Lecture 23: Sections 6.2, 7.1

Orthogonal sets

A set of vectors $\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is called orthogonal set if $\mathbf{u}_i \cdot \mathbf{u}_k = 0$ when $i \neq k$

If $\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors, then

- $\mathcal{U}$ is linearly independent
- so $\mathcal{U}$ is a basis for the subspace $H = \operatorname{Span} \mathcal{U}$
- and $\mathcal{U}$ is called an orthogonal basis for H
- any $\mathbf{x} \in H$ can be written as $\mathbf{x} = \alpha_1 \mathbf{u}_1 + \ldots + \alpha_p \mathbf{u}_p$ with

$$\alpha_k = \frac{\mathbf{u}_k \cdot \mathbf{x}}{\mathbf{u}_k \cdot \mathbf{u}_k} \qquad (k = 1, \ldots, p)$$

- If $\mathbf{u}_k \cdot \mathbf{u}_k = 1$ for $k = 1, \ldots, p$, then $\mathcal{U}$ is called an orthonormal set (or basis)

The orