma^2 \not\equiv 0, \tau = 0)$: The Lichnerowicz equation, with the conformal gauge condition $R = 8$, takes the form $\Delta\phi = \phi - \frac{1}{8}\sigma^2\phi^{-7}$, so that in terms of the lemma, $f(x,\phi) = \phi - \frac{1}{8}\sigma^2\phi^{-7}$, and $D_2 f(x,\phi) = 1 + \frac{7}{8}\sigma^2\phi^{-8} > 0$.
- $(\mathscr{Y}^+, \sigma^2 \not\equiv 0, \tau \neq 0)$: Choose the conformal gauge so that $R = +1$. Then $f(x,\phi) = \frac{1}{8}\phi - \frac{1}{8}\sigma^2\phi^{-7} + \frac{1}{12}\tau^2\phi^5$, and $D_2 f(x,\phi) = \frac{1}{8} + \frac{7}{8}\sigma^2\phi^{-8} + \frac{5}{12}\tau^2\phi^4 > 0$.
- $(\mathscr{Y}^0, \sigma^2 \not\equiv 0, \tau \neq 0)$: Choose the conformal gauge so that $R = 0$. Then $f(x,\phi) = -\frac{1}{8}\sigma^2\phi^{-7} + \frac{1}{12}\tau^2\phi^5$, and $D_2 f(x,\phi) = \frac{7}{8}\sigma^2\phi^{-8} + \frac{5}{12}\tau^2\phi^4 > 0$.
- $(\mathscr{Y}^-, \sigma^2 \equiv 0, \tau \neq 0)$: Choose the conformal gauge so that $R = -1$. Then $f(x,\phi) = -\frac{1}{8}\phi + \frac{1}{12}\tau^2\phi^5$, and $D_2 f(x,\phi) = -\frac{1}{8} + \frac{5}{12}\tau^2\phi^4$, which is *not* necessarily positive for positive ϕ. However, if ϕ is a solution of $\Delta\phi = -\frac{1}{8}\phi + \frac{1}{12}\tau^2\phi^5$, then at any point $x_{\min}$ where ϕ assumes a minimum, we have $0 \leq$

$\Delta\phi(x_{\min}) = -\frac{1}{8}\phi(x_{\min}) + \frac{1}{12}\tau^2\phi^5(x_{\min})$, and hence $\phi(x_{\min}) \geq \left(\frac{3}{2\tau^2}\right)^{1/4}$. And on $I = \left(\left(\frac{3}{2\tau^2}\right)^{1/4}, +\infty\right)$, we have $D_2 f(x,\phi) \geq -\frac{1}{8} + \frac{5}{12}\tau^2\left(\frac{3}{2\tau^2}\right) \geq \frac{1}{2}$.

- $(\mathscr{Y}^-, \sigma^2 \not\equiv 0, \tau \neq 0)$: Choose the conformal gauge so that $R = -1$. Then $f(x,\phi) = -\frac{1}{8}\phi - \frac{1}{8}\sigma^2\phi^{-7} + \frac{1}{12}\tau^2\phi^5$, and $D_2 f(x,\phi) = -\frac{1}{8} + \frac{7}{8}\sigma^2\phi^{-8} + \frac{5}{12}\tau^2\phi^4$, which is *not* necessarily positive for $\phi > 0$. However, we find that $\phi \in I = \left(\left(\frac{3}{2\tau^2}\right)^{1/4}, +\infty\right)$, and $D_2 f(x,\phi) \geq \frac{1}{2} + \frac{7}{8}\sigma^2(\frac{4}{9}\tau^4)$ on I. □

3.3 Near-CMC-results

As just seen, setting $\tau =$ constant greatly simplifies Equations (2.18) & (2.19), since the momentum constraint then becomes trivial. Letting this assumption loose, we now have to handle the coupled system. The idea behind the near-CMC setting is to impose sufficient control over the gradient of τ to determine whether or not solutions exist. The near-CMC conditions (given by Equations (2.18) & (2.19)) leave the constraint equations coupled, but ensures the coupling is weak. The first near-CMC existence results were presented in [12], but a constructive proof, which is now addressed, was given in [17].

Theorem 3.14. Consider (Σ, γ), where γ is s.t. $R = -1$, and has no conformal Killing vector fields. Let $\sigma \in W^p_2(\Sigma)$. Then for every function $\tau \in W^p_2(\Sigma)$, $\tau : \Sigma \to \mathbb{R}_+$ which satisfies the inequalities (3.41) & (3.55) on $|\nabla\tau|$, Equations (2.18) & (2.19) with data (γ, σ, τ) admit a unique solution (ϕ, W), with $\phi \in C^{2,\alpha}(\Sigma)$ and with $W \in C^{3,\alpha}(\Sigma)$ for $\alpha = 1 - 3/p$.

Moreover, for every choice of γ and σ which satisfy the above-stated hypotheses, there is an open set in $W^p_2(\Sigma)$ of nonzero functions τ which satisfy the inequalities (3.41) & (3.55).

Remark 3.15. This theorem prescribes *sufficient* conditions on conformal data (γ, σ, τ) so that Equations (2.18) & (2.19) admit a solution, but not necessary ones. △

Proof. Consider the sequence of semi-decoupled equations:

$$\begin{aligned}
\Delta\phi^{(n)} &= -\frac{1}{8}\phi^{(n)} - \frac{1}{8}\left(\sigma^{ab} + (\mathfrak{L}W^{(n)})^{ab}\right)\left(\sigma_{ab} + (\mathfrak{L}W^{(n)})_{ab}\right)(\phi^{(n)})^{-7} \\
&\quad + \frac{1}{12}\tau^2(\phi^{(n)})^5 \qquad (3.26) \\
D_a\left(\mathfrak{L}W^{(n)}\right)^a_b &= \frac{2}{3}(\phi^{(n-1)})^6 D_b\tau \qquad (3.27)
\end{aligned}$$

which are to be solved for the sequence $\{\phi^{(n)}, W^{(n)}\}_{n=1}^{\infty}$, once $\phi^{(0)}$ has been chosen.

Step 1: Existence of the sequence. Pick some positive definite $\phi^{(0)} \in W_1^p(\Sigma)$, and find $W^{(1)}$ by solving Equation (3.27). Since $\gamma \in C^3(\Sigma)$, the coefficients a^{ij}, a^i and a of the conformal Laplacian are all contained in $C^1(\Sigma)$. Since $\tau \in W_2^p(\Sigma)$, $\phi^{(0)} \in W_1^p(\Sigma)$, and $W_1^p(\Sigma)$ is closed under multiplication, we have $\frac{2}{3}(\phi^{(0)})^6 D_b\tau \in W_1^p(\Sigma)$. Therefore, Equation (3.27) has a unique solution $W^{(1)} \in W_3^p(\Sigma)$ (see Appendix C.3.1). By C.24, for $k = 0$ and $k = 1$, $\exists$ a constant $c(\gamma, p, k)$ s.t.

$$||W^{(1)}||_{W_{k+2}^p} \leq c(\gamma, p, k)||(\phi^{(0)})^6\nabla\tau||_{W_k^p} \qquad (3.28)$$

The inequality can be turned into a pointwise bound for the quantity

$$|\mathfrak{L}W^{(1)}| := \left[\left(\mathfrak{L}W^{(1)}\right)_{ab}\left(\mathfrak{L}W^{(1)}\right)^{ab}\right]^{1/2} \qquad (3.29)$$

By Sobolev embedding theorem: since $W^{(1)} \in W_{k+2}^p(\Sigma)$ (for $k = 0, 1$) with $p > 3$, we have $W^{(1)} \in C^{k+1,\alpha}(\Sigma)$, and for some constant $c(\gamma, p, k)$, we have

$$||W^{(1)}||_{C^{k+1,\alpha}} \leq c(\gamma, p, k)||W^{(1)}||_{W_{k+2}^p} \qquad (3.30)$$

It follows from the definition of these Hölder norms that

$$||\mathfrak{L}W^{(1)}||_{C^0} \leq c||DW^{(1)}||_{C^0} \leq c||W^{(1)}||_{C^{1,\alpha}} \qquad (3.31)$$

We also have

$$|\mathfrak{L}W^{(1)}(x)| \leq c||\mathfrak{L}W^{(1)}||_{C^0} \qquad \forall\ x \in \Sigma \qquad (3.32)$$

Combining inequalities (3.28)-(3.32), with $k = 0$, we have

$$|\mathfrak{L}W^{(1)}(x)| \leq C||(\phi^{(0)})^6\nabla\tau||_{W_0^p} \qquad (3.33)$$

which implies that, for some $c > 0$,

$$|\mathfrak{L}W^{(1)}(x)| \leq c \left(\max_{\Sigma} \phi^{(0)} \right)^6 \max_{\Sigma} |\nabla \tau| \tag{3.34}$$

which is the desired pointwise bound.

Once we have $W^{(1)}$, we may substitute it into Equation (3.26) with $n = 1$. We then prove that there exists a unique solution $\phi^{(1)}$ to this equation using the sub and super solution theorem: The function

$$f^{(1)}(x, \phi^{(1)}) = -\frac{1}{8}\phi^{(1)} - \frac{1}{8}\left(\sigma^{ab} + (\mathfrak{L}W^{(1)})^{ab}\right)\left(\sigma_{ab} + (\mathfrak{L}W^{(1)})_{ab}\right)(\phi^{(1)})^{-7} + \frac{1}{12}\tau^2(\phi^{(1)})^5 \tag{3.35}$$

is $C^1(\Sigma \times \mathbb{R}^+)$. Then the constants

$$\phi_-^{(1)} = \left(\frac{3}{2\ \max_\Sigma \tau^2}\right)^{1/4} \tag{3.36}$$

$$\phi_+^{(1)} = \max\left\{1, \left[\frac{3\left(1 + 2\max_\Sigma \sigma_{ab}\sigma^{ab} + 2\max_\Sigma \mathfrak{L}W_{ab}^{(1)}\mathfrak{L}W^{(1)ab}\right)}{2\ \min_\Sigma \tau^2}\right]^{1/4}\right\} \tag{3.37}$$

are sub and super solutions for Equation (3.26), hence there exists a unique solution $\phi^{(1)} \in C^{2,\alpha}$.

With $\phi^{(1)}$ determined, we may proceed on to solve Equation (3.27) with $n = 2$ for $W^{(2)}$, and then solve Equation (3.26) with $n = 2$ for $\phi^{(2)}$, etc.

Since $W^{(n)} \in W_3^p(\Sigma)$ and $\phi^{(n)} \in C^{2,\alpha}(\Sigma)$, the arguments used to show that $W^{(1)}$ and $\phi^{(1)}$ exist and are unique can be used at each stage of the iteration, and hence we can verify existence and uniqueness of the sequence $\{W^{(n)}, \phi^{(n)}\}$ for all n. The $\mathfrak{L}W^{(n)}$ satisfy the pointwise bounds:

$$|\mathfrak{L}W^{(n)}(x)| \leq c \left(\max_{\Sigma} \phi^{(n-1)} \right)^6 \max_{\Sigma} |\nabla \tau| \tag{3.38}$$

and sub and super solutions for Equations (3.26) are:

$$\phi_-^{(n)} = \left(\frac{3}{2\max_\Sigma \tau^2}\right)^{1/4} \tag{3.39}$$

$$\phi_+^{(n)} = \max\left\{1, \left[\frac{3\left(1 + 2\max_\Sigma \sigma_{ab}\sigma^{ab} + 2\max_\Sigma \mathfrak{L}W_{ab}^{(n)}\mathfrak{L}W^{(n)ab}\right)}{2\ \min_\Sigma \tau^2}\right]^{1/4}\right\} \tag{3.40}$$

Step 2: Uniform bounds on the sequence. While we have established bounds for the functions $\phi^{(n)}$ and $|\mathfrak{L}W^{(n)}|$ for each n, in principle, these bounds could go to ∞ or to 0. We now show that this does not happen:

The subsolution $\phi_-^{(n)}$ in Equation (3.39) is independent of n, and thus provides a uniform lower bound $\phi_-^{(\infty)}$.

As for the upper bound, imposing the restriction

$$\frac{c\max_\Sigma|\nabla\tau|^2}{\min_\Sigma\tau^2} < 1 \tag{3.41}$$

some algebra shows that the constant

$$\xi = \max\left\{1, \left(1 - \frac{c\max_\Sigma|\nabla\tau|^2}{\min_\Sigma\tau^2}\right)^{-1}\left(\frac{3}{2\min_\Sigma\tau^2}\right)\left(1 + 2\max_\Sigma\sigma^{ab}\sigma_{ab}\right)\right\} \tag{3.42}$$

provides a uniform upper bound: $\phi_+^{(\infty)} := \sqrt[4]{\xi}$.

Note that so far, we had not imposed any restriction on the initial guess $\phi^{(0)}$, however, we now see that we must have $\phi_-^{(\infty)} \leq \phi^{(0)} \leq \phi_+^{(\infty)}$. It follows from inequality (3.38) that $|\mathfrak{L}W^{(n)}|$ has an n-independent upper bound on Σ.

Note that for any fixed choice of the metric γ (with or without $R_{(\gamma)} = -1$), there is always an open set of functions $\tau \in W_2^p(\Sigma)$ such that inequality (3.41) is satisfied.

Step 3: C^0 convergence of the sequence. We now use a contraction mapping argument to show that the sequence converges in C^0 to a limit $(\phi^{(\infty)}, W^{(\infty)})$. Since $W^{(n)}$ is very strongly controlled by $\phi^{(n-1)}$ via the momentum equation and inequality (3.38), convergence of $W^{(n)}$ will follow from the convergence of $\phi^{(n)}$. Consider the functional:

$$\mathcal{I}\left(x, \phi^{(n-1)}, \phi^{(n)}, \phi^{(n+1)}\right) := \int_0^1 \left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\Delta\psi^{(n+1)}(t) - F\left(x, \psi^{(n)}(t), \psi^{(n+1)}(t)\right)\right]\right\}\mathrm{d}t \tag{3.43}$$

$$\begin{aligned} \text{where} \qquad & \psi^{(n)}(t) := t\phi^{(n)} + (1-t)\phi^{(n-1)} \\ \text{and} \qquad & F\left(x, \psi^{(n)}, \psi^{(n+1)}\right) := -\frac{1}{8}\psi^{(n+1)} - \frac{1}{8}\left(\sigma^{ab} + \left(\mathfrak{L}W\left[\psi^{(n)}\right]\right)^{ab}\right) \\ & \times\left(\sigma_{ab} + \left(\mathfrak{L}W\left[\psi^{(n)}\right]\right)_{ab}\right)(\psi^{(n+1)})^{-7} + \frac{1}{12}\tau^2(\psi^{(n+1)})^5 \end{aligned} \tag{3.44}$$

and $\mathfrak{L}W[\psi^{(n)}]$ is determined by the equation

$$D_a\left(\mathfrak{L}W[\psi^{(n)}]\right)^a_b = \frac{2}{3}(\psi^{(n)})^6 D_b\tau \tag{3.45}$$

Computing $\mathcal{I}$ in two ways (using the Fund. Thm. of Calc., and taking the derivative directly), one obtains:

$$\begin{aligned} 0 \quad &= \quad \int_0^1 \Big\{\Delta\left(\phi^{(n+1)}-\phi^{(n)}\right) - D_2F\left(x,\psi^{(n)},\psi^{(n+1)}\right)\left[\phi^{(n)}-\phi^{(n-1)}\right] \\ &\quad -D_3F\left(x,\psi^{(n)},\psi^{(n+1)}\right)\left[\phi^{(n+1)}-\phi^{(n)}\right]\Big\}\,\mathrm{d}t & (3.46)\\ &=: \quad \Delta\left(\phi^{(n+1)}-\phi^{(n)}\right) - \mathcal{F}\left[\phi^{(n)}-\phi^{(n-1)}\right] - \mathcal{G}\left[\phi^{(n+1)}-\phi^{(n)}\right] & (3.47)\\ \iff \quad & \mathcal{F}\left[\phi^{(n)}-\phi^{(n-1)}\right] = \Delta\left(\phi^{(n+1)}-\phi^{(n)}\right) - \mathcal{G}\left[\phi^{(n+1)}-\phi^{(n)}\right] & (3.48)\end{aligned}$$

where we defined:

$$\begin{aligned}\mathcal{F}\left[\phi^{(n)}-\phi^{(n-1)}\right] \quad &= \quad \int_0^1 D_2F\left(x,\psi^{(n)}(t),\psi^{(n+1)}(t)\right)\mathrm{d}t\left[\phi^{(n)}-\phi^{(n-1)}\right] & (3.49)\\ \mathcal{G}\left[\phi^{(n+1)}-\phi^{(n)}\right] \quad &= \quad \int_0^1 D_3F\left(x,\psi^{(n)},\psi^{(n+1)}\right)\mathrm{d}t\left[\phi^{(n+1)}-\phi^{(n)}\right] & (3.50)\end{aligned}$$

Using the lower bound on the $\phi^{(n)}$, as have that

$$\mathcal{G}\left[\phi^{(n+1)}-\phi^{(n)}\right] \geq \gamma\left(\phi^{(n+1)}-\phi^{(n)}\right) \quad \text{where} \quad \gamma := \frac{1}{8}\left[5\frac{\min_\Sigma \tau^2}{\max_\Sigma \tau^2} - 1\right] \tag{3.51}$$

Note that $\gamma > 0$ as a consequence of the mean value theorem, and the restriction of $\max_\Sigma(\nabla\tau)^2/\min_\Sigma \tau^2$.

Similarly, using the upper bound on the $\phi^{(n)}$, one can obtain:

$$|\mathcal{F}\left[\phi^{(n)}-\phi^{(n-1)}\right]| \leq \Theta \max_\Sigma |\phi^{(n)}-\phi^{(n-1)}| \qquad \text{where} \tag{3.52}$$

$$\Theta := c\left[\max_\Sigma|\sigma| + \hat{c}\left(\phi_+^{(\infty)}\right)^6 \max_\Sigma|\nabla\tau|\right]\left(\phi_+^{(\infty)}\right)^5\left(\phi_-^{(\infty)}\right)^{-7}\max_\Sigma|\nabla\tau| \tag{3.53}$$

We can now apply the maximum principle to Equation (3.48) and obtain the pointwise bound:

$$|\phi^{(n+1)}-\phi^{(n)}| \leq (\Theta/\gamma)\max_\Sigma|\phi^{(n)}-\phi^{(n-1)}| \tag{3.54}$$

Suppose that the conformal data (γ, σ, τ) satisfy the inequality

$$\frac{\Theta(\gamma,\sigma,\tau)}{\gamma(\gamma,\sigma,\tau)} < 1 \tag{3.55}$$

Claim: $\{\phi^{(n)}\}$ is a Cauchy sequence in $C^0(\Sigma)$

Proof of Claim: Let

$$\rho := ||\phi^{(1)} - \phi^{(0)}||_{C^0} \qquad \kappa := \Theta/\gamma \tag{3.56}$$

Then for $n, m \in \mathbb{N}$, $n > m$:

$$\begin{aligned} & ||\phi^{(n)} - \phi^{(m)}||_{C^0} \\ \leq\ & ||\phi^{(n)} - \phi^{(n-1)}||_{C^0} + ||\phi^{(n-1)} - \phi^{(n-2)}||_{C^0} + \ldots + ||\phi^{(m+1)} - \phi^{(m)}||_{C^0} \\ \leq\ & \left[\kappa^{n-1} + \kappa^{n-2} + \ldots + \kappa^m\right] \rho \ \leq\ \frac{\kappa^m}{1-\kappa} \end{aligned} \tag{3.57}$$

Given any $\varepsilon > 0$, we have that $\forall n, m > N$ where,

$$N > \frac{|\ln(\rho/\varepsilon(1-\kappa))|}{|\ln\kappa|} \qquad \Longrightarrow \qquad \frac{\kappa^m}{1-\kappa} < \varepsilon \tag{3.58}$$

q.e.d. proof of claim

Therefore, $\{\phi^{(n)}\}$ converges to $\phi^{(\infty)}$ in $C^0(\Sigma)$, with $\phi^{(\infty)} \geq \phi_-^{(\infty)} > 0$.

Convergence of $W^{(n)}$ in $W_2^p(\Sigma) \subset C^1(\Sigma)$ to $W^{(\infty)}$ is guaranteed by:

$$||W^{(n)} - W^{(m)}||_{W_2^p} \leq c||(\phi^{(n)})^6\nabla\tau - (\phi^{(m)})^6\nabla\tau||_{W_0^p} \tag{3.59}$$

$$\leq \tilde{c}||\nabla\tau||_{C^0}||\phi^{(n)} - \phi^{(m)}||_{C^0} \tag{3.60}$$

for constants c and $\tilde{c}$.

Step 4: Bootstrapping $\phi^{(\infty)}$ and $W^{(\infty)}$. We show that $\phi^{(\infty)}$, $W^{(\infty)} \in C^2(\Sigma)$. We make use of the following claim (stated without proof):

Claim: The functional

$$F(x,\theta,\psi) := -\frac{1}{8}\psi - \frac{1}{8}\left(\sigma^{ab} + (\mathfrak{L}W[\theta])^{ab}\right)\left(\sigma_{ab} + (\mathfrak{L}W[\theta])_{ab}\right)\psi^{-7} + \frac{1}{12}\tau^2\psi^5 \tag{3.61}$$

is a jointly Lipschitz continuous function in θ nd ψ, so long as both θ and ψ are bounded from below (by some constant $\psi^- > 0$) and from above (by some constant ψ^+).

We have:

$$\begin{aligned} & ||\phi^{(n)} - \phi^{(m)}||_{W_2^p} \\ \leq\ & c||F(\cdot, \phi^{(n-1)}, \phi^{(n)}) - F(\cdot, \phi^{(m-1)}, \phi^{(m)})||_{W_0^p} + b||\phi^{(n)} - \phi^{(m)}||_{W_0^p} \quad (3.62) \\ \leq\ & c||F(\cdot, \phi^{(n-1)}, \phi^{(n)}) - F(\cdot, \phi^{(m-1)}, \phi^{(m)})||_{C^0} + c||\phi^{(n)} - \phi^{(m)}||_{C^0} \quad (3.63) \end{aligned}$$

where compactness of Σ was used. Using the claim, and the convergence of $\{\phi^{(n)}\}$ in C^0, it follows that $\{\phi^{(n)}\}$ is Cauchy in $W_2^p(\Sigma)$, and thus converges to a unique limit $\phi^{(\infty)}$ in $W_2^p(\Sigma)$, and by Sobolev embedding theorem, $\phi^{(\infty)} \in C^{1,\alpha}(\Sigma)$ for $\alpha \in (0, 1 - 3/p)$.

To obtain one more degree of differentiability, we examine the sequence $\{\phi^{(n)}\}$ in the Holder space $C^{2,\alpha}(\Sigma)$:

$$\begin{aligned} ||\phi^{(n)} - \phi^{(m)}||_{C^{2,\alpha}} \leq\ & c\Big(||F(\cdot, \phi^{(n-1)}, \phi^{(n)}) - F(\cdot, \phi^{(m-1)}, \phi^{(m)})||_{C^{0,\alpha}} \\ & +||\phi^{(n)} - \phi^{(m)}||_{C^0}\Big) \quad (3.64) \end{aligned}$$

hence $\phi^{(\infty)} \in C^{2,\alpha}(\Sigma)$.

As for the vector field, we first show that $W^{(\infty)}$ is (at least) a weak solution of the momentum constraint. Let $\hat{W}^{(\infty)}$ denote the W_2^p solution of the equation

$$D_a \left(\mathfrak{L}\hat{W}_\infty\right)^a_b = \frac{2}{3}\phi^{(\infty)6} D_b \tau \quad (3.65)$$

and consider the quantity $||W^{(\infty)} - \hat{W}^{(\infty)}||_{W_2^p}$:

$$\begin{aligned} ||W^{(\infty)} - \hat{W}^{(\infty)}||_{W_2^p} &= \lim_{n\to\infty} ||W^n - \hat{W}^{(\infty)}||_{W_2^p} \quad (3.66) \\ &= \lim_{n\to\infty} c||(\phi^{(n)})^6 \nabla\tau - (\phi^{(\infty)})^6 \nabla\tau||_{W_0^p} \quad (3.67) \\ &= \lim_{n\to\infty} \tilde{c}||(\phi^{(n)})^6 - (\phi^{(\infty)})^6||_{W_0^p} ||\nabla\tau||_{W_0^p} \quad (3.68) \\ &= \lim_{n\to\infty} \hat{c}||(\phi^{(n)})^6 - (\phi^{(\infty)})^6||_{W_0^p} = 0 \quad (3.69) \end{aligned}$$

Hence $W^{(\infty)} = \hat{W}^{(\infty)}$ at least weakly.

Since $\frac{2}{3}(\phi^{(\infty)})^6 D_b\tau \in W_2^p(\Sigma)$, the Fredholm alternative theorem guarantees that $W^{(\infty)} \in W_4^p(\Sigma)$, and by Sobolev embedding theorem, $W^{(\infty)} \in C^{3,\alpha}(\Sigma)$.

Step 5: $(\phi^{(\infty)}, W^{(\infty)})$ is a solution Since $\phi^{(\infty)}$ and $W^{(\infty)}$ are both twice differentiable, they constitute a strong solution iff they constitute a weak solution. We already showed that the pair is a weak solution of the momentum constraint. To show that it is a weak solution of the Lichnerowicz equation, rewrite it as:

$$\Delta\phi = F(x, \phi, \phi) \tag{3.70}$$

with F defined by Equation (3.61). Consider the map

$$\mathcal{D} : W_2^p(\Sigma) \to W_0^p(\Sigma) \qquad \psi \mapsto \Delta\psi - F(x, \psi, \psi) \tag{3.71}$$

so that $\phi^{(\infty)}$ solve the Lichnerowicz equation iff $\mathcal{D}\phi^{(\infty)} = 0$. Since Δ and F are continuous maps, $\mathcal{D}$ is continuous and we can write $\mathcal{D}\phi^{(\infty)} = \lim_{n\to\infty} \mathcal{D}\phi^{(n)}$ and $||\mathcal{D}\phi^{(\infty)}||_{W_0^p} = \lim_{n\to\infty} ||\mathcal{D}\phi^{(n)}||_{W_0^p}$. Then

$$\begin{aligned}
||\mathcal{D}\phi^{(\infty)}||_{W_0^p} &= \lim_{n\to\infty} ||\mathcal{D}\phi^{(n)}||_{W_0^p} && (3.72)\\
&= \lim_{n\to\infty} ||\Delta\phi^{(n)} - F\left(\cdot, \phi^{(n)}, \phi^{(n)}\right)||_{W_0^p} && (3.73)\\
&= \lim_{n\to\infty} ||F(\cdot, \phi^{(n-1)}, \phi^{(n)}) - F(\cdot, \phi^{(n)}, \phi^{(n)})||_{W_0^p} && (3.74)\\
&= \lim_{n\to\infty} c||F(\cdot, \phi^{(n-1)}, \phi^{(n)}) - F(\cdot, \phi^{(n)}, \phi^{(n)})||_{C^0} \;=\; 0 && (3.75)
\end{aligned}$$

Step 6: Uniqueness We first need a claim:

Claim: If (ϕ, W) is a solution of Equations (2.18) & (2.19) for the conformal data (γ, σ, τ) satisfying inequality (3.41), then ϕ must satisfy $\phi_-^{(\infty)} \leq \phi(x) \leq \phi_+^{(\infty)}$

Proof of claim: Let $x_{\min} \in \Sigma$ be a local minimum for ϕ. Since $\Delta\phi(x_{\min}) \geq 0$, we can rewrite the Lichnerowicz equation as:

$$\phi^4(x_{\min}) \geq \frac{3}{2\tau^2(x_{\min})}\left(1 + [\sigma(x_{\min}) + \mathfrak{L}W(x_{\min})]^2\,\phi^{-8}(x_{\min})\right) \tag{3.76}$$

$$\iff \quad \phi(x_{\min}) \geq \left(\frac{3}{2\max_\Sigma \tau^2}\right)^{1/4} =: \phi_-^{(\infty)} \tag{3.77}$$

$$\implies \quad \phi(x) \geq \phi_-^{(\infty)} \qquad \forall\, x \in \Sigma \tag{3.78}$$

Similarly, at a local maximum $x_{\max} \in \Sigma$, $\Delta\phi(x_{\max}) \leq 0$, so that

$$\phi^{12}(x_{\max}) \leq \frac{3}{2\tau^2(x_{\max})}\phi^8(x_{\max}) + \frac{3}{2\tau^2(x_{\max})}\left[\sigma(x_{\max}) + \mathfrak{L}W(x_{\max})\right]^2 \quad (3.79)$$

Using the condition $(\mathfrak{L}W)^2 \leq c \max_\Sigma |\nabla\tau|^2 \max_\Sigma \phi^{12}$ as well as inequality (3.41), we conclude $\phi(x) \leq \phi_+^{(\infty)}$. *q.e.d. proof of claim*

To prove uniqueness, suppose that (ϕ, W) and $(\hat{\phi}, \hat{W})$ are both solutions, and consider the interpolation function $\Psi(t) := t\phi + (1-t)\hat{\phi}$. Define

$$\mathfrak{I}(x, \phi, \hat{\phi}) := \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\left[\Delta\Psi(t) - F(x, \Psi(t), \Psi(t))\right] \mathrm{d}t \quad (3.80)$$

Computing $\mathfrak{I}(x, \phi, \hat{\phi})$ in two ways, we obtain:

$$||\phi - \hat{\phi}||_{C^0} \leq \Theta/\gamma ||\phi - \hat{\phi}||_{C^0} \quad (3.81)$$

Since $\Theta/\gamma < 1$, this is only satisfied if $\phi = \hat{\phi}$, which immediately implies that $W = \hat{W}$.

Last step: We just showed that conformal data (γ, σ, τ) satisfying inequalities (3.41) & (3.55) always map to a solution of the constraint equations.

Given fixed γ and σ, one can always find a non-zero function τ satisfying both inequalities, and the set of such functions is open in $W_2^p(\Sigma)$. Indeed, given any positive function τ_0, add a constant ρ sufficiently large so that:

$$\frac{\max_\Sigma |\nabla\tau|^2}{\min_\Sigma \tau^2} = \frac{\max_\Sigma |\nabla\tau_0|^2}{\min_\Sigma (\tau_0 + \rho)^2} \quad (3.82)$$

so that inequality (3.41) is satisfied.

Alternatively, 'squeezing' τ_0 as such:

$$\tau_0 \to \nu\tau_0 + (1-\nu)\bar{\tau}_0 \qquad \nu \in (0,1) \quad \bar{\tau}_0 = \text{average of } \tau_0 \quad (3.83)$$

one can find ν so that both inequalities are satisfied.

$\square$

Note that the theorem just proved requires $R^{(\gamma)} = -1$. The following corollary extends this result to metrics in the negative Yamabe class:

Corollary 3.16. Consider (Σ, γ), where $\gamma \in C^{3,\alpha}(\Sigma)$, $\gamma \in \mathcal{Y}^-(\Sigma)$ has no conformal Killing vector fields. Let $\sigma \in W^p_2(\Sigma)$. Then for every function $\tau : \Sigma \to \mathbb{R}_+$, $\tau \in W^p_2(\Sigma)$ which satisfies a variant of the inequalities (3.41) & (3.55), there is a vector field $U \in C^{3,\alpha}(\Sigma)$ for $\alpha < 1 - 3/p$ and a pair of functions $\psi \in C^{3,\alpha}(\Sigma)$ and $\theta \in C^{2,\alpha}(\Sigma)$ such that the data

$$\gamma_{ab} = (\theta\psi)^4 \gamma_{ab} \tag{3.84}$$

$$K^{cd} = \theta^{-10}\left(\psi^{-10}\sigma^{cd} + \mathfrak{L}_{(\psi^4\gamma)} U^{cd}\right) + \frac{1}{3}\theta^{-4}(\psi^{-4}\gamma^{ab})\tau \tag{3.85}$$

are a solution of the Einstein constraint equations.

Those results were recently extended to non-negative Yamabe metrics in [1], where the following is proved:

Theorem 3.17. Consider (Σ, γ) where γ is a smooth metric which has no conformal Killing vector fields, and $\gamma \in \mathcal{Y}^0(\Sigma) \cup \mathcal{Y}^+(\Sigma)$. For each smooth $\sigma \not\equiv 0$, and for each smooth function $\tau : \Sigma \to \mathbb{R}_+$ which satisfies a variant of the inequalities (3.41) & (3.55), there exist smooth positive functions ϕ and θ and a smooth vector field W such that the data

$$\gamma_{ab} = (\phi\theta)^4 \lambda_{ab} \tag{3.86}$$

$$K^{ab} = \phi^{-10}\left(\theta^{-10}\sigma + (\mathfrak{L}W)^{(\theta^4\lambda)}\right)^{ab} + \frac{1}{3}(\phi\theta)^{-4}\lambda^{ab}\tau \tag{3.87}$$

are a solution to the Einstein constraint equations.

The main difference with the results of [18] is that the sub solutions used to apply the sub and super solution theorem to the Lichnerowicz equation are not constant.

What if we now relax the condition $\sigma \not\equiv 0$? Here is a non-existence result, proved in [18]:

Theorem 3.18. Let $(\Sigma, \gamma, \sigma, \tau)$ be a set of conformal data s.t. Σ is closed, $R^{(\gamma)} \geq 0$, $\sigma^2 \equiv 0$ and $\tau = T + \rho$ with T a nonzero constant. For $\frac{|\nabla \rho|}{|T|}$ sufficiently small, the Einstein constraint equations admit no solution.

The proof assumes the existence of a solution, and then considers the sign of the terms in the Lichnerowicz equation at a maximum of ϕ to derive a contradiction.

In a certain sense, this is a stability result for the non-existence of solutions to the conformal equations. Specifically, we recall that for CMC conformal data of the type $(\Sigma \text{ closed}, \gamma \in \mathcal{Y}^+ \cup \mathcal{Y}^0, \sigma^2 \equiv 0, \tau \neq 0)$, there exist no solutions. Restricting this result to those special cases in which $R(\lambda) \geq 0$, we see that our new results show that if we perturb the conformal data above by allowing τ to be non-constant with small gradient, then the non-existence condition still holds. We do not expect to retain non-existence if we also perturb σ^2 away from zero, since we know that CMC conformal data of the type $(\Sigma \text{ closed}, \gamma \in \mathcal{Y}^+ \cup \mathcal{Y}^0, \sigma^2 \not\equiv 0, \tau \neq 0)$ do lead to the existence of unique solutions.

3.4 Far-from-CMC case: General results

Although the results of the previous two sections characterize solutions where the mean curvature is constant or controlled, the case where the mean curvature is arbitrary, the so-called 'far-from-CMC case', remained unexplored until very recently. This is partly because it was long thought that every spacetime admitted a Cauchy surface of constant mean curvature. However, a general condition ensuring the existence of CMC Cauchy surfaces in cosmological spacetimes is given in [3], along with an example of a spacetime that does *not* satisfy the condition.

So far, we considered the Einstein constraint equations in vacuum, i.e.: the energy density ρ as well as the momentum current density j were identically vanishing. However, the first existence results in the far-from-CMC case were given in [15], and required in particular that $\rho \not\equiv 0$.

Remark 3.19. More precisely, the energy-momentum tensor in [15] satisfies the *Dominant Energy Condition*, ie: the vector $-T^{\mu\nu}v_\nu$ is timelike and future-directed, where v^μ is any timelike and future-directed vector field. There exist other types of energy conditions (weak, null, etc). Physically, energy conditions are not directly related to energy conservation: T satisfying Bianchi identity (see (A.33)) guarantees that $\nabla_\mu T^{\mu\nu} = 0$ regardless of whether we impose any additional constraints on $T^{\mu\nu}$. Rather, they serve to prevent other properties considered as 'unphysical', such as energy propagating faster than the speed of light, or empty space spontaneously decaying into compensating regions of positive and negative energy. [7] In particular, for an observer moving with four-velocity v^ν, the quantity $-T^\mu{}_\nu v^\nu$ physically represents the energy-momentum four-current density of matter as seen by him. The fact that it is timelike and future-directed in the DEC can be interpreted as saying that the speed of energy flow of matter is always less than the speed of light. [28]

The three-tensors ρ and j enter the Einstein constraint equations (1.2) & (1.3) as the pullbacks on Σ of the analogous tensors living on M. They are subject to the condition $-\rho^2 + j \cdot j \leq 0$ induced by the DEC on T. The conformal decomposition presented in Section 2.3 now has to account for the presence of matter terms. Letting $\rho = \phi^{-8}\hat{\rho}$ and $j = \phi^{-10}\hat{j}$ ensures that the inequality $-\hat{\rho}^2 + \hat{j} \cdot \hat{j} \leq 0$ holds. $\hat{\rho}$ and $\hat{j}$ are *prescribed* non-physical matter fields, so that the conformally formulated Einstein constraint equations are still to be solved for γ and W only. [15] $\triangle$

Under the additional assumptions that $\gamma \in \mathcal{Y}^+$ with Σ closed, and the matter fields and the TT-tensor be small (in some precise sense), the authors devised new advances that are free of the near-CMC assumption. Those are a topological fixed-point argument, and a global supersolution construction for the Hamiltonian constraint. Together with other tools, they were used to show that a solution could be constructed from τ taken to be an arbitrary smooth function without restrictions on the size of its partial derivatives. This is to be contrasted with the results presented in Section 3.3, where the fixed-point argument and the global barrier construction relied critically on the near-CMC assumption.

Once these results at hand, there are two remaining open problems regarding existence of solutions:

- Existence of near-CMC-free global super solutions for the Hamiltonian constraint equations when $\gamma \in \mathcal{Y}^0 \cup \mathcal{Y}^+$ and for large data,

- Existence of near-CMC-free global sub solutions for the Hamiltonian constraint equation when $\gamma \in \mathcal{Y}^+$ in vacuum.

The weakness in the construction pointed out by the second problem is resolved in [23]. In this article, two proofs are presented, that show that the conformal method can be used to construct a corresponding set of vacuum solutions. The first proof builds on the work presented in [15]. It uses the facts that it is only the global subsolution that requires the presence of matter. The global supersolution is applicable in vacuum, and requires that the matter fields, if present, be weak. The first proof shows that solutions exist, under certain mild technical conditions, whenever a global supersolution can be found. The proof relies on an apriori estimate that replaces the need for a global subsolution. The second proof considers a sequence of non-vacuum solutions (constructed using [15]) where the matter fields are converging to zero. Again, a lower bound is found for the sequence and is used to obtain a corresponding subsequence converging to a vacuum far-from-CMC solution.

Before summarizing the main results of [15] & [23], we give the definitions of global sub/supersolution, as understood in those articles:

Definition 3.20. Consider the equations

$$-8\Delta\phi + R\phi = -\frac{2}{3}\tau^2\phi^5 + |\sigma + \mathfrak{L}W|^2\phi^{-7} + 2\rho\phi^{-3} \tag{3.88}$$

$$\Delta_{\text{conf}} W = \frac{2}{3}\phi^6\gamma \mathrm{d}\tau + J \tag{3.89}$$

Given a function ϕ, let $W_{(\phi)}$ be the corresponding solution of Equation (3.89). We say ϕ_+ is a *global supersolution* if whenever $0 < \phi \leq \phi_+$, then

$$-8\Delta\phi_+ + R\phi_+ \geq -\frac{2}{3}\tau^2\phi_+^5 + |\sigma + \mathfrak{L}W_{(\phi)}|^2\phi_+^{-7} + 2\rho\phi_+^{-3} \tag{3.90}$$

We say $\phi_- > 0$ is a *global subsolution* if whenever $\phi \geq \phi_-$, then

$$-8\Delta\phi_- + R\phi_- \leq -\frac{2}{3}\tau^2\phi_-^5 + |\sigma + \mathfrak{L}W_{(\phi)}|^2\phi_-^{-7} + 2\rho\phi_-^{-3} \tag{3.91}$$

The main existence result of [15] is the following:

Theorem 3.21. [15] Let (Σ, γ) be a 3-dimensional closed Riemannian manifold. Let $\gamma \in W_s^p$ admit no conformal Killing field, and be in $\mathcal{Y}^+(\Sigma)$, where $p \in (1, \infty)$ and $s \in (1 + \frac{3}{p}, \infty)$ are given. Select q and e to satisfy:

- $\frac{1}{q} \in (0,1) \cap (0, \frac{s-1}{3}) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}]$,
- $e \in (1 + \frac{3}{q}, \infty) \cap [s-1, s] \cap [\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}]$

Assume that the data satisfies:

- $\tau \in W_{e-1}^q$ if $e \geq 2$, and $\tau \in W_1^z$ otherwise, with $z = \frac{3q}{3 + \max\{0, 2-e\}q}$,
- $\sigma \in W_{e-1}^q$, with $||\sigma^2||_\infty$ sufficiently small,
- $\rho \in W_{s-2}^{p,+} \cap L^\infty$ 0, with $||\rho||_\infty$ sufficiently small,
- $j \in W_{e-2}^q$, with $||j||_{W_{e-2}^q}$ sufficiently small.

Then, there exist $\phi \in W_s^p$ with $\phi > 0$ and $W \in W_e^q$ solving the Einstein constraint equations.

Results on existence of solutions in vacuum are given in [23] as follows: Considering the system

$$-8\Delta\phi + R\phi = -\frac{2}{3}\tau^2\phi^5 + |\sigma + \mathfrak{L}W|^2\phi^{-7} \tag{3.92}$$

$$\Delta_{\text{conf}} W = \frac{2}{3}\phi^6 d\tau \tag{3.93}$$

Theorem 3.22. Let $\gamma \in W_2^p$ with $p > 3$ be a metric on a smooth, compact 3-manifold. Suppose γ has no conformal Killing fields and that one of the following conditions holds for a TT-tensor $\sigma \in W_1^p$ and a function $\tau \in W_1^p$.

1. $\gamma \in \mathcal{Y}^+(\Sigma)$, $\sigma \not\equiv 0$
2. $\gamma \in \mathcal{Y}^0(\Sigma)$, $\sigma \not\equiv 0$, $\tau \not\equiv 0$
3. $\gamma \in \mathcal{Y}^-(\Sigma)$ and $\exists\ \hat{\gamma} \in [\gamma]$ s.t. $R^{(\hat{\gamma})} = -\frac{2}{3}\tau^2$

If $\phi_+ \in W_2^{p,+}$ is a global supersolution for (γ, σ, τ), then there exists a solution $(\phi, W) \in W_2^{p,+} \times W_2^p$ to system (3.92)-(3.93) s.t. $\phi \le \phi_+$.

The following proposition extends the results of [15] to the vacuum setting:

Proposition 3.23. Suppose $\gamma \in W_2^p$ with $p > 3$, and that $\gamma \in \mathcal{Y}^+(\Sigma)$, $\tau \in W_1^p$, and $\sigma \in W_1^p$. If $||\sigma||_\infty$ is sufficiently small, then there exists a global supersolution of the system.

Remark 3.24. In [23], the proofs make use of *solution operators*, defined as follows:

- Let $\beta = \sigma + \mathfrak{L}W$ in Equation (3.92). The following result can be gathered from several articles, and is summarized in [21]:

Proposition 3.25. Suppose β, $\tau \in L^{2p}$ and $g \in W_2^p$, where $p > 3$. Then there exists a positive solution $\phi \in W_2^{p,+}$ of Equation (3.92) iff one of the following is true:

1. $\gamma \in \mathcal{Y}^+(\Sigma)$, $\beta \not\equiv 0$
2. $\gamma \in \mathcal{Y}^0(\Sigma)$, $\beta \not\equiv 0$, $\tau \not\equiv 0$,
3. $\gamma \in \mathcal{Y}^-(\Sigma)$ and $\exists\ \hat{\gamma} \in [\gamma]$ s.t. $R^{(\hat{\gamma})} = -\frac{2}{3}\tau^2$,
4. $\gamma \in \mathcal{Y}^0(\Sigma)$, $\beta \equiv 0$, $\tau \equiv 0$.

In Cases 1-3 the solution is unique. In Case 4 any two solutions are related by scaling by a constant multiple.

We say that g and τ are *Lichnerowicz compatible* if they satisfy one of the conditions of Cases 1-3 and we say that β is *admissible* if it further satisfies the same condition.

If g and τ are Lichnerowicz compatible, we define the *Lichnerowicz operator* $\mathcal{L}_\tau$ to be the map taking β to the unique solution of Equation (3.92).

- Similarly, for a scalar field $\tau \in W_1^p$ with $p > n$, define $\mathcal{W}_\tau : L^\infty \to W_2^p$ by $\mathcal{W}_\tau(\phi) = W$ where W is a solution of $\Delta_{\text{conf}} W = \frac{2}{3}\phi^6 d\tau$.

Let us now assume that $g \in W_p^2$ and $\tau \in W_p^1$ (with $p > 3$) are Lichnerowicz compatible and that $\sigma \in W_p^1$ is admissible (i.e.: $\sigma \not\equiv 0$ if $\gamma \in \mathcal{Y}^+(\Sigma)$). This is exactly the hypothesis that g, τ and σ satisfy one of Cases 1-3 of Proposition 3.25.

Define $\mathcal{N}_{\sigma,\tau} : L_+^\infty \to W_p^{2,+}$ by $\mathcal{N}_{\sigma,\tau} = \mathcal{L}_\tau \left(\sigma + \mathfrak{L}\mathcal{W}_\tau(\phi)\right)$. $\mathcal{N}_{\sigma,\tau}$ is well-defined provided that g has no conformal Killing fields, so that the domain of $\mathcal{W}_\tau$ is all of L^∞. It can be verified that $\sigma + \mathfrak{L}\mathcal{W}_\tau(\phi)$ belongs to the domain of $\mathcal{L}_\tau$ for any choice of $\phi \in L_+^\infty$. Finding solutions to the system (3.92)-(3.93) then amounts to finding fixed points of $\mathcal{N}_{\sigma,\tau}$. $\triangle$

3.5 Toy-models in the far-from-CMC case

3.5.1 Conformally flat torus & the CTS method.

The remaining limitations of the construction presented in [15] are now as follows:

- The near-CMC hypothesis is replaced by a smallness assumption on the TT-tensor (i.e.: a small-TT hypothesis)

- The construction only works on Yamabe-positive compact manifolds.

- It is not known if small-TT conformal data determine a unique solution.

In order to address the first two issues, [22] makes use of a variation of the conformal method, which we now briefly present: the *conformal thin sandwich (CTS) method*, first introduced in [30]. Applying York splitting gave (Equation (2.21), at the end of Section 2.4):

$$K = \phi^{-10}(\sigma + \mathfrak{L}_{\text{conf}} W) + \frac{1}{3}\phi^{-4}\gamma\tau \tag{3.94}$$

However, the formulation now depends on the choice of representative γ for the class of conformally related metrics. Indeed, if $\gamma' = \theta^4\gamma$, then in order to have that $\tilde{K}' = \theta^{-10}\tilde{K}$, we must impose $\sigma' = \theta^{-10}\sigma$. But then $(\mathfrak{L}_{(\gamma'),\mathrm{conf}}W) = \theta^{-4}(\mathfrak{L}_{(\gamma),\mathrm{conf}}W)$.

A remedy is to look for a particular solution to Equation (2.10) of the form:

$$\tilde{\mathfrak{L}}_{(\gamma),\mathrm{conf}}W := N^{-1}\mathfrak{L}_{(\gamma),\mathrm{conf}}W \tag{3.95}$$

where N is a given scalar function, which is such that $N = \theta^{-6}N'$. Then defining

$$\tilde{K} := \sigma + (\tilde{\mathfrak{L}}_{(\gamma),\mathrm{conf}}W) \tag{3.96}$$

yields the desired decomposition, since

$$(\tilde{\mathfrak{L}}_{(\gamma'),\mathrm{conf}}W) = \theta^{-10}\ (\tilde{\mathfrak{L}}_{(\gamma),\mathrm{conf}}W) \tag{3.97}$$

Letting $(\tilde{\Delta}_{(\gamma),\mathrm{conf}}W)^b := D_a(\tilde{\mathfrak{L}}_{(\gamma),\mathrm{conf}}W)^{ab}$, we obtain the CTS momentum constraint:

$$(\tilde{\Delta}_{\gamma,\mathrm{conf}}W)^b = \frac{2}{3}\varphi^6\gamma^{ab}\tau_a \tag{3.98}$$

Remark 3.26. [22] From the perspective of working with a fixed background metric g, the standard conformal method simply corresponds to the CTS method, with the choice $N = 1/2$. We can think of the CTS approach as providing many different parameterizations, one for each choice of N. It is not known if certain choices of N are superior for the purposes of finding a parameterization. △

Using different (although equivalent) decompositions for the conformal method, York splitting, and the CTS method, [22] derives the CTS equations for the conformally flat torus $\Sigma = S^1_{r_1} \times \ldots \times S^1_{r_n}$, where $S^1_{r_i}$ denotes the circle of radius r_i. Equipped with the product metric γ chosen s.t. when restricted to a single circle, $\gamma_{ij} = \delta_{ij}$, Σ is a Yamabe-null manifold. It is required that γ and K be periodic functions of x_n only, the coordinate of the n^{th}-circle, and that $\pounds_{\partial_k} g = \pounds_{\partial_k} K = 0$ for $1 \leq k \leq n-1$. Letting $x = x_n$, $r_n = 1$, and restricting to the specific case

$n = 3$ the resulting equations read

$$-6\phi'' - \eta^2\phi^{-7} - \left(\mu + \frac{w'}{2N}\right)^2 \phi^{-7} + \tau^2\phi^5 = 0 \tag{3.99}$$

$$\left(\frac{w'}{2N}\right)' - \phi^6\tau' = 0 \tag{3.100}$$

where $' := \frac{d}{dx}$. The constants η and μ constitute the constant part of the TT-tensor, and the unknown function w is related to the vector field W of the conformal method via $2W = w\partial_n$. If we let

$$\tau = \tau_t = t + \lambda(x) := t + \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases} \tag{3.101}$$

we have that the prescribed data set $\{\eta, \mu, t\}$ for the CTS equations allows for violation of both the near-CMC and small-TT conditions on the manifold. Decreasing the value of t makes the discontinuity in τ more significant (hence large t corresponds to the near-CMC regime), and increasing η and μ make the TT-tensor large.

This simple model has the advantages that it comprises important issues: in particular, the nonlinear coupling between the equations is present, and ∂_x is a nontrivial conformal Killing vector field (see Appendix D.3). However, the convenient choice of curvature (ie: piecewise constant) gets rid of that last issue, since it introduces a Dirac distribution in Equation (3.100), which can then easily be solved to yield: $\frac{w'}{2N} = \phi(0)^q[\lambda + \gamma_N]$, where γ_N is defined as

$$\gamma_N = -\frac{\int_{S^1} \lambda N}{\int_{S^1} N} \tag{3.102}$$

The resulting existence/uniqueness statements for Equation (3.99) are summarized in a series of theorems, which we now state, and partly illustrate in Table 3. It is understood that (ϕ, w) is a solution of Equations (3.99) & (3.100) if $(\phi, w + c)$ is a solution, for any constant c.

Theorem 3.27. *(Near-CMC Results)* If $|t - \gamma_N| > 2$, $\exists$ a solution (ϕ, w) of Equations (3.99) & (3.100) iff $\eta \neq 0$ or $\mu \neq 0$. Solutions are unique if $\mu = 0$.

Theorem 3.28. *(Exceptional case: $t = \gamma_N$)* If $t = \gamma_N$ and $\mu = \eta = 0$, then $\exists$ a one-parameter family of solutions to Equations (3.99) & (3.100). If $\mu = 0$ and $\eta \neq 0$, $\nexists$ a solution.

Theorem 3.29. *(Small-TT results)* Suppose $|t| > |\gamma_N|$ and $|t| \neq 1$. If $\mu \neq 0$ or $\eta \neq 0$, and if μ and η are sufficiently small, then $\exists$ at least one solution of Equations (3.99) & (3.100).

Theorem 3.30. *(Non-vanishing mean curvature)* Suppose $|t| > 1$ and either $\mu \neq 0$ or $\eta \neq 0$. Then $\exists$ at least one solution of Equations (3.99) & (3.100).

Theorem 3.31. *(Non-existence/Non-uniqueness)* Suppose $|t| < 1$ and $\mu = 0$. There exists a critical value $\eta_0 \geq 0$ s.t. if $|\eta| < \eta_0$, $\exists$ at least two solutions of Equations (3.99) & (3.100), and if $|\eta| \geq \eta_0$, there are no solutions. If in addition $|t| > |\gamma_N|$, then $\eta_0 > 0$.

Table 3: Existence & Uniqueness results on $\Sigma = n$-torus, in the CTS formulation.

	$\eta = 0$, $\mu = 0$	$\eta \neq 0$, $\mu = 0$	$\eta = 0$, $\mu \neq 0$
Near-CMC: $\lvert t - \gamma_N\rvert > 2$	$\nexists$	$\exists\,!$	$\exists$
Exceptional case: $t = \gamma_N$	$\exists$		$\nexists$
Small-TT results: $\lvert t\rvert > \lvert\gamma_N\rvert$ & $\lvert t\rvert \neq 1$ with μ & η suff. small		$\exists$	$\exists$
$\tau \neq 0$: $\lvert t\rvert > 1$		$\exists$	$\exists$
τ changing sign: $\lvert t\rvert < 1$ & $\lvert t\rvert > \lvert\gamma_N\rvert$		$\exists$ $\not!$ if $\lvert\eta\rvert < \eta_0$ $\nexists$ if $\eta_0 < \lvert\eta\rvert$	

Theorem 3.31 is the first nontrivial non-uniqueness result for the vacuum conformal method. It arises from the nonlinear coupling of the equations, and suggests that the conformal and CTS methods might contain poorly behaved terms.

3.5.2 $S^2 \times S^1$

Based on the treatment of the toy-model just discussed, we consider now the case where $\Sigma = S^2 \times S^1$ is equipped with the product of the round metrics. This is a Yamabe-positive manifold, with $R^{(\gamma)} = 2$. The following is original work, which closely follows [22].

We choose to parametrize $S^2 \times S^1$ using the coordinates: (ψ, θ, φ), where

$$\psi \in [-\pi, \pi] \qquad \theta \in [0, \pi] \qquad \varphi \in [-\pi, \pi] \tag{3.103}$$

and where φ corresponds to the coordinate on S^1. Imposing the restrictions $\pounds_{\partial_\psi}\gamma = \pounds_{\partial_\theta}\gamma = 0$ implies to work with $\psi = \frac{\pi}{2}$. Using the same kind of decompositions as in [22] yields the CTS equations:

$$\begin{aligned} -12\phi'' + \phi^5\tau^2 + 3\phi - \left[3\eta^2 + \left(\mu + \frac{w'}{2N}\right)^2\right]\phi^{-7} &= 0 \qquad (3.104)\\ (\frac{w'}{2N})' - \phi^6\tau' &= 0 \qquad (3.105) \end{aligned}$$

where $' := \frac{d}{d\varphi}$. With $\tau = \tau_t$ given by Equation (3.101), we also get $\frac{w'}{2N} = \phi^6(0)[\lambda + \gamma_N]$ with γ_N defined as before.

The C.T.S. Lichnerowicz equation becomes

$$-12\phi'' + \tau_t^2\phi^5 + 3\phi - \left[3\eta^2 + \left(\mu + \phi^6(0)[\lambda + \gamma_N]\right)^2\right]\phi^{-7} = 0 \tag{3.106}$$

Following [22], let us introduce a family of Lichnerowicz equations depending on $d \in \mathbb{R}^+$:

$$-12\phi_d'' + \tau_t^2\phi_d^5 + 3\phi_d - \left[3\eta^2 + \left(\mu + d^6[\lambda + \gamma_N]\right)^2\right]\phi_d^{-7} = 0 \tag{3.107}$$

Defining $u = u_d := d^{-1}\phi_d$ gives:

$$-12u_d'' + \tau_t^2 d^4 u_d^5 + 3u_d - \left[3\eta^2 d^{-8} + \left(\mu d^{-6} + [\lambda + \gamma_N]\right)^2 d^4\right]u_d^{-7} = 0 \tag{3.108}$$

Claim 3.32. The solutions to Equations (3.104) & (3.105) with mean curvature given by Equation (3.101) are in one-to-one correspondence with the functions u_d satisfying

$$\begin{aligned} -12u_d'' + \tau_t^2 d^4 u_d^5 + 3u_d - \left[3\eta^2 d^{-8} + \left(\mu d^{-6} + [\lambda + \gamma_N]\right)^2 d^4\right] u_d^{-7} &= 0 \\ u_d(0) &= 1 \end{aligned} \tag{3.109}$$

for some $d > 0$.

Consider

$$-12u'' + \alpha_1 u^5 + 3u - \alpha_2 u^{-7} = 0 \tag{3.110}$$

$$\text{where} \qquad \alpha_1 = \tau_t^2 d^4 \quad \& \quad \alpha_2 = \left[3\eta^2 d^{-8} + \left(\mu d^{-6} + [\lambda + \gamma_N]\right)^2 d^4\right] \tag{3.111}$$

Proposition 3.33. Suppose $p > 1$, α_1, $\alpha_2 \in L^\infty([-\pi, \pi])$ and $\alpha_1 \not\equiv 0$, $\alpha_2 \not\equiv 0$

1. There exists a unique solution $u \in W_2^{p,+}([-\pi, \pi])$.

2. If $w \in W_2^{\infty,+}$ is a subsolution (resp. supersolution), i.e.

$$-12w'' + \alpha_1 w^5 + 3w - \alpha_2 w^{-7} \leq 0 \qquad (\text{resp. } \geq 0) \tag{3.112}$$

then $w \leq u$ (resp. $w \geq u$).

3. The solution $u \in W_2^{p,+}$ depends continuously on $(\alpha_1, \alpha_2) \in (L^\infty, L^\infty)$

Proof. **1.** The statement directly follows from results presented in [11]:

Theorem: The Lichnerowicz equation

$$\Delta_{(\gamma)}\varphi = r\varphi - a\varphi^{-\frac{3n-2}{n-2}} - q_1\varphi^{-\frac{n}{n-2}} + (b - q_2)\varphi^{\frac{n+2}{n-2}} \tag{3.113}$$

on a compact n-manifold (M, e) with given Riemannian metric $\gamma \in W_2^p$ properly Riemannian, $p > \frac{n}{2}$, and a, b, $q_1 \in L^\infty$ admits a solution $\varphi > 0$, $\varphi \in W_2^p$ when $\gamma \in \mathcal{Y}^+$ if $a + q_1 \not\equiv 0$.

2. Follows from the sub and super solution theorem (See Appendix C.3.3).

3. Consider the map $\mathcal{N} : W_2^{p,+} \times (L^\infty \times L^\infty) \to L^p$ taking

$$(u, \alpha_1, \alpha_2) \longmapsto -12u_d'' + \alpha_1 \ u_d^5 + \ 3u_d - \alpha_2 \ u_d^{-7} \tag{3.114}$$

This map is continuous, since $\Delta_{(\gamma)}$ is a continuous operator, and W_2^p is an algebra. The Fréchet derivative at (u, α_1, α_2) with respect to u in the direction h is

$$\mathcal{N}'[u, \alpha_1, \alpha_2]h = -4h'' + \left[5 \cdot \alpha_1 \ u_d^4 + 3 + 7 \cdot \alpha_2 \ u_d^{-8}\right] h \tag{3.115}$$

Continuity of the map $(u, \alpha_1, \alpha_2) \to \mathcal{N}'[u, \alpha_1, \alpha_2]$ follows from:

- $W_2^p([-\pi, \pi])$ is an algebra continuously embedded in $C^0([-\pi, \pi])$ (See Propositions C.13 & C.15)
- the following lemma, proved in [22]:

Lemma For constant $0 < m < M$ and $p > 1$, define the slab $S^p_{m,M} = \{u \in W_p^2(S^1) : m \leq u \leq M\}$. For $u \in W_2^{p,+}(S^1)$, let $F_r(u) = u^r$. There exists a constant $K(m, M, r)$ s.t. $||F_r(u) - F_r(v)||_{L^p(S^1)} \leq K(m, M, r)||u - v||_{L^p(S^1)}$ $\forall$ $u, v \in S^p_{m,M}$. Let $L_{u,r} : W_2^p \to L^p$ be the linear function $L_{u,r}v = F_r(u)v$. The map $u \mapsto L_{u,r}$ is Lipschitz continuous on $S^p_{m,M}$.

We can then make use of the following theorem, presented in [11]:

Theorem The Poisson operator $\Delta_{(\gamma)} - a$ on scalar functions in a metric γ on a smooth compact Riemannian manifold (M, e), with $\gamma \in W_2^p$ properly Riemannian, $p > \frac{n}{2}$, $a \in L^p$, is an isomorphism from W_2^p onto L^p if $a \geq 0$, $a \not\equiv 0$.

$V = 5 \cdot \alpha_1 \ u^4 + 3 + 7 \cdot \alpha_2 \ u^{-8} \not\equiv 0$ since $\alpha_1 \not\equiv 0$, $\alpha_2 \not\equiv 0$ and $u \not\equiv 0$. Also, $V \geq 0$. Therefore, $-\Delta_{(\gamma)} + V : W_2^p \longrightarrow L^p$ is an isomorphism.
The Implicit Function Theorem implies that if u_0 is a solution for data (α_0, β_0), there is a continuous map defined near (α_0, β_0) taking (α, β) to the corresponding solution of Equation (3.110). $\square$

Definition 3.34. Define $\mathcal{F}(d) : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ by $\mathcal{F}(d) = u_d(0)$. When $\mu = \eta = 0$, we denote $\mathcal{F}$ by $\mathcal{F}_0$.

Elementary estimates for $\mathcal{F}$

The following claims are proved by manipulating inequalities and limits, assuming $|t| \neq 1$. The qualitative results and techniques for the proofs are similar to those in [22].

Claim 3.35. *(Constant supersolution)* If $M \geq M_d := \max\{M_{d,+}, M_{d,-}\} > 0$ where

$$M_{d,\pm} = \left[\frac{3\eta^2 d^{-12} + (\mu d^{-6} + [\gamma_N \pm 1])^2}{(t \pm 1)^2}\right]^{\frac{1}{12}} \tag{3.116}$$

then M is a constant supersolution to Equation (3.110).

Claim 3.36. *(Constant subsolution)* If $0 < m \leq m_d := \min\{m_{d,+}, m_{d,-}, 1\}$ where

$$m_{d,\pm} = \left[\frac{3\eta^2 d^{-12} + (\mu d^{-6} + [\gamma_N \pm 1])^2}{(t \pm 1)^2 + \frac{3}{d^4}}\right]^{\frac{1}{8}} \tag{3.117}$$

then m is a constant subsolution to Equation (3.110).

Remark 3.37. We always have $m_d \leq M_d$. $\triangle$

Lemma 3.38. Let

$$M_\infty := \max\left[\left|\frac{\gamma+1}{t+1}\right|^{\frac{1}{6}}, \left|\frac{\gamma-1}{t-1}\right|^{\frac{1}{6}}\right] \ \& \ m_\infty := \max\left[\left|\frac{\gamma+1}{t+1}\right|^{\frac{1}{4}}, \left|\frac{\gamma-1}{t-1}\right|^{\frac{1}{4}}\right] \tag{3.118}$$

Given $\epsilon > 0$, $m_\infty - \epsilon \leq u \leq M_\infty + \epsilon$ holds for d sufficiently large.

If $\mu = \eta = 0$ then $m_\infty - \epsilon \leq u \leq M_\infty$ for all $d > 0$.

Lemma 3.39. If $\mu = \eta = 0$ then $\mathcal{F}_0(d) \leq M_\infty$ for all $d > 0$. Otherwise there is a positive constant c such that $\mathcal{F}(d) \geq cd^{-\frac{2}{3}}$ for d sufficiently small..

We can now state a criterion for existence of a solution:

Lemma 3.40. *(Existence criterion)* Suppose $\eta \neq 0$ and $\mu \neq 0$. There exists a solution of $\mathcal{F}(d) = 1$ if and only if for some $d > 0$, $\mathcal{F}(d) \leq 1$.

Proof. From the preceding lemma, we have that for d sufficiently small, $\mathcal{F}(d) > 1$. Fixing $p > 1$, we have from Proposition 3.33 that the map $d \mapsto u_d$ from $(0, \infty)$ to $W^p_2(S^1)$ is continuous. Since $W^p_2(S^1) \hookrightarrow C(S^1)$, it follows that $\mathcal{F}$ is continuous, so the result is obtained from the Intermediate Value Theorem. $\square$

Near-CMC results

Lemma 3.41, Lemma 3.42 and its proof follow [22].

Lemma 3.41. Suppose $|t-\gamma_N| > 2$. Then $M_\infty < 1$. If $\eta \neq 0$ or $\mu \neq 0$, $\exists$ a solution of $\mathcal{F}(d) = 1$. If $\eta = \mu = 0$, then $\mathcal{F}_0(d) < 1$ for all $d > 0$. In particular, $\nexists$ a solution of $\mathcal{F}_0(d) = 1$.

Lemma 3.42. *(Differentiability of $\mathcal{F}$)* The function $\mathcal{F}$ is differentiable. Moreover $\mathcal{F}'(d) = h(0)$ where $h \in W_2^p([0,\pi])$ solves $-\Delta_g h + V(d,u)h = -R(d,u)$ and where

$$\begin{aligned} V(d,u) &= 5\tau_t^2 d^4 u^4 + 3 + 7\left[3\eta^2 d^{-8} + \left(\mu d^{-6} + [\lambda+\gamma_N]\right)^2 d^4\right] u^{-8} \quad (3.119) \\ R(d,u) &= 4\tau_t^2 d^3 u^5 + \Big[24\eta^2 d^{-9} - 4\left(\mu d^{-6} + [\lambda+\gamma_N]\right)^2 d^3 \\ &\quad +12\left(\mu d^{-6} + [\lambda+\gamma_N]\right)\mu d^{-3}\Big] u^{-7} \quad (3.120) \end{aligned}$$

Proof. Consider the function $\mathcal{M} : \mathbb{R}_{>0} \times W_2^{p,+}([-\pi,\pi]) \to L^p([-\pi,\pi])$ defined by

$$\mathcal{M}(d,v) = -12v'' + \tau_t^2 d^4 v^5 + 3v - \left[3\eta^2 d^{-8} + \left(\mu d^{-6} + [\lambda+\gamma_N]\right)^2 d^4\right] v^{-7} \quad (3.121)$$

so that $\mathcal{M}(d,u) = 0$ for all $d > 0$. $\mathcal{M}$ is Fréchet differentiable:

$$\begin{aligned} & \mathcal{M}'[d,v](\delta,h) \\ = \;& -12h'' + \left(5\tau_t^2 d^4 v^4 + 3 + 7\left[3\eta^2 d^{-8} + \left(\mu d^{-6} + [\lambda+\gamma_N]\right)^2 d^4\right] v^{-8}\right)\ h \\ & + \Big(4\tau_t^2 d^3 v^5 + \Big[24\eta^2 d^{-9} - 4\left(\mu d^{-6} + [\lambda+\gamma_N]\right)^2 d^3 \\ & +12\left(\mu d^{-6} + [\lambda+\gamma_N]\right)\mu d^{-3}\Big] v^{-7}\Big)\ \delta \quad (3.122) \\ =: \;& -12h'' + V(d,v)h + R(d,v)\delta \quad (3.123) \end{aligned}$$

By an argument similar to the one used in the third part of Proposition 3.33 ($W_2^p([-\pi,\pi]) \hookrightarrow C([-\pi,\pi])$ and the lemma) the operators $V(v,d)$ and $R(v,d)$ are continuous. So the map $(d,v) \mapsto \mathcal{M}'[d,v]$ is continuous. The operator from $W_2^p([-\pi,\pi]) \to L^p([-\pi,\pi])$, $h \mapsto -\Delta_{(\gamma)}h + Vh$ has a continuous inverse as $V \in L^\infty$, $V \geq 0$ and $V \not\equiv 0$. The Implicit Function Theorem then implies that given a solution of $\mathcal{M}(d_0, u_0) = 0$ there is a unique function G defined near d_0 such that $\mathcal{M}(d, G(d)) = 0$, and G is continuously differentiable. But $\mathcal{M}(d, u) = 0$ for all d, so by the uniqueness of G we have $G(d) = u$. Let $h = G'(d)$. Then by the chain rule

$$0 = \frac{d}{dd}\,\mathcal{M}(d, G(d)) = -h'' + Vh + R \tag{3.124}$$

Since the map $u_d \mapsto u_d(0)$ is linear and continuous on $W_2^p(S^1)$, and $\mathcal{F}(d) = u_d(0)$, it follows that $\mathcal{F}$ is continuously differentiable and $\mathcal{F}'(d) = G'(d)(0)$. That is, $\mathcal{F}'(d) = h(0)$ where h solves Equation (3.124). □

The following proposition differs from the analogous proposition in [22]: here, we have a restriction on η.

Proposition 3.43. Suppose $|t - \gamma| > 2$, $\mu = 0$, $\eta \geq 0$. Then for η sufficiently large, there exists at most one solution of $\mathcal{F}(d) = 1$.

Proof. Suppose $\mathcal{F}(d) = 1$. We show that for η sufficiently large, $\mathcal{F}'(d) < 0$ and hence there can be at most one solution.
If $\mu = 0$, $\eta \geq 0$, R defined in the above lemma becomes:

$$R = 24\eta^2 d^{-9} u^{-7} + 4d^3\left((t+\lambda)^2 u^5 - (\gamma_N + \lambda)^2\, u^{-7}\right) \tag{3.125}$$

Noticing that R is then an increasing function of u, we have that:

$$R \geq 24\eta^2 d^{-9} m_d^{-7} + 4d^3\left((t+\lambda)^2 m_d^5 - (\gamma_N + \lambda)^2\, m_d^{-7}\right) \tag{3.126}$$

Requiring that $R \geq 0$ will give us a condition on how large η should be. After some arithmetic, we get:

$$\eta^2 \quad \geq \quad \frac{1}{6} d^{12}(t+\lambda)^2 \left(\left(\frac{\gamma_N + \lambda}{t+\lambda}\right)^2 - \left[\frac{3\eta^2 d^{-12} + (\gamma_N + \lambda)^2}{(t+\lambda)^2 + \frac{3}{d^4}}\right]^{\frac{3}{2}}\right) \tag{3.127}$$

This can easily be solved numerically to find out η_0 such that the above inequality holds for any $\eta \geq \eta_0$. Then for such η, we have that on $I_+ = (0, \pi)$ and $I_- =$

$(-\pi, 0)$, $R \geq 0$, $R \not\equiv 0$. Note that over $I_+ \cup I_-$, the coefficients of Equation (3.110) are smooth. Since it was proved that $V \geq 0$, $V \not\equiv 0$, we can apply the strong maximum principle to $\Delta_g h = Vh + R$ to conclude that $h < 0$ on $I_+ \cup I_-$. In particular, $h(0) = \mathcal{F}'(d) < 0$. □

The case $t = \gamma$

Note that the following results are different from the ones in [22]: on the n-torus, we have a one-parameter family of solutions when $\mu = \eta = 0$, and no solution when $\mu = 0$ and $\eta \neq 0$.

- <u>CASE $\mu = \eta = 0$:</u> We have the following non-existence result:

Lemma 3.44. Suppose $t = \gamma$ and $\mu = \eta = 0$. Then System (3.109) doesn't have a solution.

Proof. Rearranged, Equation (3.110) with $\eta = \mu = 0$ reads:

$$-12u_d'' + d^4(t+\lambda)^2 \left(u^5 - u^{-7}\right) + 3u = 0 \tag{3.128}$$

$M = 1 - \epsilon$ is a constant supersolution if

$$d^4(t+\lambda)^2 \left(M^5 - M^{-7}\right) + 3M \geq 0 \tag{3.129}$$

or equivalently

$$\frac{(1-\epsilon)^8}{1-(1-\epsilon)^{12}} \cdot \frac{3}{(t+\lambda)^2} \geq d^4 \tag{3.130}$$

Given d, it is always possible to find $\epsilon > 0$ such that the last inequality is satisfied. Therefore $u < 1$, and in particular, $u(0) = 1$ cannot have a solution. □

- <u>CASE $\mu = 0,\ \eta > 0$:</u> We have the following non-existence result:

Lemma 3.45. Suppose $t = \gamma$ and $\mu = 0$, $\eta > 0$. Then System (3.109) doesn't have a solution.

Proof. Proceeding as in the previous lemma, we show that the condition $1 > \eta^2 d^{-8}$ is sufficient to ensure that $M = (1-\epsilon)$ is a constant supersolution, for $\epsilon > 0$ appropriately chosen. Then we show that even in the case where $1 < \eta^2 d^{-8}$, $M = (1-\epsilon)$ is a constant supersolution.

Equation (3.110) reads:

$$-12u_d'' + d^4 u_d^5 (t+\lambda)^2 + 3u_d - \left[3\eta^2 d^{-8} + (t+\lambda)^2 d^4\right] u_d^{-7} \;=\; 0 \tag{3.131}$$

$M = 1-\epsilon$ is a constant supersolution if

$$\frac{1}{3} d^4 (t+\lambda)^2 \left(M^5 - M^{-7} + \frac{3M}{d^4(t+\lambda)^2} - \frac{3\eta^2 M^{-7}}{d^{12}(t+\lambda)^2} \right) \geq 0 \tag{3.132}$$

or equivalently

$$\frac{3(1-\epsilon)^8}{d^4(t+\lambda)^2} \geq \frac{3\eta^2}{d^{12}(t+\lambda)^2} + \left(1-(1-\epsilon)^{12}\right) \tag{3.133}$$

If

$$\frac{3(1-\epsilon)^8}{d^4(t+\lambda)^2} > \frac{3\eta^2}{d^{12}(t+\lambda)^2} \quad \Longleftrightarrow \quad (1-\epsilon)^8 > \frac{\eta^2}{d^8} \tag{3.134}$$

is satisfied, then it will be possible to find an appropriate $\epsilon_1 > 0$ such that the previous inequality holds. If $1 > \eta^2 d^{-8}$, it is possible to find $\epsilon_2 > 0$ s.t. the above inequality to hold. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$.

We now consider the case $\eta^2 d^{-8} > 1$. $M = (1-\epsilon)$ is a supersolution if

$$d^4 M^5 (t+\lambda)^2 + 3M - 3\eta^2 d^{-8} M^{-7} - (t+\lambda)^2 d^4 M^{-7} \geq 0 \tag{3.135}$$

or equivalently

$$d^4(t+\lambda)^2 \geq -3 \frac{\overbrace{\eta^2 d^{-8} - M^8}^{>0}}{\underbrace{1-M^{12}}_{>0}} \tag{3.136}$$

which is always true. $\square$

- <u>CASE $\mu \neq 0,\ \eta = 0$:</u> We have the following non-existence result:

Lemma 3.46. Suppose $t = \gamma$ and $\mu \neq 0$, $\eta = 0$. Then System (3.109) doesn't have a solution.

Proof. Proceeding as in the previous lemma, we show that $M = (1-\epsilon)$ is a constant supersolution, for $\epsilon > 0$ appropriately chosen. With $\mu \neq 0$ and $\eta = 0$, Equation (3.110) reads:

$$-12u_d'' + d^4 u_d^5 (t+\lambda)^2 + 3u_d - \left(\mu d^{-6} + (t+\lambda)\right)^2 d^4 u_d^{-7} \;=\; 0 \tag{3.137}$$

$M := 1 - \epsilon$ is a supersolution if:

$$d^4 M^5 (t+\lambda)^2 + 3M - \left(\mu d^{-6} + (t+\lambda)\right)^2 d^4 M^{-7} \geq 0 \quad ? \tag{3.138}$$

Let us derive two stronger conditions, by suppressing the linear term in the above inequality:

Condition 1: $(t+\lambda)^2 \geq -\left[\mu d^{-6} + 2(t+\lambda)\right] \mu d^{-6} \frac{1}{1-M^{12}}$

Condition 2: $(1-\epsilon)^{12} \geq \left[\frac{\mu}{(t+\lambda)d^6} + 1\right]^2$

Clearly, it is possible to find $\epsilon > 0$ satisfying Condition 2 iff $\frac{\mu}{(t+\lambda)d^6} < 0$, ie: $\mu < 0$ and $t > 1$, or $\mu > 0$ and $t < 1$. But in the other two possible cases, namely $\mu < 0$ and $t < 1$, or $\mu > 0$ and $t > 1$, Condition 1 holds trivially, since the RHS is then negative. Therefore, $M = (1-\epsilon)$ is a constant supersolution. □

Mean curvature of constant sign

First, consider the following *perturbed* Lichnerowicz equation:

$$-12\epsilon^2 u_\epsilon'' - \left[3\eta^2 d^{-8} + (\mu d^{-6} + \gamma_N + \lambda)^2 d^4\right] u_\epsilon^{-7} + 3\epsilon^2 u_\epsilon + \tau_t^2 d^4 u_\epsilon^5 = 0 \tag{3.139}$$

$$\iff \qquad -12\epsilon^2 u_\epsilon'' - \alpha_\pm^2 u_\epsilon^{-7} + 3\epsilon^2 u_\epsilon + \beta_\pm^2 u_\epsilon^5 = 0 \tag{3.140}$$

$$\text{where} \quad \alpha_\pm^2 = 3\eta^2 d^{-8} + (\mu d^{-6} + \gamma_N \pm 1)^2 d^4 \quad \& \quad \beta_\pm^2 = (t \pm 1)^2 d^4 \tag{3.141}$$

Definition 3.47. We say that $f(x) \to L$ *rapidly* at infinity if

$$\lim_{x\to\infty} |f(x) - L|x^n = 0 \qquad \forall\, n \in \mathbb{N} \tag{3.142}$$

We say that $f(x) \to L$ *rapidly* at 0 if

$$\lim_{x\to 0} |f(x) - L|x^{-n} = 0 \qquad \forall\, n \in \mathbb{N} \tag{3.143}$$

The statement of the following theorem is identical to the one in [22], with a slightly different proof.

Theorem 3.48. Suppose that $\beta_\pm \neq 0$, then

$$\lim_{\epsilon\to 0} u_\epsilon(0) = \left[\frac{|\alpha_-| + |\alpha_+|}{|\beta_+| + |\beta_-|}\right]^{\frac{1}{6}} \tag{3.144}$$

and this convergence is rapid.

Proof. We shall make use of the following results, proved in [22]:

Proposition: Consider the following boundary value problem on $\mathbb{R}$, with piecewise constant coefficients:

$$-\nu'' - \alpha^2\nu^{-7} + \beta^2\nu^5 = 0 \tag{3.145}$$

where α and β are equal to the constants $\alpha_\pm$ and $\beta_\pm$ on $(-\infty, 0)$ and $(0, +\infty)$. Suppose $\beta_\pm \not\equiv 0$. Let $L_\pm = |\alpha_\pm/\beta_\pm|^{1/6}$. $\exists$ a solution $\nu \in W^\infty_{2,\mathrm{loc}}(\mathbb{R})$ to Equation (3.145) satisfying $\lim_{x\to\pm\infty} \nu(x) = L_\pm$. Moreover, ν converges rapidly to its limits at $\pm\infty$, ν' converges rapidly to 0 at $\pm\infty$, and

$$\nu(0) = \left[\frac{|\alpha_+| + |\alpha_-|}{|\beta_+| + |\beta_-|}\right]^{1/6} \tag{3.146}$$

Following [22], we use this function to build approximate solutions of Equation (3.140). On $[-\frac{\pi}{2}, \frac{\pi}{2}]$, define:

$$w_\epsilon(x) = \nu(x/\epsilon) + h_\epsilon(x) \tag{3.147}$$

where h_ϵ is small correction term defined as follows: Let

$$\zeta(x) = \begin{cases} \frac{1}{\pi}x^2 & 0 \le x \le \pi/2 \\ 0 & -\pi/2 < x \le 0 \end{cases} \tag{3.148}$$

$$h_\epsilon(x) = -d_{\epsilon,+}\zeta(x) - d_{\epsilon,-}\zeta(-x) \quad \text{where} \quad d_{\epsilon,\pm} = \frac{1}{\epsilon}\nu'\left(\pm\frac{\pi}{2\epsilon}\right) \tag{3.149}$$

With this choice of h_ϵ, $w'_\epsilon(\pm\pi/2) = 0$.

Then, define the nonlinear Lichnerowicz operator $\mathscr{N}_\epsilon : W^p_2(S^1) \to L^p(S^1)$ by:

$$\mathscr{N}_\epsilon(w) = -12\epsilon^2 w'' - \alpha_\pm^2 w^{-7} + 3\epsilon^2 w + \beta_\pm^2 w^5 = 0 \tag{3.150}$$

and compute the resulting 'error':

$$\begin{aligned} \mathscr{E}_\epsilon &= \mathscr{N}_\epsilon(w_\epsilon(x)) - \mathscr{N}_\epsilon(\nu_\epsilon(x)) && (3.151) \\ &= \frac{2}{\pi}\cdot 12\epsilon^2\left[d_{\epsilon,+}\chi_+ + d_{\epsilon,-}\chi_-\right] - \alpha_\pm^2\left[(\nu(x/\epsilon) + h_\epsilon(x))^{-7} - \nu(x/\epsilon)^{-7}\right] \\ &\quad +3\epsilon^2 h_\epsilon(x) + \beta_\pm^2\left[(\nu(x/\epsilon) + h_\epsilon(x))^5 - \nu(x/\epsilon)^5\right] && (3.152) \end{aligned}$$

where $\chi_\pm$ are the characteristic functions on $(-\pi/2, 0)$ and $(0, \pi/2)$.

Once again, we make use of a lemma proved in [22]:

Lemma: Consider

$$\begin{aligned} \mathscr{B}_\epsilon &= \frac{2}{\pi}\epsilon^2\left[d_+\chi_+ + d_-\chi_-\right] - \alpha^2\left[(\nu(x/\epsilon) + h_\epsilon(x))^{-7} - \nu(x/\epsilon)^{-7}\right] \\ &\quad +\beta^2\left[(\nu(x/\epsilon) + h_\epsilon(x))^5 - \nu(x/\epsilon)^5\right] && (3.153) \end{aligned}$$

$||\mathscr{B}_\epsilon||_{L^\infty(S^1)} \to 0$ rapidly as $\epsilon \to 0$.

Comparing $\mathscr{E}_\epsilon$ and $\mathscr{B}_\epsilon$, they differ by $3\epsilon^2 h_\epsilon(x)$. Since we easily have that $||3\epsilon^2 h_\epsilon(x)||_{L^\infty(S^1)} \to 0$ rapidly as $\epsilon \to 0$, we conclude that $||\mathscr{E}_\epsilon||_{L^\infty(S^1)} \to 0$ rapidly as $\epsilon \to 0$.

The other arguments needed to show that the limit in (3.144) holds are all taken from [22]. Their proofs carry out without changes, therefore we just state the main steps.

Corollary: As a corollary of the lemma stated in the proof of Proposition 3.33, we have that if $0 < m < M$, there exists a constant $C(m, M)$ s.t. $\forall\ v, w \in S^p_{m,M}$

$$||\mathscr{N}_\epsilon'[v] - \mathscr{N}_\epsilon'[w]||_{L(W^p_2(S^1), L^p(S^1))} < C(m, M)||v - w||_{W^p_2(S^1)} \tag{3.154}$$

Proposition: Let $V \in L^\infty(S^1)$ and consider the operator $\mathscr{L}_\epsilon = -\epsilon^2\Delta + V$ as a map from $W^2_p(S^1)$ to $L^p(S^1)$, where $p > 1$. Suppose there is a constant m s.t. $V \geq m > 0$. Then $\mathscr{L}_\epsilon$ is continuously invertible. Moreover, there is a constant C s.t. if ϵ is sufficiently small, $||\mathscr{L}_\epsilon^{-1}|| \leq C\epsilon^{-4}$.

The proof ends using Newton's method: Let X and Y be Banach spaces, $x \in X$, $r > 0$. Let $\mathscr{N} : B_r(x) \to Y$ be a differentiable map with Lipschitz continuous derivative, ie: there exists $k > 0$ s.t.

$$||\mathscr{N}'[x_1] - \mathscr{N}'[x_2]||_{L(X,Y)} \leq k||x_1 - x_2||_X \qquad \forall\ x_1, x_2 \in B_r(x) \tag{3.155}$$

Suppose x is a point where $\mathscr{N}'[x]$ has a continuous inverse. If

$$2k \cdot ||\mathscr{N}[x]||^2 \cdot ||\mathscr{N}'[x]^{-1}|| < 1 \quad \text{and} \quad 2 \cdot ||\mathscr{N}[x]|| \cdot ||\mathscr{N}'[x]^{-1}|| < r \tag{3.156}$$

then $\exists$ a solution of $\mathscr{N}[u] = 0$ satisfying $||u - x||_X \leq 2 \cdot ||\mathscr{N}[x]|| \cdot ||\mathscr{N}'[x]^{-1}||$. $\square$

We can now get results similar to [22]. A direct application of Theorem 3.48 is the following proposition:

Proposition 3.49. Let $\eta = \mu = 0$ and $|t| \neq 1$, and denote by u_0 the corresponding solution to Equation (3.110). Then

$$\lim_{d\to\infty} u_0(0) = \begin{cases} 1 & |t| < 1 \\ |t|^{-\frac{1}{6}} & |t| > 1 \end{cases} \tag{3.157}$$

and the convergence is rapid.

To obtain an analogous result without $\eta = \mu = 0$, we show that small perturbations of $u_{0,d}$ are sub and super solutions of the equation for u_d. To this end, first define the nonlinear operator $\mathscr{N}_d : W^{p,+}_2(S^1) \to L^p(S^1)$:

$$\mathscr{N}_d(v) = -12v'' - \left[3\eta^2 d^{-8} + (\mu d^{-6} + \gamma_N + \lambda)^2 d^4\right] v^{-7} + 3v + (t+\lambda)^2 d^4 v^5 \tag{3.158}$$

and the perturbation operator $\mathscr{G}_d : [-m_\infty/2, M_\infty] \to L^\infty$ as $\mathscr{G}_d(K) = \mathscr{N}_d(u_0+K)$. So that $u_{0,d} + K$ is a sub or super solution iff $\mathscr{G}(K) \leq 0$ or ≥ 0. We may write $\mathscr{G}_d(K) = \mathscr{D}(K) + \mathscr{E}(K)$ where:

$$\mathscr{D}(K) = (t+\lambda)^2 d^4 \left[(u_0+K)^5 - u_0^5\right] \tag{3.159}$$

$$\begin{aligned}\mathscr{E}(K) &= (\gamma_N+\lambda)^2 d^4 u_0^{-7} + K \\ &\quad - \left[3\eta^2 d^{-8} + \left[\mu d^{-6} + \lambda + \gamma_N\right]^2 d^4\right](u_0+K)^{-7}\end{aligned} \tag{3.160}$$

We have the following lemma [22]:

Lemma 3.50. There exist positive constants D_-, D_+, E_- and E_+ s.t.

$$E_-K \leq (\gamma_N+\lambda)\left[u_0^{-7} - (u_0+K)^{-7}\right] \leq E_+K \qquad K \geq 0 \tag{3.161}$$

$$E_+K \leq (\gamma_N+\lambda)\left[u_0^{-7} - (u_0+K)^{-7}\right] \leq E_-K \qquad K \leq 0 \tag{3.162}$$

and

$$D_-K \leq \mathscr{D}(K) \leq D_+K \qquad K \geq 0 \tag{3.163}$$

$$D_+K \leq \mathscr{D}(K) \leq D_-K \qquad K \leq 0 \tag{3.164}$$

$\forall\ d > 1$ and $\forall\ K \in [-m_\infty/2, M_\infty]$.

which we use to prove the following proposition:

Proposition 3.51. There exists a constant $c > 0$ s.t. $||u_0 - u||_{L^\infty} < cd^{-2}$ $\forall\ d$ sufficiently large. In particular,

$$\lim_{d\to\infty} \mathcal{F}(d) = \lim_{d\to\infty} u(0) = \lim_{d\to\infty} u_0(0) \tag{3.165}$$

Proof. The idea is to find constants $K_-(d)$ and $K_+(d)$ that are $O(d^{-2})$, and satisfying $\mathscr{G}(K_-(d)) < 0$ and $\mathscr{G}(K_+(d)) > 0$. It will then follow that $u_0 + K_-(d)$ and $u_0 + K_+(d)$ are sub and super solutions of Equation (3.110), implying the asymptotic result.

First let $0 < K \leq M_\infty$:

$$\begin{aligned}
\mathscr{E}(K) &= (\gamma_N + \lambda)^2 d^4 u_{0,d}^{-7} + 3K \\
&\quad - \left[3\eta^2 d^{-8} + \left[\mu d^{-6} + \lambda + \gamma_N\right]^2 d^4\right](u_{0,d} + K)^{-7} \qquad (3.166)\\
&= (\gamma_N + \lambda)^2 d^4 \left[u_{0,d}^{-7} - (u_{0,d} + K)^{-7}\right] + 3K \\
&\quad - \left[3\eta^2 d^{-8} + \mu^2 d^{-8} + 2\mu d^{-6}(\lambda + \gamma_N) d^4\right](u_{0,d} + K)^{-7} \qquad (3.167)\\
&\geq E_- K - \left[(3\eta^2 + \mu^2) d^{-6} + 2\mu(\lambda + \gamma_N)\right](m_\infty/2)^{-7} d^{-2} \qquad (3.168)
\end{aligned}$$

Picking

$$K_+(d) = \frac{\left[(3\eta^2 + \mu^2) d^{-6} + 2\mu(\lambda + \gamma_N)\right](m_\infty/2)^{-7}}{E_-} d^{-2} \qquad (3.169)$$

ensures $\mathscr{G}_d(K_+(d)) \geq 0$, for d sufficiently large.

Then let $-m_\infty/2 \leq K < 0$:

$$\mathscr{E}(K) \leq E_+ K - \left(3\eta^2 + \mu^2\right) d^{-8}(2M_\infty)^{-7} + 2\mu(\lambda + \gamma_N) d^{-2}(m_\infty/2)^{-7} + 3K \qquad (3.170)$$

Picking

$$K_-(d) = -\frac{\frac{2}{3}\mu(\lambda + \gamma_N)(m_\infty/2)^{-7}}{E_+} d^{-2} \qquad (3.171)$$

ensures $\mathscr{G}_d(K_-(d)) \leq 0$, for d sufficiently large. □

This proposition now easily follows from the previous results:

Proposition 3.52. Suppose $|t| > 1$. If $\eta \neq 0$ or $\mu \neq 0$, there exists at least one solution of $\mathcal{F}(d) = 1$.

Non-existence / Non-uniqueness

As in [22], we obtain non-existence and non-uniqueness results, using the same techniques. We now let $\mu = 0$, and $|t| < 1$, so that the mean curvature changes sign. Since we are concerned with the behaviour of $\mathcal{F}$ in terms of η, we let $\mathcal{F} = \mathcal{F}_{[\eta]}$.

Proposition 3.53. For fixed d, the value of $\mathscr{F}_{[\eta]}(d)$ is strictly increasing in η. Moreover,

$$\mathscr{F}_{[\eta]}(d) \geq \left[\frac{3\eta^2 + (\gamma_N \pm 1)^2 d^8}{(t\pm 1)^2 d^4 + 3}\right]^{\frac{1}{8}} d^{-1} \tag{3.172}$$

Proof. Fix $d > 0$ and suppose $0 \leq \eta_1 \leq \eta_2$. Let $u_{d,1}$ and $u_{d,2}$ be the corresponding solutions of Equation (3.110). Substituting $u_{d,1}$ into the equation for $u_{d,2}$, we get:

$$\begin{aligned} & -12u''_{d,1} + \tau_t^2 d^4 u_{d,1}^5 + 3u_{d,1} - \left[3\eta_2^2 d^{-8} + (\lambda+\gamma_N)^2 d^4\right] u_{d,1}^{-7} \\ = \;& 3\left(\eta_1^2 - \eta_2^2\right) d^{-8} u_{d,1}^{-7} < 0 \end{aligned} \tag{3.173}$$

So $u_{d,1}$ is a subsolution of the equation for $u_{d,2}$, and $u_{d,1} \leq u_{d,2}$. Since a similar computation shows that $u_{d,1} + \epsilon$ is also a subsolution, $u_{d,1} < u_{d,2}$ everywhere, $\mathcal{F}_{[\eta_1]}(d) < \mathcal{F}_{[\eta_2]}(d)$.

For the estimate (3.172), recall the result $0 < m \leq m_d := \min\{m_{d,+}, m_{d,-}, 1\}$ is a constant subsolution of Equation (3.110), where:

$$m_{d,\pm} = \left[\frac{3\eta^2 d^{-12} + (\mu d^{-6} + [\gamma_N \pm 1])^2}{(t\pm 1)^2 + \frac{3}{d^4}}\right]^{\frac{1}{8}} \tag{3.174}$$

Setting $\mu = 0$ and rearranging gives the desired estimate:

$$m_{d,\pm} = \left[\frac{3\eta^2 + (\gamma_N \pm 1)^2 d^8}{(t\pm 1)^2 d^4 + 3}\right]^{\frac{1}{8}} d^{-1} \tag{3.175}$$

□

Proposition 3.54. Suppose $\mu = 0$ and $\eta \neq 0$. Then there exists a constant $c > 0$ such that

$$u_d \geq u_{0,d} + cd^{-8} \tag{3.176}$$

for all d sufficiently large.

Proof. We make use of the perturbation operator $\mathscr{G}_d$ and Lemma 3.50, presented in the previous section. With $\mu = 0$, we have:

$$\begin{aligned} \mathscr{E}(K) &= \frac{1}{3}(\gamma_N+\lambda)^2 d^4\left[u_{0,d}^{-7} - (u_{0,d}+K)^{-7}\right] - \eta^2 d^{-8}(u_{0,d}+K)^{-7} + K \\ &\leq E_+ K - \eta^2 (2M_\infty)^{-7} d^{-8} + K \end{aligned} \tag{3.177}$$

Let

$$K_- = \frac{\eta^2(2M_\infty)^{-7}}{D_+ + E_+ + 1} d^{-8} \tag{3.178}$$

Then we have:

$$\begin{aligned} \mathscr{G}_d(K_-) &= \mathscr{D}(K_-) + \mathscr{E}(K_-) && (3.179) \\ &\leq D_+K_- + E_+K_- - \eta^2(2M_\infty)^{-7}d^{-8} + K_- && (3.180) \\ &= (D_+ + E_+ + 1)\, K_- - \eta^2(2M_\infty)^{-7}d^{-8} = 0 && (3.181) \end{aligned}$$

Therefore, $u_{0,d} + K_-$ is a subsolution, and inequality (3.176) is thus obtained with $c = \frac{\eta^2(2M_\infty)^{-7}}{D_+ + E_+ + 1}$. □

We conclude with a proposition on non-existence/non-uniqueness of solutions:

Proposition 3.55. Suppose $\mu = 0$ and $|t| < 1$. $\exists\ \eta_0 \geq 0$ s.t. if $0 < |\eta| < \eta_0$, $\exists$ at least two solutions of $\mathcal{F}(d) = 1$, while if $|\eta| > \eta_0$ there are no solutions. If $|t| > \gamma_N$ then $\eta_0 > 0$.

Proof. We make use of Theorem 3.48 and Proposition 3.54 to show that $u_d(0) > 1$ for d sufficiently large.

$$u_d(0) - 1 \geq (u_{0,d}(0) - 1) + cd^{-8} \to 0 \quad \text{as } d \to \infty \tag{3.182}$$

Fix η_1, and pick d_0 s.t. $\mathcal{F}_{[\eta_1]}(d) > 1$ if $d > d_0$. From Proposition 3.53, we can find η_2 s.t. $\mathcal{F}_{[\eta_2]}(d) > 1\ \forall\ d \in (0, d_0]$. Letting $\eta = \max\{\eta_1, \eta_2\}$, we must have $\mathcal{F}_{[\eta]}(d) > 1\ \forall\ d > 1$, hence for η sufficiently large, there are no solutions to System 3.109.

Let $A = \{\eta \geq 0 : \mathcal{F}_{[\eta]}(d) > 1 \forall\ d > 0\}$. We have just shown that $A \neq \varnothing$. Let $\eta_0 = \inf A$, and pick $\eta < \eta_0$. Then for some d_0, $\mathcal{F}_{[\eta]}(d_0) < 1$. Since $\mathcal{F}_{[\eta]}(d) > 1$ for d sufficiently small, and d sufficiently large, it follows from the continuity of $\mathcal{F}$ that there are at least two solutions of $\mathcal{F}_{[\eta]}(d) = 1$, one for $d < d_0$ and one for $d_0 < d$. □

Summary of the results

As was done with the previous toy-model, the non-/existence and non-/uniqueness results for $\Sigma = S^2 \times S^1$ are summarized in Table 4.

Table 4: Existence & Uniqueness results on $\Sigma = S^2 \times S^1$, in the CTS formulation.

	$\eta = 0$, $\mu = 0$	$\eta \neq 0$, $\mu = 0$	$\eta = 0$, $\mu \neq 0$
Near-CMC: $\lvert t - \gamma_N \rvert > 2$	$\nexists$	$\exists$ $!$ if $\eta_0 < \lvert\eta\rvert$	$\exists$
Exceptional case: $t = \gamma_N$	$\nexists$	$\nexists$	$\nexists$
$\tau \neq 0$: $\lvert t\rvert > 1$		$\exists$	$\exists$
τ changing sign: $\lvert t\rvert < 1$ & $\lvert t\rvert > \lvert\gamma_N\rvert$		$\exists$ $\not!$if $\lvert\eta\rvert < \eta_0$ $\nexists$ if $\eta_0 < \lvert\eta\rvert$	

Comparison with Rendall's result

It is interesting to compare those results to the following theorem proved by A.Rendall, and presented in [18]:

Theorem 3.56. Let $(\Sigma, \gamma, \sigma, \tau)$ be a set of conformal data with $\Sigma = S^2 \times S^1$, $\gamma =$ (round sphere metric) $\times$ (circle metric), $\sigma^2 \equiv 0$ and $\tau = f(x)$, where x is the coordinate on the S^1 factor, and $f(-x) = -f(x)$. For such data, the conformal Einstein constraint equations either admit no solution, or admit more than one solution.

Proof. Define a pair of groups:

- $\mathcal{S} = SO(3)$ acts on the S^2-component of Σ, leaving S^1 invariant. Note that $(\gamma, \sigma + \gamma\tau) \mapsto (\gamma, \sigma + \gamma\tau)$ under the action of $\mathcal{S}$.
- $\mathcal{Z}^2$ is the reflection group, where $\Psi \in \mathcal{Z}^2$ reflects S^1 across some central

point $p_0 \in S^1$, leaving S^2 invariant. $(\gamma, \sigma+\gamma\tau) \mapsto (\gamma, -(\sigma+\gamma\tau))$ under the action of Ψ.

It follows that if $\exists!$ solution (ϕ, W) to the conformal Einstein constraint equations for these conformal data, then the reconstituted data are s.t. $(h,K) \mapsto (h,K)$ under the $\mathcal{S}$ action, and $(h,K) \mapsto (h,-K)$ under the Ψ action. Let us work by contradiction, and assume $\exists!$ solution to the conformal ECE, with resulting initial data set (h,K). Consider (h,K) and its local spacetime development g, we define

- $\mathcal{R}(x,t) =$ radius of S^2 at (x,t), where $x \in S^1$, and $x = 0$ at p_0;
- $m(x,t) = \frac{1}{2}\mathcal{R}(x,t)\,(1 - g(\nabla\mathcal{R}(x,t), \nabla\mathcal{R}(x,t)))$.

From the vacuum Einstein equations (Equation (1.1)), m and $\mathcal{R}$ must satisfy

$$\nabla_\alpha \nabla_\beta \mathcal{R} = \frac{m}{\mathcal{R}^2}\, g_{\alpha\beta} \tag{3.183}$$

$$\nabla_\alpha m = 0 \implies m = \hat{m}, \text{ a constant} \tag{3.184}$$

where the indices α, β take on the two values x and t.

Claim: At the point $(x,t) = (0,0)$: (a) $\nabla\mathcal{R}(0,0) = 0$ (b) $\mathcal{R}(0,0) = 2\hat{m}$.

Proof of Claim: (a) follows from the facts that

- Since $K \mapsto -K$ under Ψ, we have $K_{cd}(0,0) = 0$ and therefore $\partial_t\mathcal{R} = 0$;
- Since $\gamma \mapsto \gamma$ under Ψ, $\partial_x\mathcal{R}(0,0) = 0$.

(b) follows immediately from (a), along with the definition of m:

$$\hat{m} = m(0,0) = \frac{1}{2}\mathcal{R}(0,0)(1 - g(\nabla\mathcal{R}(0,0), \nabla\mathcal{R}(0,0))) = \frac{1}{2}\mathcal{R}(0,0) \tag{3.185}$$

q.e.d. proof of claim.

We next consider a global maximum point $x_m \in S^1$ of the function $\mathcal{R}(x,0)$. Since the data are presumed to be smooth, we have $\partial_x\mathcal{R}(x_m,0) = 0$. Thus we find that

$$g(\nabla\mathcal{R}(x_m,0), \nabla\mathcal{R}(x_m,0)) = g^{tt}(\partial_t\mathcal{R}(x_m,0), \partial_t\mathcal{R}(x_m,0))^2 \leq 0 \tag{3.186}$$

So we have, from the definition of m,

$$\hat{m} = m(x_m, 0) = \frac{1}{2}\mathcal{R}(x_m, 0)(1 - g(\nabla\mathcal{R}(x_m, 0), \nabla\mathcal{R}(x_m, 0))) \geq \frac{1}{2}\mathcal{R}(x_m, 0) \quad (3.187)$$

Since x_m is a global maximum for $\mathcal{R}(x, 0)$, it follows from this result that $\forall\, x \in S^1$,

$$\mathcal{R}(x, 0) \leq \mathcal{R}(x_m, 0) \leq 2\hat{m} \quad (3.188)$$

Now comparing this inequality with $\mathcal{R}(0, 0) = 2\hat{m}$, we verify that $x = 0$ is a global maximum for $\mathcal{R}(x, 0)$. However, using that $K_{cd}(0, 0) = 0$, together with Equation (3.183), we get

$$\partial_x\partial_x\mathcal{R}(0, 0) = \nabla_x\nabla_x\mathcal{R}(0, 0) = \frac{\hat{m}}{\mathcal{R}^2}g_{xx}(0, 0) > 0 \quad (3.189)$$

This contradicts the assumption that $(0, 0)$ is a global maximum for $\mathcal{R}$, completing the proof by contradiction. □

The proof does not tell which of existence or uniqueness fails in this specific case. Comparing the CTS formulation used to derive the results with the conditions imposed by Rendall, we see that we must consider the case where:

- $t = 0$ since we want τ to be an odd function,

- $\gamma_N = 0$ since, as pointed out by Remark 3.26, the CTS formulation reduces to the conformal formulation, provided that $N = 1/2$. Then $\gamma_N = \frac{\int_\Sigma \lambda N}{\int_\Sigma N} = \int_\Sigma \lambda = 0$.

Therefore, this corresponds to the exceptional case $t = \gamma_N$, for which we have non-existence results. What Rendall's result tells us is that there cannot be a unique solution exhibiting symmetry. We have found that there is no solution with symmetry, however, there could still exist a solution which does not exhibit symmetry. In order to rule out this possibility, other kinds of arguments must be used.

Chapter 4

Conclusion

The goal of the thesis was three-fold. We first introduced the initial-value formulation of general relativity, and the conformal method, which to this day has been the most successful approach to the problem. We then provided the reader with a clear review of the main achievements regarding the parametrization of the space of solutions in the CMC and near-CMC cases. Lastly, we explained the most recent results in the far-from-CMC case, and pointed out the remaining open questions.

It is hoped that toy-models such as the ones presented at the end of Chapter 3 help to gain a better understanding of the issues that are faced when dealing with the far-from-CMC case. Carrying on with this idea, there are several possible directions for further investigations of the new $S^2 \times S^1$ model presented above. One can try to understand the nature of the multiplicity of solutions when the mean curvature changes sign, as in [29]. It would also be interesting to find out whether this is caused by 'poorly behaved terms', such as the ones present in the extended CTS method (see [5]). The role played by the presence of Killing vector fields also has to be clarified.

Appendix A

Calculus on Manifolds

Unless otherwise stated, in this appendix, we will assume that M and N are manifolds endowed with *smooth* (C^∞-)differentiable structures, of respective dimensions m and n. For practical purposes, we assume that a point on M belongs to some chart (U, φ), and has local coordinates x^μ, and that a point on N belongs to some chart (V, ψ), and has local coordinates y^ν.

A.1 Vectors, Dual vectors, and Tensors

Definition A.1. [25] Let $f : M \to N; f : p \mapsto f(p)$ be a map. Then f and $f(p)$ have the following representations:

$$\psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n \qquad y = \psi \circ f \circ \varphi^{-1}(x) \tag{A.1}$$

abbreviated as $y^\nu = f^\nu(x^\mu)$. f is *differentiable* or *smooth* at p if is it C^∞ with respect to each x^μ. $f \in C^\infty(M)$ if it is smooth $\forall\, p \in M$.

A.1.1 Vectors

Let $c : (a,b) \subset \mathbb{R} \to M$ be a curve on M such that $t = 0 \in (a,b)$, and $f : M \to \mathbb{R}$, $f \in C^\infty(M)$. Then [25]:

$$\left.\frac{\mathrm{d}f(c(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}\right|_{t=0} \partial_\mu f =: X^\mu \partial_\mu f =: X[f] \tag{A.2}$$

where we used $\partial_\mu f := \partial_\mu(f \circ \varphi^{-1}(x))$, and defined $X^\mu = \frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}|_{t=0}$. The last equality defines the action of the differential operator X on f.

Consider the following equivalence class of curves:

$$[c(t)] = \left\{ \tilde{c}(t) : \tilde{c}(0), \left.\frac{\mathrm{d}x^\mu(\tilde{c}(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}\right|_{t=0} \right\} \tag{A.3}$$

Definition A.2. [25] Equation (A.2) defines X, the *tangent vector* to M at $p = c(0)$, along the direction given by the curves in $[c(t)]$.

Remark A.3. Based on this definition, a tangent vector appears as a generalization of the directional derivative in Euclidian space. The following alternate definition puts the emphasis on viewing a tangent vector as a differential operator. △

Definition A.4. [6] Consider the set of C^∞-functions whose domain includes $p \in M$, and define an equivalence relation on this set as follows: $f \sim g$ if $\exists$ and open neighborhood U containing p s.t. $f = g$ on U. Then the *germ* of f at p is the equivalence class $[f]$ of f under $\sim$. The set of germs at p is denoted by $C^\infty(M, \{p\}, \mathbb{R})$.

Definition A.5. A tangent vector at $p \in M$ is a map $X : C^\infty(M, \{p\}, \mathbb{R}) \to \mathbb{R}$ s.t. for any chart (U, φ) about p, $\exists\ \{a^\mu\} \in \mathbb{R}^m$ s.t. $X([f]) = a^\mu \partial_\mu([f \circ \varphi^{-1}])$.

Definition A.6. The set of all tangent vectors at $p \in M$ forms the *tangent space* T_pM. The tangent vectors $e^\mu = \partial_\mu$ form a basis for T_pM known as the *canonical basis*. Therefore, T_pM has dimension m, like M.

A.1.2 Dual vectors

Definition A.7. [25] The dual vector space to T_pM is denoted by T_p^*M and is called the *cotangent space* at p. The element $\omega : T_pM \to \mathbb{R}$ of T_p^*M is a *dual vector* or a *one-form*.

Definition A.8. Consider the natural inner product $\langle\ ,\ \rangle : T_p^*M \times T_pM \to \mathbb{R}$. In local coordinates, the *dual basis* of T_p^*M associated to the canonical basis of T_pM is denoted by $\{\mathrm{d}x^\mu\}$ and defined as:

$$\langle \mathrm{d}x^\nu, \partial_\mu \rangle := \partial_\mu x^\nu = \delta^\nu_\mu \tag{A.4}$$

In local coordinates, a dual vector can be written as $\omega = \omega_\nu \mathrm{d}x^\nu$. From the last definition, the inner product between a vector and a dual vector can be inferred [25]:

$$\langle \omega, V \rangle = \langle \omega_\nu \mathrm{d}x^\nu, V^\mu \partial_\mu \rangle = \omega_\nu V^\mu \langle \mathrm{d}x^\nu, \partial_\mu \rangle = \omega_\nu V^\mu \delta^\nu_\mu = \omega_\mu V^\mu \tag{A.5}$$

A.1.3 Tensors

Definition A.9. [25] A (q, r)-*tensor* is a map $T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} : \otimes^q T_p^*M \otimes^r T_pM \to \mathbb{R}$.

Remark A.10. By definition, dual vectors are $(0, 1)$-tensors. $\triangle$

In local coordinates, a (q, r)-tensor can be written as [25]

$$T = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_q} \otimes \mathrm{d}x^{\nu_1} \otimes \dots \otimes \mathrm{d}x^{\nu_r} \tag{A.6}$$

and it acts on q dual vectors $\omega_{i\mu} \mathrm{d}x^\mu$, $(1 \leq i \leq q)$ and r vectors $V_j = V_j^\nu \partial_\nu$, $(1 \leq j \leq r)$ as:

$$T(\omega_1, \dots, \omega_q; V_1, \dots V_r) = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \omega_{1\mu_1} \dots \omega_{q\mu_q} V_1^{\nu_1} \dots V_r^{\nu_r} \tag{A.7}$$

A.2 Pushforward & Pullback

A smooth map $f : M \to N$ naturally induces two maps. [25]

Definition A.11. The *differential map* or *pushforward* $f_* : T_pM \to T_{f(p)}N$ is induced by f, in the sense that if $V \in T_pM$ and $g \in C^\infty(N)$, then we define

$$(f_*V)[g] := V[g \circ f] \quad \Longleftrightarrow \quad (f_*V)[g \circ \psi^{-1}(y)] := V[g \circ f \circ \varphi^{-1}(x)] \tag{A.8}$$

where $x = \varphi(p)$ and $y = \psi(f(p))$. Let $V = V^\mu \frac{\partial}{\partial x^\mu}$ and $f_*V = W^\nu \frac{\partial}{\partial y^\nu}$, then

$$W^\nu \frac{\partial}{\partial y^\nu}[g \circ \psi^{-1}(y)] = V^\mu \frac{\partial}{\partial x^\mu}[g \circ f \circ \varphi^{-1}(x)] \tag{A.9}$$

or taking $g = y^\nu$: $W^\nu = V^\mu \frac{\partial}{\partial x^\mu} y^\nu(x) =: V^\mu J$ since $J = \frac{\partial y^\nu(x)}{\partial x^\mu}$ is the Jacobian of f.

Definition A.12. The *pullback* $f^* : T^*_{f(p)}N \to T^*_pM$ is induced by f, in the sense that if $V \in T_pM$ and $\omega \in T^*_{f(p)}N$, the pullback of ω by f^* is defined by:

$$\langle f^*\omega, V\rangle = \langle \omega, f_*N\rangle \tag{A.10}$$

Remark A.13. • The pullback map f^* naturally extends to $(0, r)$-tensors.

• The differential map f_* naturally extends to $(q, 0)$-tensors.

• There is no natural extension of those maps to tensors of mixed type (ie: (q, r)-tensors, $q \geq 1$, $r \geq 1$), unless f is a diffeomorphism. $\triangle$

A.3 Submanifolds

Definition A.14. [25] Let $f : M \to N$, $f \in C^\infty(M)$, and $\dim M \leq \dim N$.

• f is an *immersion* of M into N if $f_* : T_pM \to T_{f(p)}N$ is an injection, ie: $\mathrm{rank} f_* = \dim M$.

• f is an *embedding* if f is an *injection* and an immersion. $f(M)$ is then called a *submanifold* of N.

Definition A.15. [24] A *hypersurface* is a submanifold of codimension 1.

A.4 Flows & Lie derivatives

A.4.1 Vector & Tensor fields

Definition A.16. [25] A *vector field* is a smooth assignment of vectors to each point of M. In other words, X is a vector field if $X(f) \in C^\infty(M)$ for any $f \in C^\infty(M)$. Define $\chi(M) := \{X : X \text{ is a vector field on } M\}$.

Remark A.17. X_x denotes the vector field evaluated at $x \in M$. $\triangle$

Definition A.18. [25] Similarly, a (q,r)-*tensor field* is a smooth assignment of (q,r)-tensors to each point of M. Define $\mathscr{T}^{q,r}(M) := \{T : T \text{ is a } (q,r)-\text{ tensor field on } M\}$.

A.4.2 Flows

Let $X = X^\mu \partial_\mu$ be a vector field in M.

Definition A.19. [25] An *integral curve* $x(t)$ of X is a curve on M, whose tangent vector at $x(t)$ is X_x. In local coordinates:

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = X^\mu(x(t)) \tag{A.11}$$

In other words, solving Equation A.11 gives the integral curves of X: existence and uniqueness results for such ODEs guarantee that $\exists!$ local solution, provided that $x_0^\mu = x^\mu(0)$. Moreover, for compact M, the integral curves are known to exist for all $t \in \mathbb{R}$. [25]

Let $\sigma(t, x_0)$ be an integral curve of X passing through the point x_0 at $t = 0$. Then it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^\mu(t, x_0) = X^\mu(\sigma^\mu(t, x_0)) \qquad \sigma^\mu(0, x_0) = x_0^\mu \tag{A.12}$$

Definition A.20. The map $\sigma : \mathbb{R} \times M \to M$ is called a *flow* generated by X.

A.4.3 Lie derivatives

Let $\sigma(t,x)$ and $\tau(t,x)$ be two flows generated by the vector fields X and Y:

$$\frac{\mathrm{d}\sigma^\mu(s,x)}{\mathrm{d}s} = X^\mu(\sigma(s,x)) \qquad \frac{\mathrm{d}\tau^\mu(t,x)}{\mathrm{d}t} = Y^\mu(\tau(t,x)) \tag{A.13}$$

Suppose we wish to evaluate the change of the vector field Y along $\sigma(s,x)$. This implies that we must compare the vector Y at a point x and at a nearby point $x' = \sigma_\varepsilon(x)$. For the difference of those two vectors to be well-defined, they must belong to the same tangent space, so that $Y|_{\sigma_\varepsilon(x)}$ first has to pushforwarded by the map $(\sigma_{-\varepsilon})_*$.

Definition A.21. The *Lie derivative* of a vector field Y along the flow σ of X is defined by

$$\mathfrak{L}_X Y = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[(\sigma_{-\varepsilon})_* Y|_{\sigma_\varepsilon(x)} - Y|_x] \tag{A.14}$$

Definition A.22. Let $f \in C^\infty(M)$, then the *Lie bracket* $[X,Y]$ is defined s.t. $[X,Y]f = X[Y[f]] - Y[X[f]]$.

Claim A.23.

$$\mathfrak{L}_X Y = (X^\mu\partial_\mu Y^\nu - Y^\mu\partial_\mu X^\nu)e_\nu = [X,Y] \tag{A.15}$$

Provide proof?

Remark A.24. Geometrically, the Lie bracket shows the non-commutativity of two flows. Indeed, let ε and δ be small parameters, then it can be shown

$$\tau^\mu(\delta,\sigma(\varepsilon,x)) - \sigma^\mu(\varepsilon,\tau(\delta,x)) = \varepsilon\delta[X,Y]^\mu \tag{A.16}$$

△

Definition A.25. Similarly, we may define the Lie derivative of a dual vector ω along X as:

$$\mathfrak{L}_X\omega = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[(\sigma_\varepsilon)^*\omega|_{\sigma_\varepsilon(x)} - \omega_x\right] \tag{A.17}$$

which yields

$$\mathfrak{L}_X \omega = \left(X^\nu \partial_\nu \omega_\mu + \partial_\mu \left(X^\nu \omega_\mu \right) \right) \mathrm{d}x^\mu \tag{A.18}$$

The Lie derivative of $f \in C^\infty(M)$ along the flow generated by X is the usual directional derivative of f along X: $\mathfrak{L}_X f = X[f]$.

Proposition A.26. The Lie derivative satisfies

$$\mathfrak{L}_X (t_1 + t_2) = \mathfrak{L}_X t_1 + \mathfrak{L}_X t_2 \tag{A.19}$$

where t_1 and t_2 are tensor fields of the same type and

$$\mathfrak{L}_X (t_1 \otimes t_2) = (\mathfrak{L}_X t_1) \otimes t_2 + t_1 \otimes (\mathfrak{L}_X t_2) \tag{A.20}$$

where t_1 and t_2 are tensor fields of arbitrary types.

From Definitions A.21 & A.25, and Proposition A.26, one can now get the Lie derivative for an arbitrary tensor:

$$\begin{aligned} & \mathfrak{L}_X T^{\mu_1 \mu_2 \ldots \mu_k}{}_{\nu_1 \nu_2 \ldots \nu_l} \\ = \; & X^\sigma D_\sigma T^{\mu_1 \mu_2 \ldots \mu_k}{}_{\nu_1 \nu_2 \ldots \nu_l} \\ & - (D_\lambda V^{\mu_1}) T^{\lambda \mu_2 \ldots \mu_k}{}_{\nu_1 \nu_2 \ldots \nu_l} - (D_\lambda V^{\mu_2}) T^{\mu_1 \lambda \ldots \mu_k}{}_{\nu_1 \nu_2 \ldots \nu_l} - \cdots \\ & + (D_{\nu_1} V^\lambda) T^{\mu_1 \mu_2 \ldots \mu_k}{}_{\lambda \nu_2 \ldots \nu_l} + (D_{\nu_2} V^\lambda) T^{\mu_1 \mu_2 \ldots \mu_k}{}_{\nu_1 \lambda \ldots \nu_l} + \cdots \end{aligned} \tag{A.21}$$

A.5 The metric

Definition A.27. [9] A *pseudo-Riemannian metric* g on M is a symmetric $(0,2)$-tensor field on M s.t. the quadratic form it defines on vectors is non-degenerate.

In local coordinates, $g_p = g_{\mu\nu}(p) \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ at $p \in M$, which is usually abbreviated as $g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$. Therefore, it is common to regard $(g_{\mu\nu})$ as a matrix. If it has maximal rank, its *inverse* is defined, and satisfies $g_{\mu\nu} g^{\nu\lambda} = g^{\lambda\nu} g_{\nu\mu} = \delta^\lambda_\mu$. We denote $g \equiv \det(g_{\mu\nu})$. The non-degeneracy condition in the above definition then says that for any chart, $g(X,X) = g_{\mu\nu}X^\mu X^\nu$ is s.t. g does not vanish. [25]

Since $g_{\mu\nu}$ is a symmetric matrix, its eigenvalues are real.

Definition A.28. If there are i positive and j negative eigenvalues, the pair (i,j) is called the *index* or the *signature* of the metric.

The definition of a pseudo-Riemannian metric is very general. In the Cauchy problem, here are two kinds of metric we are interested in:

Definition A.29. g is a *Riemannian metric* if the quadratic form it defines is positive-definite, ie: $g(U,U) \geq 0$, with equality holding iff $U = 0$. It follows that it has index $(i,0)$.

Definition A.30. g is a *Lorentzian metric* if it has index $(i,1)$.

Let $M \subset N$ be a submanifold, (N,g) be a pseudo-Riemannian manifold, and g have constant index over N. If $f : M \to N$ is the embedding which induces the submanifold structure of M, then $h = f^*g$ is the *induced metric* on M, with components given by [25]:

$$h_{ab} = g_{\alpha\beta}(f(x))\frac{\partial f^\alpha(x)}{\partial x^a}\frac{\partial f^\beta(x)}{\partial x^b} \tag{A.22}$$

A.6 Covariant differentiation

Heuristically, we would like to have an analog of the directional derivative for tensors. But in order to be able to compare a tensor from one point on the manifold to another, we must transport it on the manifold. This requires the introduction of an extra-structure [13]:

Definition A.31. An *affine connection* ∇ is a map $\nabla : \chi(M) \times \chi(M) \to \chi(M)$, $\nabla : (X,Y) \mapsto \nabla_X Y$ which satisfies:

- $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$
- $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$
- $\nabla_X(fY) = X[f]Y + f\nabla_X Y$

for $X, Y, Z \in \chi(M)$, and $f, g \in C^\infty(M)$.

The connection allows us to obtain a new kind of differentiation:

Proposition A.32. [13] Let M be a differentiable manifold with an affine connection ∇. Let V and W be vector fields along the differentiable curve $c : I \to M$, and $f \in C^1(I)$. Then $\exists!$ correspondence which associates to V another vector field $\frac{DV}{dt}$ along c, called the *covariant derivative* of V along c such that:

- $\frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$
- $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$
- If V is induced by a vector field Y, ie: $V(t) = Y(c(t))$, then $\frac{DV}{dt} = \nabla_{dc/dt}Y$.

A.7 The Levi-Civita connection

The Levi-Civita connection is the only connection that satisfies some very specific properties, which we now describe. In the main body of the thesis, ∇ is always assumed to be a Levi-Civita connection.

Definition A.33. Let ∇ be an affine connection on M. A vector field V along a curve $c : I \to M$ is called *parallel* when $\frac{DV}{dt} = 0$, for all $t \in I$.

Definition A.34. Let (M, g) be a Riemannian manifold with an affine connection ∇. A connection is said to be *compatible* with g when for any smooth curve c and any pair of parallel vector fields X and Y along c, we have $g(X, Y) = \mathrm{C}^{\mathrm{st}}$.

Proposition A.35 and Corollary A.36 provide alternate definitions of compatibility.

Proposition A.35. Let (M, g) be a Riemannian manifold. A connection ∇ on M is compatible with a metric iff for any vector fields V and W along the differentiable curve $c : I \to M$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g(V, W) = g(\frac{DV}{\mathrm{d}t}, W) + g(V, \frac{DW}{\mathrm{d}t}) \qquad t \in I \tag{A.23}$$

Corollary A.36. A connection ∇ on a Riemannian manifold M is compatible with the metric iff $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad \forall\ X, Y, Z \in \chi(M)$

Definition A.37. An affine connection ∇ is said to be *symmetric* when

$$\nabla_X Y - \nabla_Y X = [X, Y] \qquad \forall\ X, Y \in \chi(M) \tag{A.24}$$

Theorem A.38. Given a Riemannian manifold M, $\exists!$ affine connection ∇ on M satisfying the conditions:

- ∇ is symmetric,
- ∇ is compatible with the Riemannian metric.

∇ is then referred to as the *Levi-Civita* or the *Riemannian connection*.

The following provide pratical ways of working with connections, and in particular with the Levi-Civita connection:

Definition A.39. [25] Define m^3 functions ${\Gamma^\lambda}_{\nu\mu}$ by $\nabla_{e_\nu} e_\mu = e_\lambda {\Gamma^\lambda}_{\nu\mu}$ called the *connection coefficients* or the *Christoffel symbols of the connection*. They specify how the basis vectors change from point to point.

Let $X = X^\mu e_\mu$ and $Y = Y^\nu e_\nu$, then

$$\begin{aligned} \nabla_X Y &= X^\mu \nabla_{e_\mu}(Y^\nu e_\nu) = X^\mu Y^\nu \nabla_{e_\mu} e_\nu + X^\mu e_\mu(Y^\nu) e_\nu && \text{(A.25)} \\ &= \left(X^\mu Y^\nu {\Gamma^\lambda}_{\mu\nu} + X^\mu e_\mu(Y^\lambda) \right) e_\lambda && \text{(A.26)} \end{aligned}$$

Claim A.40. If ∇ is symmetric, then ${\Gamma^k}_{ij} = {\Gamma^k}_{ji}$.

The Christoffel symbols of the Levi-Civita connection can be computed in local coordinates as:

$$\Gamma^m{}_{ij} = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\right) g^{km} \tag{A.27}$$

Given a vector field X, we may view ∇_X as an operator acting on vector fields. We can also define its action on C^∞-functions by remembering that since it has the meaning of a directional derivative, the most natural definition is $\nabla_X f := X[f]$. Then ∇_X satisfies the Leibnitz rule, since

$$\nabla_X(fY) = X[f]Y + f\nabla_X Y = (\nabla_X f)Y + f\nabla_X Y \tag{A.28}$$

We may require that ∇_X follows the Leibnitz rule when it acts on tensors of arbitrary type, ie: $\nabla_X(T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2)$.

Let ω be a dual vector field on M and Y a vector field on M. Using that $\langle \omega, Y\rangle \in C^\infty(M)$, we can compute $\nabla_X \omega$:

$$X\left[\langle\, \omega, Y\rangle\right] = \nabla_X\left(\langle\, \omega, Y\rangle\right) = \langle \nabla_X \omega, Y\rangle + \langle \omega, \nabla_X Y\rangle \tag{A.29}$$

$$\Longleftrightarrow \qquad (\nabla_X \omega)_\nu = X^\mu \partial_\mu \omega_\nu - X^\mu \Gamma^\lambda{}_{\mu\nu}\omega_\lambda \tag{A.30}$$

We can now generalize these results:

$$\begin{aligned} \nabla_{e_\nu} T^{\lambda_1\ldots\lambda_p}{}_{\mu_1\ldots\mu_q} = {} & \partial_\nu T^{\lambda_1\ldots\lambda_p}{}_{\mu_1\ldots\mu_q} + \Gamma^{\lambda_1}{}_{\nu\kappa} T^{\kappa\lambda_2\ldots\lambda_p}{}_{\mu_1\ldots\mu_q} + \ldots \\ + \Gamma^{\lambda_p}{}_{\nu\kappa} T^{\lambda_1\ldots\lambda_{p-1}\kappa}{}_{\mu_1\ldots\mu_q} & - \Gamma^\kappa{}_{\nu\mu_1} T^{\lambda_1\ldots\lambda_p}{}_{\kappa\mu_2\ldots\mu_q} - \ldots - \Gamma^\kappa{}_{\nu\mu_q} T^{\lambda_1\ldots\lambda_p}{}_{\mu_1\ldots\mu_{q-1}\kappa} \end{aligned} \tag{A.31}$$

A.8 Curvature

In this appendix and the next, we describe some very important objects that live on manifolds, and characterize them geometrically. We follow the rather old-fashioned treatment of [13], which hopefully gives a more complete mathematical picture than most modern presentations.

Definition A.41. The curvature R of a Riemannian manifold (M, g) is a correspondence that associates to every pair $X, Y \in \chi(M)$ a mapping $R(X, Y) : \chi(M) \to \chi(M)$ given by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \qquad Z \in \chi(M) \tag{A.32}$$

where ∇ is the Levi-Civita connection on M.

One way to think of R is as a measure of how much M deviates from being Euclidian. Since in local coordinates $R(e_i, e_j)e_k = (\nabla_{e_j} \nabla_{e_i} - \nabla_{e_i} \nabla_{e_j})e_k$, we may also think of R as measuring the non-commutativity of the covariant derivative.

Proposition A.42. We have the following identities, the first of which is known as *Bianchi identity*:

$$\begin{aligned}
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \iff && R_{ijks} + R_{jkis} + R_{kijs} = 0\\
&g(R(X,Y)Z,T) = -g(R(Y,X)Z,T) && R_{ijks} = -R_{jiks}\\
&g(R(X,Y)Z,T) = -g(R(X,Y)T,Z) && R_{ijks} = -R_{ijsk}\\
&g(R(X,Y)Z,T) = g(R(Z,T)X,Y) && R_{ijks} = R_{ksij}
\end{aligned} \tag{A.33}$$

Definition A.43. Consider $R : \chi(M) \times \chi(M) \times \chi(M) \times \chi(M) \to D(M)$, where $D(M) := \{f : f \in C^\infty(M)\}$, defined by $R : (X, Y, Z, W) \mapsto g(R(X, Y)Z, W)$. This is the *curvature* or *Riemann tensor*.

Tracing over the curvature tensor still provides some useful information on the geometry of the manifold:

Given a vector space V, define, for $x, y \in V$:

$$|x \wedge y| := \sqrt{|x|^2|y|^2 - \langle x, y\rangle^2} \tag{A.34}$$

Definition A.44. Let $p \in M$ and $\sigma \subset T_pM$ be a two-dimensional subspace of the tangent space T_pM. Let $X, Y \in \sigma$ be two linearly independent vectors. Then the real number

$$K(\sigma) = K(X, Y) = \frac{R(X, Y, X, Y)}{|X \wedge Y|^2} \tag{A.35}$$

(which does not depend on the choice of the vectors $X, Y \in \sigma$) is called the *sectional curvature* of σ at p.

Definition A.45. Let $X = Z_n$ be a unit vector in T_pM; we take an orthonormal basis $\{Z_1, Z_2, \ldots, Z_{n-1}\}$ of the hyperplane in T_pM orthogonal to X and consider:

$$\mathrm{Ric}_p(X) = \frac{1}{n-1}\sum_{i=1}^{n-1} R(X, Z_i, X, Z_i) \tag{A.36}$$

$$R(p) = \frac{1}{n}\sum_{j=1}^{n} \mathrm{Ric}_p(Z_j) = \frac{1}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=1}^{n} R(Z_i, Z_j, Z_i, Z_j) \tag{A.37}$$

Those expressions define the *Ricci curvature* in the direction x and the *scalar curvature* or the *Ricci scalar* at p, respectively.

Claim A.46. These expressions do not depend on the choice of the corresponding orthonormal basis.

Proof. The proof makes use of a bilinear form defined on T_pM as follows: Let $X, Y \in T_pM$, and put $Q(X, Y) =$ (trace of the mapping $Z \mapsto R(X, Z)Y$). By definition, Q is symmetric, and satisfies

$$Q(X, X) = (n-1)\mathrm{Ric}_p(X) \tag{A.38}$$

$$\sum_{j=1}^{n} Q(Z_j, Z_j) = (n-1)\sum_{j=1}^{n} \mathrm{Ric}_p(Z_j) = n(n-1)R(p) \tag{A.39}$$

which shows that $\mathrm{Ric}_p(X)$ and $K(p)$ are defined intrinsically. □

Definition A.47. The bilinear form $\frac{1}{n-1}Q$ is the *Ricci tensor*, denoted as Ric.

In local coordinates: $\mathrm{Ric}_{ik} = R^j{}_{ijk} = g^{sj}R_{sijk}$ and $R = \frac{1}{n(n-1)}g^{ik}\mathrm{Ric}_{ik}$.

A.9 The extrinsic curvature

Let $f : M \to \tilde{M}$ be an immersion, with (M, g) a Riemannian manifold, $\dim M = m$ and $\dim\tilde{M} = m + n = k$. For each $p \in M$, the inner product on $T_p\tilde{M}$ splits $T_p\tilde{M}$

into the direct sum $T_p\tilde{M} = T_pM \oplus (T_pM)^\perp$ where $(T_pM)^\perp$ is the orthogonal complement of T_pM in $T_p\tilde{M}$. If $v \in T_p\tilde{M}$, $p \in M$, we can write

$$v = v^T + v^N \quad \text{with} \quad v^T \in T_pM, \quad v^N \in (T_pM)^\perp \tag{A.40}$$

ie: we can decompose v into *tangential* and *normal* components. Let us denote by $\tilde{\nabla}$ and ∇ the Riemannian connections on $\tilde{M}$ and M, respectively. If X and Y are local vector fields on M, and $\tilde{X}$ and $\tilde{Y}$ are local extensions on $\tilde{M}$, define

$$\nabla_X Y := (\tilde{\nabla}_{\tilde{X}}\tilde{Y})^T \qquad B(X,Y) := \tilde{\nabla}_{\tilde{X}}\tilde{Y} - \nabla_X Y \tag{A.41}$$

Claim A.48. $B(X,Y)$ is a local vector field on $\bar{M}$ normal to M, and does not depend on the extensions $\tilde{X}$, $\tilde{Y}$. Moreover, B is bilinear and symmetric.

Proof. Let $\tilde{X}_1$ and $\tilde{Y}_1$ be other extensions of X and Y, then

$$(\tilde{\nabla}_{\tilde{X}}\tilde{Y} - \nabla_X Y) - (\tilde{\nabla}_{\tilde{X}_1}\tilde{Y} - \nabla_X Y) = \tilde{\nabla}_{\tilde{X}-\tilde{X}_1}\tilde{Y} \quad = \quad 0 \tag{A.42}$$

$$(\tilde{\nabla}_{\tilde{X}}\tilde{Y} - \nabla_X Y) - (\tilde{\nabla}_{\tilde{X}}\tilde{Y}_1 - \nabla_X Y) = \tilde{\nabla}_{\tilde{X}}(\tilde{Y} - \tilde{Y}_1) \quad = \quad 0 \tag{A.43}$$

since $\tilde{X} - \tilde{X}_1 = 0$ and $\tilde{Y} - \tilde{Y}_1 = 0$ on M. Bilinearity and symmetry easily follow from the definitions. □

Definition A.49. Let $p \in M$ and $\eta \in (T_pM)^\perp$, and define the symmetric bilinear form: $H_\eta : T_pM \times T_pM \to \mathbb{R}$ by $H_\eta(X,Y) = g(B(X,Y),\eta)$, $X, Y \in T_pM$.

The quadratic form defined on $T_pM \times T_pM$ by $H_\eta(X,Y)$ is called the *extrinsic curvature* of f at p along the normal vector η.

Remark A.50. Note that heuristically, $B(X,Y)$ has no components in the embedded manifold. The extrinsic curvature effects the projection of $B(X,Y)$ onto some vector, normal to the embedded manifold. This motivates the representation of the extrinsic curvature in local coordinates used in the next appendix. △

The bilinear mapping H_η is associated to a linear self-adjoint operator $S_\eta : T_pM \to T_pM$ by

$$g(S_\eta(X), Y) = H_\eta(X, Y) = g(B(X, Y), \eta) \tag{A.44}$$

Note that in the special case where $f : M^n \to \tilde{M}^{n+1}$, ie: $f(M) \subset \tilde{M}$ is a hypersurface, we may choose as our basis of T_pM the eigenbasis $\{e_1, \ldots, e_n\}$, where $S_\eta(e_i) = \lambda_i e_i$. We then define the *mean curvature* of f by $\frac{1}{n}(\lambda_1 + \ldots + \lambda_n)$, which is the trace of S_η.

A.10 The Gauss & Codazzi equations

Given an isometric immersion, let us denote the space of differentiable vector fields normal to M by $\chi(M)^\perp$. The extrinsic curvature of the immersion can then be considered as a tensor $B : \chi(M) \times \chi(M) \times \chi(M)^\perp \to \mathbb{R}$ defined by $B(X, Y, \eta) = g(B(X, Y), \eta)$. This allows us to define the covariant derivative for this tensor:

$$(\tilde{\nabla}_X B)(Y, Z, \eta) = X(B(Y, Z, \eta)) - B(\nabla_Z Y, Z, \eta) - B(Y, \nabla_X Z, \eta) - B(Y, Z, \nabla^\perp_X \eta) \tag{A.45}$$

Proposition A.51. The following equations hold:

$$\tilde{R}(X, Y, Z, T) = R(X, Y, Z, T) - g(B(Y, T), B(X, Z)) + g(B(X, T), B(Y, Z)) \tag{A.46}$$

$$\tilde{R}(X, Y, Z, \eta) = (\tilde{\nabla}_Y B)(X, Z, \eta) - (\tilde{\nabla}_X B)(Y, Z, \eta) \tag{A.47}$$

and are known are the *Gauss* and *Codazzi equations*, respectively.

Appendix B

Hypersurfaces in Lorentzian spacetime

This appendix makes use of the material presented in Appendix A, in the context of general relativity. Its goal is to introduce the definitions and notations used when addressing the Einstein constraint equations. We start with a remark:

Remark B.1. Although all the definitions in Appendices A.6 - A.10 were given on Riemannian manifolds, equivalent definitions hold when the metric is pseudo-Riemannian, and in particular, Lorentzian. $\triangle$

Definition B.2. If (M, g) is Lorentzian, the elements of T_pM are divided into three classes:

- if $g(X, X) > 0$, X is *spacelike*
- if $g(X, X) = 0$, X is *lightlike* or *null*
- if $g(X, X) < 0$, X is *timelike*

Definition B.3. A vector is called *causal* if it is timelike or null. A curve is causal if its tangent vector is everywhere causal. The terminology is due to the fact that the causal curves are precisely those along which causal influences can propagate.

B.1 Hypersurfaces

A hypersurface Σ can be represented by giving parametric equations of the form $x^\alpha = x^\alpha(y^a)$, where y^a $(a = 1, 2, 3)$ are coordinates intrinsic to the hypersurface. The vector $\partial_\alpha \Phi$, which is normal to the hypersurface, can be used to define a *unit normal* n_α such that:

$$n^\alpha n_\alpha \equiv \epsilon = \begin{cases} -1 & \text{if } \Sigma \text{ is spacelike} \\ +1 & \text{if } \Sigma \text{ is timelike} \end{cases} \tag{B.1}$$

Remark B.4. Since the initial-value problem of general relativity is mostly concerned with spacelike hypersurfaces, we will not treat the special case of null hypersurfaces, which needs to be handled with a lot of care. For a reference, see [24]. △

Definition B.5. [26] A spacelike hypersurface is called a *Cauchy surface* if each (inextendible) causal curve hits it precisely once.

Remark B.6. Not every spacetime possesses a Cauchy surface. Those which do are called *globally hyperbolic*. However, if the strong cosmic censorship hypothesis is true, then only globally hyperbolic spacetimes are physically significant; other spacetimes are then just mathematical artefacts. △

B.2 The induced metric

Let (M, g) be a four-dimensional spacetime manifold, and $\Sigma \subset M$ a hypersurface. h, the metric on Σ, is obtained by pulling back g (cf. Appendix A.5), and is given in local coordinates by $h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b$, where the vectors $e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$ are tangent to curves contained in Σ. The completeness relation for the inverse metric is given by:

$$g^{\alpha\beta} = \epsilon n^\alpha n^\beta + h^{ab} e^\alpha_a e^\beta_b \tag{B.2}$$

where h^{ab} is the inverse of the induced metric.

B.3 The extrinsic curvature

We note that an arbitrary tensor T defined on M can always be projected down to the hypersurface, so that only its tangential components survive. The quantity that effects the projection is $h^{\alpha\beta} \equiv h^{ab} e^\alpha_a e^\beta_b = g^{\alpha\beta} - \epsilon n^\alpha n^\beta$.

Let us now examine how tangent tensor fields are differentiated. For simplicity, consider the case of a tangent vector field:

$$A^\alpha = A^a e^\alpha_a, \qquad A^\alpha n_\alpha = 0, \qquad A_a = A_\alpha e^\alpha_a \tag{B.3}$$

We want to relate the covariant derivative of $A^{\alpha\beta\cdots}$, defined with respect to a connection $\nabla^{(g)}$ compatible with $g_{\alpha\beta}$ to the covariant derivative of $A^{ab\cdots}$, defined in terms of a connection $\nabla^{(h)}$ compatible with h_{ab}.

We define the *intrinsic covariant derivative* of A_a to be the projection of $D^{(g)}_\beta A_\alpha$ onto the hypersurface:

$$A_{a|b} := \left(D^{(g)}_\beta A_\alpha\right) e^\alpha_a e^\beta_b \tag{B.4}$$

$A_{a|b}$ are the tangential components of the vector $\left(D^{(g)}_\beta A^\alpha\right) e^\beta_b$. We would now like to see whether this vector possesses also a normal component:

$$\begin{aligned}
& \left(D^{(g)}_\beta A^\alpha\right) e^\beta_b && \\
= \;& g^\alpha{}_\mu \left(D^{(g)}_\beta A^\mu\right) e^\beta_b && \\
= \;& (\epsilon n^\alpha n_\mu + h^{am} e^\alpha_a e_{m\mu}) \left(D^{(g)}_\beta A^\mu\right) e^\beta_b && \text{using (B.2)} \\
= \;& \epsilon \left(n_\mu \left(D^{(g)}_\beta A^\mu\right) e^\beta_b\right) n^\alpha && \text{normal} \\
& + h^{am} \left(\left(D^{(g)}_\beta A_\mu\right) e^\mu_m e^\beta_b\right) e^\alpha_a && + \text{ tangential part} \\
= \;& -\epsilon \left(\left(D^{(g)}_\beta n_\mu\right) A_\mu e^\beta_b\right) n^\alpha + h^{am} A_{m|b} e^\alpha_a && \textstyle\int \text{by parts \& using (B.4)} \\
= \;& A^a{}_{|b} e^\alpha_a - \epsilon A^a \left(\left(D^{(g)}_\beta n_\mu\right) e^\mu_a e^\beta_b\right) n^\alpha && \\
= \;& A^a{}_{|b} e^\alpha_a - \epsilon A^a K_{ab} n^\alpha &&
\end{aligned}$$

where we used a definition of the *second fundamental form* of the hypersurface Σ

$$K_{ab} \equiv \left(D^{(g)}_\beta n_\alpha\right) e^\alpha_a e^\beta_b \tag{B.5}$$

Thus, the normal components of $\left(D^{(g)}_{\beta} A^{\alpha}\right) e^{\beta}_{b}$ vanish iff K_{ab} vanishes.

Claim B.7. [24] $K_{ab} = \frac{1}{2}\left(\pounds_{n} g_{\alpha\beta}\right) e^{\alpha}_{a} e^{\beta}_{b}$

Proof. Note that applying the formula for the Lie derivative of an arbitrary tensor (Equation (A.21)) to the metric, we get:

$$\mathfrak{L}_X g_{\mu\nu} = X^{\sigma} D_{\sigma} g_{\mu\nu} + (D_{\mu} X^{\lambda}) g_{\lambda\nu} + (D_{\nu} X^{\lambda}) g_{\mu\lambda} = D_{\mu} V_{\nu} + D_{\nu} V_{\mu} \tag{B.6}$$

Using that K_{ab} is symmetric:

$$\begin{aligned} K_{ab} &= \frac{1}{2}\left(K_{ab} + K_{ba}\right) = \frac{1}{2}\left(D^{(g)}_{\beta} n_{\alpha} e^{\alpha}_{a} e^{\beta}_{b} + D^{(g)}_{\alpha} n_{\beta} e^{\beta}_{b} e^{\alpha}_{a}\right) \\ &= \frac{1}{2}\left(D^{(g)}_{\beta} n_{\alpha} + D^{(g)}_{\alpha} n_{\beta}\right) e^{\alpha}_{a} e^{\beta}_{b} = \frac{1}{2}\left(\mathfrak{L}_n g_{\alpha\beta}\right) e^{\alpha}_{a} e^{\beta}_{b} \end{aligned} \tag{B.7}$$

□

While h is concerned with the purely intrinsic aspects of a hypersurface's geometry, K is concerned with the extrinsic aspects, ie: the way in which the hypersurface is embedded in the enveloping spacetime manifold. Those tensors thus provide a complete characterization of the hypersurface.

B.4 The Gauss & Codazzi equations

We would now like to express the intrinsic Riemann tensor $R^{a}{}_{bcd}$ in terms of the Riemann tensor of the four-dimensional manifold, $R^{\gamma}{}_{\delta\alpha\beta}$, evaluated on Σ. This is partially done by the *Gauss & Codazzi equations* (See Equations (A.46) & (A.47) in Appendix A.10), which in local coordinates read:

$$R_{\alpha\beta\gamma\delta} e^{\alpha}_{a} e^{\beta}_{b} e^{\gamma}_{c} e^{\delta}_{d} = R_{abcd} + \epsilon\left(K_{ad} K_{bc} - K_{ac} K_{bd}\right) \tag{B.8}$$

$$R_{\mu\alpha\beta\gamma} n^{\mu} e^{\alpha}_{a} e^{\beta}_{b} e^{\gamma}_{c} = K_{ab|c} - K_{ac|b} \tag{B.9}$$

The missing components are $R_{\mu\alpha\nu\beta} n^{\mu} e^{\alpha}_{a} n^{\nu} e^{\beta}_{b}$, and these cannot be expressed solely in terms of h_{ab}, K_{ab}, and related quantities.

Remark B.8. The Mainardi equations make use of the Lie derivative of the extrinsic curvature, and other quantities to express those missing components, but they are not needed for the initial-value formulation [4]. △

We can use Equations (B.8) & (B.9) to rewrite $G \equiv \mathrm{Ric}^{(g)} - \frac{1}{2}R^{(g)}g$. A little algebra gives:

$$-2\epsilon G_{\alpha\beta}n^\alpha n^\beta = R^{(h)} + \epsilon\left(K^{ab}K_{ab} - (\mathrm{tr}_h K)^2\right) \tag{B.10}$$

$$G_{\alpha\beta}e^\alpha_a n^\beta = K^b{}_{a|b} - \partial_a(\mathrm{tr}_h K) \tag{B.11}$$

where $R^{(h)} = h^{ab}R^m{}_{amb}$ is the intrinsic Ricci scalar, and $(\mathrm{tr}_h K) = K^a{}_a = h^{ab}K_{ba}$ is the trace of K.

Finally, using Einstein's field equation to rewrite Equations (B.10) & (B.11) in terms of the energy-momentum tensor, we obtain the Einstein constraint equations:

$$R^{(h)} + (\mathrm{tr}_h K)^2 - K^{ab}K_{ab} = 16\pi\rho \tag{B.12}$$

$$D_b K^{ab} - D^a(\mathrm{tr}_h K) = 8\pi j^a \tag{B.13}$$

where we set $\epsilon = +1$ since the problem is formulated on a Cauchy surface, and where $D = D^{(h)}$ in the second line.

Appendix C

PDEs on Riemannian manifolds

To introduce definitions, we endow M with a *smooth* Riemannian metric e. We make use of the volume element μ_e when integrating over M. $\hookrightarrow$ denotes a continuous embedding. Constants are generically denoted by C, and only depend on their subscript.

C.1 Functional spaces

We first introduce the functional spaces of relevance for the study of existence and uniqueness of solutions to the constraint equations.

C.1.1 Differentiability class spaces $C^k(M, e)$

Definition C.1. We use $C^k(M, e)$ to denote the space of k-times continuously differentiable functions f, or tensor fields of some given type, with finite norm:

$$||f||_{C^k} = \sum_{|l| \leq k} |D^l f| \tag{C.1}$$

where $|\cdot|$ is the pointwise norm of tensors with the metric e. For example, for a continuous function $u : M \to \mathbb{R}$, $|u| = ||u||_{C(M,e)} = \sup_{x\in M}|u(x)|$, whereas for an (r,s)-tensor $|T| = \left(T^{a_1\ldots a_r b_1\ldots b_s}T_{a_1\ldots a_r b_1\ldots b_s}\right)^{1/2}$.

C.1.2 Hölder spaces $C^{k,\alpha}(M,e)$

Definition C.2. Let $f \in C^k(M,e)$. The α^{th}-Hölder semi-norm of f is

$$[f]_{C^{0,\alpha}(M,e)} := \sup_{x,y\in M, x\neq y}\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha}\right\} \tag{C.2}$$

$C^{k,\alpha}(M,e)$ is the space of k-times continuously differentiable functions f with finite Hölder semi-norm:

$$||f||_{C^{k,\alpha}} = \sum_{|l|\leq k}|D^l f|_{C(M,e)} + \sum_{|l|=k}[D^l f]_{C^{0,\alpha}} \tag{C.3}$$

where $|\cdot|$ is the pointwise norm of tensors with the metric e. The definition for tensors of some given type is similar.

Remark C.3. If (M,e) has finite volume, then $L^q \subset L^p$ for all $p \leq q$. Indeed by the Hölder inequality it holds that

$$||u||_{L^p} \leq (\mathrm{Vol}(M,e))^{1/q'}\,||u||_{L^q} \qquad \frac{1}{q}+\frac{1}{q'}=\frac{1}{p} \tag{C.4}$$

$\triangle$

Remark C.4. It follows from definitions that for positive integers k and l and for numbers α, $\beta \in (0,1)$ such that $k+\alpha \geq l+\beta$, one has that $f \in C^{k,\alpha}(M^n)$ implies $f \in C^{l,\beta}(M^n)$, and one has $||f||_{C^{l,\beta}} \leq ||f||_{C^{k,\alpha}}$. $\triangle$

Proposition C.5. *(Closedness under multiplication)* The Hölder spaces as well as the C^k spaces are closed under multiplication: $f,g \in C^{k,\alpha}(M,e)$ implies that $fg \in C^{k,\alpha}(M,e)$.

C.1.3 Sobolev spaces $W^p_s(M,e)$ and $H_s(M,e)$

Definition C.6. [9] The Sobolev space W^p_s is the space of functions, or tensor fields of some given type, with L^p integrable weak derivatives of order $\leq s$ in the metric e. It is a Banach space with norm:

$$||f||_{W^p_s} := \left(\int_M \sum_{0 \leq k \leq s} |D^k f|^p \mu_e \right)^{1/p} \qquad 1 \leq p < \infty \tag{C.5}$$

where $|\cdot|$ is the pointwise norm of tensors in the metric e.

For example [17], for a vector field X we have

$$\begin{aligned} ||X||_{W^p_s} \;\; := \;\; & \left\{ \int_M \left[\left(X^c X^d e_{cd} \right)^{p/2} + \left(D^a X^c D^b X^d e_{ab} e_{cd} \right)^{p/2} + \dots \right.\right. \\ & \left.\left. + \left(D^{a_1} \dots D^{a_s} D^{b_1} \dots D^{b_s} X^d e_{a_1 b_1} \dots e_{a_s b_s} e_{cd} \right)^{p/2} \right] \mu_e \right\} \end{aligned} \tag{C.6}$$

Remark C.7. [17] While this definition depends explicitly on the choice of the metric e, one readily verifies that for any pair of C^∞ metrics e and $\tilde{e}$, the corresponding norms $||\;||^{(e)}_{W^p_s}$ and $||\;||^{(\tilde{e})}_{W^p_s}$ are *compatible* in the sense that

- for a given vector field X, $||X||^{(e)}_{W^p_s}$ exists iff $||X||^{(\tilde{e})}_{W^p_s}$ exists as well,
- there exists a positive pair of constants c_1 and c_2 such that

$$c_1 ||X||^{(e)}_{W^p_s} \leq ||X||^{(\tilde{e})}_{W^p_s} \leq c_2 ||X||^{(e)}_{W^p_s} \tag{C.7}$$

for every vector field for which these norms exist. Therefore, for practical purposes (ie: making estimates), the norms are equivalent. △

Definition C.8. We define $H_s := W^2_s$, which is a *Hilbert* space, with inner product $(u,v) = \int_M u_{a_1 \dots a_m} v^{a_1 \dots a_m}$.

Definition C.9. $\bar{W}^p_s \subset W^p_s$ is the closure of $C^\infty(M)$, or tensors fields of some given type, with compact support in M, w.r.t. the norm (C.5).[9]

Remark C.10. It follows from definitions that for any manifold M, if $f \in W^p_k$, then $f \in W^p_l$ for $l \leq k$, and $||f||_{W^p_l} \leq ||f||_{W^p_k}$. △

Proposition C.11. If M is closed, let $k \geq 0$ be an integer, and $p \geq 1$. If $f \in W^p_k(M)$, then $f \in W^q_k(M)$ for any q s.t. $1 \leq q \leq p$. Further, $\exists\, C > 0$ (related to $\mathrm{Vol}(M, e)$) s.t. $\forall\, f \in W^p_k$, $||f||_{W^q_k} \leq C||f||_{W^p_k}$.

Proposition C.12. *(Schauder ring property)* Let M be closed, $\dim(M) = n$, and let $k' \geq k$, k, k' positive integers, and $k' > n/p$ for some $p \geq 1$. If $f \in W^p_k(M)$ and $W^p_{k'}(M)$, then $fg \in W^p_k(M)$.

C.2 Relations between function spaces

The following are very useful to make estimates.

C.2.1 W^p_s in L^r and C^k spaces

Proposition C.13. *(Embedding property)*

1. Let $U \subset \mathbb{R}^n$ be open. If $s > \frac{n}{p}$ then $W^p_s \hookrightarrow \bar{C}^0$ (continuous and bounded functions); and $\sup_M |u| \leq C_U ||u||_{W^p_s}$. Also $W^1_n \in \bar{C}^0$.
2. If $s = \frac{n}{p}$, then $W^p_s \hookrightarrow L^q\ \forall\, q$ s.t. $p \leq q < \infty$.
3. If $s < \frac{n}{p}$, then $W^p_s \hookrightarrow L^q\ \forall\, q$ s.t. $p \leq q \leq \frac{np}{n-sp}$

In cases 2 and 3, it holds that $||u||_{L^q} \leq C_U ||u||_{W^p_s}$.

Corollary C.14. An easy induction shows the continuous embeddings:

$$W^p_{s+m} \subset \bar{C}^m \qquad \text{if} \qquad s > \frac{n}{p} \tag{C.8}$$

$$W^p_{s+m} \subset W^q_m \qquad \text{if} \qquad s < \frac{n}{p},\ p \leq q \leq \frac{np}{n - sp} \tag{C.9}$$

Proposition C.15. Let $U \subset \mathbb{R}^n$ be open. W^p_s has the continuous *multiplication property*: if $s_1 + s_2 > s + \frac{n}{p}$ and $s_1, s_2 \geq s$, then

$$W^p_{s_1} \times W^p_{s_2} \to W^p_s \quad \text{by} \quad (u, v) \mapsto u \otimes v \tag{C.10}$$

$$\text{and} \qquad ||u \otimes v||_s \leq C_U ||u||_{W^p_{s_1}} ||u||_{W^p_{s_2}} \tag{C.11}$$

In particular, W^p_s is an algebra if $s > \frac{n}{p}$.

C.2.2 Sobolev and Hölder spaces

Theorem C.16. *(Sobolev embedding theorem)* [17] Let $n \geq 1$, $k \geq 0$ and $p \geq 1$. If $f \in W_k^p(M)$, then for any integer l and any number $\alpha \in (0,1)$ which satisfy $l + \alpha < k - n/p$ one has $f \in C^{l,\alpha}(M)$ and $f \in C^l(M)$. Further, $\exists\ C > 0$ s.t. for $f \in W_k^p(M)$, one has $||f||_{C^{l,\alpha}} \leq C||f||_{W_k^p}$.

Theorem C.17. *('converse' of Sobolev embedding theorem, on manifold with finite measure)* If for any integers $k \geq 0$ and any $\alpha \in (0,1)$, one has $f \in C^{k,\alpha}(M)$, then for any $p \geq 1$, one has $f \in H_k^p(M)$. Further, $\exists$ a constant $C > 0$ such that $\forall\ f \in C^{k+\alpha}(M)$, $||f||_{H_k^p} \leq C||f||_{C^{k,\alpha}}$.

C.3 Elliptic operators

[9] A linear differential operator of order m from sections u of a tensor bundle E over (M, e) (ie: u is a tensor field) into sections of another such bundle F reads

$$Lu \equiv \sum_{k=0}^{m} a_k D^k u \tag{C.12}$$

with a_k a linear map from tensor fields to tensor fields, given also by tensor fields over M. Note that this expression is not in local coordinates: in particular, D^k here means 'apply D k times'.

Definition C.18. [9] The *principal symbol* of the operator L at a point $p \in M$, for a dual vector $\xi \in T_p^*M$, is the linear map $\sigma(\xi)$ from E_p to F_p (tensor fields evaluated at p) determined by the contraction of a_m with $(\otimes\xi)^m$. The operator is said to be *elliptic* if for each $p \in M$ and $\xi \in T_p^*M$, its principal symbol is an isomorphism from E_p onto F_p for all $\xi \neq 0$.

Let e now be uniformly equivalent in each chart to the Euclidian metric, and M be compact.

Theorem C.19. [9] Let $Lu \equiv \sum_{k=0}^{2} a_k D^k u$ be elliptic, with coefficients s.t. $a_2 \in W_2^p$, $a_1 \in W_1^p$, and $a_0 \in L^p$, where $p > \frac{n}{2}$. Then L is a continuous mapping $W_2^q \to L^q$ for any $1 < q \leq p$.

Proof. • By Sobolev embedding theorem, $a_2 \in W_2^p \in C^{0,\alpha}$ with $0 < \alpha \leq 2 - \frac{n}{p}$, so that if $u \in W_2^q$, then $a_2 D^2 u \in L^q$.

• By Sobolev embedding theorem, if $p < n$, $q < n$, $p \leq p_1 \leq \frac{np}{n-p}$ and $q \leq q_1 \leq \frac{nq}{n-q}$, then $a_1 \in L^{p_1}$ and $Du \in L^{q_1}$. Then, by Hölder inequality, $a_1 Du \in L^r$ with

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} \geq \frac{n(p+q) - 2pq}{npq} =: s \tag{C.13}$$

Therefore, if $p \geq \frac{n}{2}$, then for any q, we have $a_1 Du \in L^q$ if $s \geq \frac{1}{q}$.

• Assume $a_0 \in L^p$ and $u \in W_2^q$, then $u \in L^{q_2}$, with

$$q_2 = +\infty \text{ if } q > \frac{n}{2} \qquad q_2 \text{ arb. large if } q_2 = \frac{n}{2} \qquad \frac{1}{q_2} = \frac{1}{q} - \frac{2}{n} \text{ if } q < \frac{n}{2}$$

So $a_0 u \in L^r$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q_2}$, hence $a_0 u \in L^q$ if $q \leq p$, $p > \frac{n}{2}$. □

Theorem C.20. [9] Under the hypotheses of the above theorem, it holds that

1. The operator $L : W_2^q \to L^q$ has finite dimensional kernel and closed range.

2. If L is injective on W_2^q, then there is a number C_L such that for each u in W_2^q the following inequality holds: $||u||_{W_2^q} \leq C_L ||Lu||_{L^q}$

3. If the formal adjoint L^* of L satisfies the same hypothesis as L and is injective, then L is surjective from W_2^q onto L^q, and hence is an isomorphism if also injective.

Proof. 1. Recall that a Banach space is finite dimensional if the unit sphere is a compact subset; or equivalently, if every sequence of elements of norm 1 has a convergent subsequence. Consider a sequence $u_n \in W_2^q$, with $||u_n||_{W_2^q} = 1$. The unit ball of W_2^q is compact in the W_1^q topology. The sequence u_n admits therefore a subsequence, still denoted u_n, which converges in the W_1^q topology. If

$u_n \in \mathrm{Ker}(L)\ \forall n$, it satisfies

$$||u_n - u_m||_{W_2^q} \leq C_L ||u_n - u_m||_{W_1^q} \tag{C.14}$$

(stated without proof) which shows that the subsequence converges also in the original topology of W_2^q. Hence $\dim(\mathrm{Ker}(L)) < \infty$.

Denote by E a vector space complementary to $\mathrm{Ker}(L)$: E is closed since $\mathrm{Ker}(L)$ is finite dimensional. Suppose there is no constant C s.t. $||u||_{W_2^q} \leq C||Lu||_{L^q}$ $\forall u \in E$. Then $\exists$ a sequence $\{u_n\} \subset E$ s.t. $||u_n||_{W_2^q} = 1$ and $||Lu_n||_{W_1^q}$ tends to 0; hence Lu_n admits a Cauchy subsequence Lu_n in W_1^q. The subsequence u_n is then a Cauchy sequence in W_2^q by the above inequality; it converges to some function $\bar{u} \in E$, with $||\bar{u}||_{W_2^q} = 1$. By continuity $L\bar{u} = 0$. The existence of $\bar{u}$ contradicts the hypothesis that E is complementary to the kernel of L. The inequality (C.14) shows that the range of L as mapping from E, equivalently as mapping from W_2^q, is a closed subspace of L^q.

2. This is a particular case of inequality (C.14) when $\mathrm{Ker}(L) = \{0\}$.

3. Let ϕ, $\psi \in C^\infty(M)$ be arbitrary. By definition: $\int_M L\psi\phi\mu_e = \int_M \psi L^*\phi\mu_e$. To show that the range of $L : W_2^q \to L^q$ is the entire space L^q, it is sufficient to show that if $\int_M L\psi\phi\mu_e = 0\ \forall\psi$, then $\phi \equiv 0$, because the closure of $L\psi$ in L^q, that is the range of Lu, $u \in W_2^q$, is then identical to L^q, since L^q is the dual of a space $L^{q'}$, and C^∞ is dense in $L^{q'}$ since M is compact. This is equivalent to the injectivity of L^*, which says that $L^*\phi = 0 \Rightarrow \phi = 0$. $\square$

C.3.1 Δ and Δ_{conf}

[17] In the conformal formulation, the constraint equations involve two differential operators: the Laplacian Δ and the conformal Laplacian Δ_{conf}, defined as:

$$\Delta f = D_a D^a f \qquad f \in C^\infty(M) \tag{C.15}$$

$$(\Delta_{\mathrm{conf}} W)^b = D_a (\mathfrak{L}_{\mathrm{conf}} W)^{ab} \qquad W \in \chi(M) \tag{C.16}$$

$$\text{where} \quad (\mathfrak{L}_{\mathrm{conf}} W)^{ab} = D^a W^b + D^b W^a - \frac{2}{n}\gamma^{ab} D_c W^c \tag{C.17}$$

If we work on a closed manifold Σ, both operators are elliptic, self-adjoint and second-order. We can therefore use Theorems C.19 & C.20:

Theorem C.21. [9] Suppose that $\gamma \in W_2^p$ is properly Riemannian, $p > \frac{3}{2}$, $q > 1$, $\frac{6}{5} \leq q \leq p$. Then:

1. The kernel of Δ_{conf} in the space of W_2^p vector fields is the space of W_2^q conformal Killing vector fields.

2. If (Σ, γ) admits no W_2^q conformal Killing vector field, then Δ_{conf} is an isomorphism from W_2^q onto L^q.

Proof. 1. On a compact manifold, integration by parts shows that the following identity holds for smooth vector fields X:

$$\int_M X_j \left(\Delta_{\text{conf}} X\right)^j \mu_\gamma \equiv \int_M X_j D_i \left(D^i X^j + D^j X^i - \frac{2}{3}\gamma^{ij} D_k X^k \right) \mu_\gamma$$

$$\equiv -\frac{1}{2} \int_M \left(D^i X^j + D^j X^i - \frac{2}{3}\gamma^{ij} D_k X^k \right) \left(D_i X_j + D_j X_i - \frac{2}{3}\gamma_{ij} D_l X^l \right) \mu_\gamma \tag{C.18}$$

Hence if $\Delta_{\text{conf}} X = 0$ for W_2^p vector fields,

$$D_i X_j + D_j X_i - \frac{2}{n}\gamma_{ij} D_l X^l = 0 \tag{C.19}$$

2. Since Δ_{conf} is self-adjoint, the results follow from Theorem C.20. $\square$

Proposition C.22. *(Existence, uniqueness and regularity)* [17] If (Σ, γ) has no conformal Killing fields (ie: $\text{Ker}(\Delta_{\text{conf}}) = \{0\}$) and $\gamma \in C^{k+2}$ with $k \geq 0$ (so the coefficients a^{ij}, a^i, a of Δ_{conf} are all C^k) then $\exists!$ solution to $\Delta_{\text{conf}} W = Z$ for any $Z \in W_k^p(\Sigma)$ so long as $p \geq 2$. Moreover $W \in W_{k+2}^p(\Sigma)$.

However, the hypothesis that (Σ, γ) has no conformal Killing fields can be relaxed. For smooth metrics, we have a similar existence theorem, and the estimate follows even in the presence of conformal Killing fields, so long as we take the RHS to be L^2 orthogonal to the subspace of conformal Killing fields. This is shown in [18], where they consider the model equation $D^a(\mathfrak{L}W)_{ab} = J_b$.

Lemma C.23. Let Σ be a closed manifold and let γ be a smooth Riemannian metric on Σ. If the 1-form field J_b is continuous on Σ and satisfies the condition $\int_\Sigma V^b J_b = 0$ for any conformal Killing field V of (Σ, γ), then there exists a solution W of the model equation, and every such solution satisfies $|LW(x)| \leq C \max_\Sigma |J|$.

Proof. The condition $\int_\Sigma V^b J_b = 0$ for any conformal Killing field V of (Σ, γ) is equivalent to saying that J_b is L^2-orthogonal to $\mathrm{Ker}(\Delta_{\mathrm{conf}})$. The existence result then follows from standard elliptic theory. □

Lastly, we use Theorems C.19 & C.20 to make statements about $L\psi = \rho$ where $L = a^{ij} D_i D_j + a^i D_i + a$ and ρ is some specified function or tensor field. Those are used in particular in Section 3.3

Proposition C.24. *(Sobolev norm of the solution)* [17] For any given a^{ij}, a^i, $a \in C^k(\Sigma)$ with $k \geq 0$, $\exists$ constants C_1 and C_2 (generally depending upon a^{ij}, a^i and a) s.t. $\forall\ q \in W^p_k(\Sigma)$ for which a solution exists, one has $||\psi||_{W^p_{k+2}} \leq C_1 ||q||_{W^p_k} + C_2 ||\psi||_{L^1}$. Moreover, if ψ is orthogonal to $\mathrm{Ker}(L)$ or if $\mathrm{Ker}(L) = \{0\}$, then one may replace the previous inequality by $||\psi||_{W^p_{k+2}} \leq C_3 ||q||_{W^p_k}$ for some constant C_3. Generally, C_1, C_2 and C_3 depend upon a^{ij}, a^i and a.

Proposition C.25. *(Hölder norm of the solution)* [17] For the Hölder case: For a^{ij}, a^i, $a \in C^k(\Sigma^3)$ with $k \geq 0$, there exist constants C_1 and C_2 such that for any $q \in C^{k,\alpha}(\Sigma^3)$ for which a solution exists (with $0 < \alpha < 1$), one has $||\psi||_{C^{k+2,\alpha}} \leq C_1 ||q||_{C^{k,\alpha}} + C_2 ||\psi||_{C^0}$. And, if ψ is orthogonal to $\mathrm{Ker}(L)$ or if $\mathrm{Ker}(L) = \{0\}$ then one has instead $||\psi||_{C^{k+2,\alpha}} \leq C_3 ||q||_{C^{k,\alpha}}$.

C.3.2 Maximum principles

The maximum principle is a standard result for elliptic operators. For practical purposes, we state here several versions, following [16]. Let $\psi \in C^2(M)$

Version 1: If ψ satisfies either $\Delta\psi \leq 0$ or $\Delta\psi \geq 0$, then ψ must be a constant. Hence there is no solutions to the equation $\Delta\psi = F(x,\psi)$ with $F(x,\psi) \geq 0$ or $F(x,\psi) \leq 0$, unless, in fact $F(x,\psi) = 0$.

Version 2: Let $f : M \to \mathbb{R}$ be a positive definite ($f(x) > 0$) function.

If $-\Delta\psi + f\psi \geq 0$ then $\psi \geq 0$. If $-\Delta\psi + f\psi \geq 0, \not\equiv 0$ then $\psi > 0$.

If $-\Delta\psi + f\psi \leq 0$ then $\psi \leq 0$. If $-\Delta\psi + f\psi \leq 0, \not\equiv 0$ then $\psi < 0$.

Version 3: Let μ and κ be positive constants. If $-\Delta\psi + \mu\psi \leq \kappa$, then $\psi(x) \leq \frac{\kappa}{\mu}$.

C.3.3 The sub and super solution theorem

One of the most important tools used to prove existence of solution to the Lichnerowicz equation is the sub and super solution method. We prove it here in detail, following [16].

Proposition C.26. Let Σ be closed manifold. Let $f : \Sigma \times \mathbb{R}^+ \to \mathbb{R}$ be in $C^1(\Sigma \times \mathbb{R})$. Assume that $\exists\ \phi_- : \Sigma \to \mathbb{R}^+$ and $\phi_- : \Sigma \to \mathbb{R}^+$ s.t. for $p > 3$:

i) $\phi_-,\ \phi_+ \in W_2^p(\Sigma)$ ii) $0 < \phi_-(x) \leq \phi_+(x) \quad \forall\ x \in \Sigma$

iii) $\nabla^2\phi_- \geq f(x,\phi_-)$ iv) $\nabla^2\phi_+ \leq f(x,\phi_+)$

Then $\exists$ a function $\phi : \Sigma \to \mathbb{R}^+$ s.t.:

(a) $\phi \in C^{2,\alpha}\ \forall\ \alpha \in (1 - \dfrac{3}{p})$ (b) $\phi_-(x) \leq \phi(x) \leq \phi_+(x)$ (c) $\nabla^2\phi = f(x,\phi)$

Proof. The basic idea of the proof of this result is to use ϕ_- and ϕ_+ to construct a sequence of functions $\{\phi_n\}$ and to show that this sequence converges to a solution of $\nabla^2\phi = f(x,\phi)$. First note that by Sobolev embedding theorem, if $h \in W_k^p(M^n)$ and if $k - n/p > m + \alpha,\ m \in \mathbb{N},\ \alpha \in (0,1)$, then $h \in C^{1,\alpha}(\Sigma)$ for $\alpha \in (0, 1 - 3/p)$.

Step 1: Consider $\frac{\partial}{\partial s}f(x,s)$ on the compact set $D := \Sigma \times [\min_\Sigma \phi_-, \max_\Sigma \phi_+]$. Let $\rho \in \mathbb{R}^+$ be s.t.: $\rho - \frac{\partial}{\partial s}f(x,s) > 0$ on D, and define $L := -\nabla^2 + \rho$ and

$F(x,s) := -f(x,s) + \rho s$ so that $\frac{\partial}{\partial s}F(x,s) > 0$ on D. Note that if $L\psi \geq 0$, then by the maximum principle $\psi \geq 0$ on Σ.

Step 2: Consider the sequence of PDE

$$L\phi_{n+1} = F(x,\phi_n) \qquad L\phi_1 = F(x,\phi_+) \tag{C.20}$$

Claim: The $\{\phi_n\}$ exists, in $W_3^p(\Sigma)$, and satisfies:

$$\phi_+(x) \geq \phi_1(x) \geq \phi_2(x) \geq \ldots \geq \phi_j(x) \geq \phi_{j+1}(x) \geq \ldots \geq \phi_-(x) \tag{C.21}$$

Proof of claim: Since L is self-adjoint, it has trivial kernel. Therefore, Equations (C.20) have unique solutions, thus defining the sequence $\{\phi_n\}$. Since $f \in C^1$ on D, so is F, and since $\phi_+ \in C^1$ on Σ, $F(x,\phi_+(x)) \in C^1$ on Σ. The compactness of Σ then implies that $F(x,\phi_+(x)) \in W_1^p(\Sigma)$. Hence we may show inductively that $\phi_n \in W_3^p(\Sigma)$. To show that $\phi_1(x) \leq \phi_+(x)$, we calculate

$$L(\phi_+ - \phi_1) \geq F(x,\phi_+) - F(x,\phi_+) = 0 \quad \overbrace{\Longrightarrow}^{\text{max. principle}} \quad \phi_+(x) - \phi_1(x) \geq 0 \tag{C.22}$$

Similarly

$$L(\phi_1 - \phi_2) = F(x,\phi_+) - F(x,\phi_1) \geq 0 \implies \phi_1(x) - \phi_2(x) \geq 0 \tag{C.23}$$

$$L(\phi_j - \phi_{j+1}) = F(x,\phi_{j-1}) - F(x,\phi_{j-2}) \geq 0 \implies \phi_j(x) - \phi_{j+1}(x) \geq 0 \tag{C.24}$$

We show that $\phi_j(x) \geq \phi_-(x)$ inductively. For $j = 1$:

$$L(\phi_1 - \phi_-) \geq F(x,\phi_+) - F(x,\phi_-) \geq 0 \implies \phi_1(x) \geq \phi_-(x) \tag{C.25}$$

Then if $\phi_{j-1}(x) \geq \phi_-(x)$

$$L(\phi_j - \phi_-) \geq F(x,\phi_{j-1}) - F(x,\phi_j) \geq 0 \implies \phi_j(x) \geq \phi_-(x) \tag{C.26}$$

q.e.d. Claim

Step 3: From standard elliptic estimates results, we have

$$||\phi_n||_{W_2^p} \leq a_1 ||L\phi_n||_{W_0^p} = a_1 ||F(x,\phi_{n-1})||_{W_0^p} \leq \zeta \tag{C.27}$$

for some constant a_1, and where in the last inequality, we used that $F \in C^1(D)$, so that ζ is a fixed constant, independent of n. By Sobolev embedding theorem: $||\phi_n||_{C^1} \leq a_2||\phi_n||_{W_2^p}$, for some constant a_2. Since we have a uniform bound on the first derivative of the functions in $\{\phi_n\}$, equicontinuity of the sequence follows. Since we previously showed that $\{\phi_n\}$ is bounded, it follows from Arzela-Ascoli theorem that $\{\phi_n\}$ has a uniformly converging (in C^0) subsequence. From the monotonicity of the sequence, we conclude that $\{\phi_n\}$ itself converges uniformly. Denote by ϕ_∞ the limit function. Then $\phi_-(x) \leq \phi_\infty(x) \leq \phi_+(x)$.

Step 4: It remains to show that $\phi_\infty(x)$ is a solution of $\nabla^2\phi = f(x, \phi)$, and is in $C^{2,\alpha}(\Sigma)$. We do this via a series of three bootstrap claims:

Claim A: $\phi_\infty \in W_2^p(\Sigma) \subset C^{1,\alpha}(\Sigma)$.

Proof of Claim A: For some positive constant a_3 and a_4, we have:

$$\begin{aligned} ||\phi_l - \phi_j||_{W_2^p} &\leq a_3||L(\phi_l - \phi_j||_{W_0^p} = a_3||F(x, \phi_{l-1}) - F(x, \phi_{j-1})||_{W_0^p} && \text{(C.28)} \\ &\leq a_4||F(x, \phi_{l-1}) - F(x, \phi_{j-1})||_{C^0} && \text{(C.29)} \end{aligned}$$

(Last inequality is because Σ is compact, and hence has finite measure. See Remark C.3.) Therefore, $\{\phi_n\}$ is a Cauchy sequence in W_2^p, which is a complete space. It follows that $\{\phi_n\}$ converges to some limit, which has to be the same as in C^0, namely $\phi_\infty(x)$. By Sobolev embedding theorem, since $2 - 3/p > 1$, $\phi_\infty \in C^{1,\alpha}$ for small $\alpha \in (0, 1 - 3/p)$. *q.e.d. Claim A*

Claim B: $\phi_\infty \in C^{2,\alpha}(\Sigma)$

Proof of Claim B: For some positive constant a_5, we have:

$$||\phi_l - \phi_j||_{C^{2,\alpha}} \leq a_5||L(\phi_l - \phi_j||_{C^{0,\alpha}} = a_5||F(x, \phi_{l-1}) - F(x, \phi_{j-1})||_{C^{0,\alpha}} \quad \text{(C.30)}$$

Since $F \in C^1$, and $\{\phi_n\}$ converges in C^1, $\{\phi_n\}$ is a Cauchy sequence in $C^{2,\alpha}$. Since $C^{2,\alpha}$ is complete, $\{\phi_n\}$ converges in $C^{2,\alpha}$, and $\phi_\infty \in C^{2,\alpha}$. *q.e.d. Claim B*

Claim C: ϕ_∞ is a solution of $\nabla^2\phi_\infty = f(x, \phi_\infty(x))$.

Proof of Claim C: Since ϕ_∞ is twice differentiable, all we need to verify is that ϕ_∞ is a weak solution of this equation. Consider the continuous map

$$(L - F) : W^p_{s+2}(\Sigma) \to W^p_s(\Sigma) \quad \psi \mapsto L\psi - F(x, \psi) \qquad s \leq 2 \tag{C.31}$$

Since $\{\phi_n\}$ converges to ϕ_∞ in $W^p_2(\Sigma)$, it follows that $\{L\phi_n - F(x, \phi_n)\}$ converges to $L\phi_\infty - F(x, \phi_\infty)$ in W^p_0. We now calculate

$$||L\phi_\infty - F(x, \phi_\infty)||_{W^p_0} = \lim_{n\to 0} ||L\phi_n - F(x, \phi_n)||_{W^p_0} \tag{C.32}$$

$$= \lim_{n\to 0} ||F(x, \phi_{n-1}) - F(x, \phi_n)||_{W^p_0} \tag{C.33}$$

$$\leq a_6 \lim_{n\to 0} ||F(x, \phi_{n-1}) - F(x, \phi_n)||_{C^0} = 0 \tag{C.34}$$

Hence ϕ_∞ is a weak, and therefore a strong solution of $\nabla^2\phi_\infty = f(x, \phi_\infty(x))$. *q.e.d. Claim C* $\square$

Appendix D

Conformal methods

D.1 Isometries

Definition D.1. [25] Let (M, g) be a pseudo-Riemannian manifold. A diffeomorphism $f : M \to M$ is an *isometry* if it preserves the metric

$$f^* g_{f(p)} = g_p \quad \Longleftrightarrow \quad g_{f(p)}(f_* X, f_* Y) = g_p(X, Y) \quad X, Y \in T_p M \tag{D.1}$$

or in local coordinates:

$$\frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(f(p)) = g_{\mu\nu}(p) \tag{D.2}$$

where x and y are the coordinates of p and $f(p)$ respectively.

Definition D.2. [25] Let (M, g) be a pseudo-Riemannian manifold. A diffeomorphism $f : M \to M$ is called a *conformal transformation* if it preserves the metric *up to a scale*,

$$f^* g_{f(p)} = e^{2\sigma} g_p \quad \Longleftrightarrow \quad g_{f(p)}(f_* X, f_* Y) = e^{2\sigma} g_p(X, Y) \tag{D.3}$$

where $\sigma \in C^\infty(M)$ and $X, Y \in T_p M$, or in components

$$\frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(f(p)) = e^{2\sigma(p)} g_{\mu\nu}(p) \tag{D.4}$$

Proposition D.3. [25] Let $f : M \to M$ be a conformal transformation on a Lorentzian manifold (M, g). Then $f_* : T_pM \to T_{f(p)}M$ preserves the local light cone structure, namely:

$$f_* = \begin{cases} \text{timelike vector} & \mapsto & \text{timelike vector} \\ \text{null vector} & \mapsto & \text{null vector} \\ \text{spacelike vector} & \mapsto & \text{spacelike vector} \end{cases} \tag{D.5}$$

Definition D.4. [25] Let g and $\tilde{g}$ be metrics on a manifold M. $\tilde{g}$ is said to be *conformally related* to g if

$$\tilde{g}_p = e^{2\sigma(p)} g_p \tag{D.6}$$

This defines an equivalence relation on the set of all metric on M. The equivalence class is called the *conformal structure*. The transformation $g \mapsto e^{2\sigma} g$ is called a *Weyl rescaling*.

Proposition D.5. [9] The difference of the connections of the metrics g and $\tilde{g}$ is a tensor, given by:

$$\tilde{\Gamma}^{\lambda}_{\beta\mu} - \Gamma^{\lambda}_{\beta\mu} = \delta^{\lambda}_{\mu}\partial_{\beta}\sigma + \delta^{\lambda}_{\beta}\partial_{\mu}\sigma - g_{\beta\mu}g^{\lambda\tau}\partial_{\tau}\sigma =: S^{\lambda}_{\beta\mu} \tag{D.7}$$

Recalling the expression for the Riemann tensor: [9]

$$R^{\lambda}_{\mu\alpha\beta} = \partial_{\alpha}\Gamma^{\lambda}_{\beta\mu} - \partial_{\beta}\Gamma^{\lambda}_{\alpha\mu} + \Gamma^{\lambda}_{\alpha\rho}\Gamma^{\rho}_{\beta\mu} - \Gamma^{\lambda}_{\beta\rho}\Gamma^{\rho}_{\alpha\mu} \tag{D.8}$$

we have that since the coefficients of the connection (for instance Γ) can always be made zero at one point,

$$\tilde{R}^{\lambda}_{\mu\alpha\beta} - R^{\lambda}_{\mu\alpha\beta} = D_{\alpha}S^{\lambda}_{\beta\mu} - D_{\beta}S^{\lambda}_{\alpha\mu} + S^{\lambda}_{\alpha\rho}S^{\rho}_{\beta\mu} - S^{\lambda}_{\beta\rho}S^{\rho}_{\alpha\mu} \tag{D.9}$$

From there, we can get the expression for the difference of the Ricci tensors:

$$\tilde{R}_{\beta\mu} - R_{\beta\mu} = -g_{\beta\mu}D^{\alpha}\partial_{\alpha}\sigma - (m-2)\nabla_{\mu}\partial_{\beta}\sigma \tag{D.10}$$

$$+(m-2)\left(\partial_{\mu}\sigma\partial_{\beta}\sigma - g_{\beta\mu}\partial^{\alpha}\sigma\partial_{\lambda}\sigma\right) \tag{D.11}$$

and the difference in the scalar curvatures is:

$$e^{2\sigma}\tilde{R} - R = -2(d-1)D^{\alpha}\partial_{\alpha}\sigma - (d-2)(d-1)\partial^{\lambda}\sigma\partial_{\lambda}\sigma \tag{D.12}$$

D.2 Killing vector fields

Let (M, g) be a Riemannian manifold and $X \in \chi(M)$.

Definition D.6. If a displacement εX, ε being infinitesimal, generates an isometry, the vector field X is called a *Killing vector field*. In components, $f : x^\mu \mapsto x^\mu + \varepsilon X^\mu$ satisfies:

$$\partial_\mu(x^\kappa + \varepsilon X^\kappa)\partial_\nu(x^\lambda + \varepsilon X^\lambda)g_{\kappa\lambda}(x + \varepsilon X) = g_{\mu\nu}(x) \tag{D.13}$$

From Equation (D.13), it can be deduced that $g_{\mu\nu}$ and X^μ satisfy the *Killing equation*:

$$X^\xi \partial_\xi g_{\mu\nu} + \partial_\mu(X^\kappa g_{\kappa\nu}) + \partial_\nu(X^\lambda g_{\mu\lambda}) = 0 \quad \Longleftrightarrow \quad (\mathfrak{L}_X g)_{\mu\nu} = 0 \tag{D.14}$$

Let $\phi_t : M \to M$ be the one-parameter group of transformations which generates the Killing vector field X. Equation (D.14) shows that the local geometry does not change as we move along ϕ_t. In this sense, a Killing vector field represents a direction of symmetry of the manifold.

D.3 Conformal Killing vector fields

Definition D.7. If a displacement εX, ε being infinitesimal, generates a conformal transformation, the vector field X is called a *conformal Killing vector field*. In components, $f : x^\mu \mapsto x^\mu + \varepsilon X^\mu$ satisfies:

$$\partial_\mu(x^\kappa + \varepsilon X^\kappa)\partial_\nu(x^\lambda + \varepsilon X^\lambda)g_{\kappa\lambda}(x + \varepsilon X) = e^{2\sigma} g_{\mu\nu}(x) \tag{D.15}$$

Or if we let $\sigma = \frac{\varepsilon\psi}{2}$, where ψ is a C^∞-function on M, then $g_{\mu\nu}$ and X^μ satisfy:

$$\mathfrak{L}_X g_{\mu\nu} = X^\xi \partial_\xi g_{\mu\nu} + \partial_\mu(X^\kappa g_{\kappa\nu}) + \partial_\nu(X^\lambda g_{\mu\lambda}) = \psi g_{\mu\nu} \tag{D.16}$$

which can be solved for ψ to give:

$$\psi = \frac{1}{m}\left(X^\xi g^{\mu\nu}\partial_\xi g_{\mu\nu} + 2\partial_\mu X^\mu\right) \tag{D.17}$$

It is possible to check that a vector is a conformal Killing vector field by computing its *conformal Lie derivative*, which is derived from the above as follows:

$$
\begin{aligned}
& X^\xi \partial_\xi g_{\mu\nu} + \partial_\mu (X^\kappa g_{\kappa\nu}) + \partial_\nu (X^\lambda g_{\mu\lambda}) - \psi g_{\mu\nu} \\
= \;& \frac{g_{\mu\nu} g^{\mu\nu}}{m} \left(X^\xi \partial_\xi g_{\mu\nu} + \partial_\mu (X^\kappa g_{\kappa\nu}) + \partial_\nu (X^\lambda g_{\mu\lambda}) - \psi g_{\mu\nu} \right) \\
= \;& \frac{g_{\mu\nu}}{m} \left(g^{\mu\nu} X^\xi \partial_\xi g_{\mu\nu} + g^{\mu\nu} \partial_\mu X_\nu + g^{\mu\nu} \partial_\nu X_\mu - \left(\frac{1}{m} \left(X^\rho g^{\alpha\beta} \partial_\rho g_{\alpha\beta} + 2\partial_\alpha X^\alpha \right) \right) g^{\mu\nu} g_{\mu\nu} \right) \\
= \;& \frac{g_{\mu\nu}}{m} \left(g^{\mu\nu} X^\xi \partial_\xi g_{\mu\nu} + g^{\mu\nu} \partial_\mu X_\nu + g^{\mu\nu} \partial_\nu X_\mu - X^\rho g^{\alpha\beta} \partial_\rho g_{\alpha\beta} - 2\partial_\alpha X^\alpha \right) \\
= \;& \frac{g_{\mu\nu}}{m} \left(g^{\mu\nu} \partial_\mu X_\nu + g^{\mu\nu} \partial_\nu X_\mu - 2\partial_\alpha X^\alpha \right) \\
= \;& \partial_\mu X_\nu + \partial_\nu X_\mu - \frac{2}{m} \, g_{\mu\nu} \partial_\alpha X^\alpha \\
= \;& D_\mu X_\nu + D_\nu X_\mu - \frac{2}{m} \, g_{\mu\nu} D_\alpha X^\alpha =: (\mathfrak{L}_{\mathrm{conf}} X)_{\mu\nu}
\end{aligned}
\tag{D.18}
$$

By construction, $(\mathfrak{L}_{\mathrm{conf}} X)_{\mu\nu} = 0$ iff X is a conformal Killing vector field.

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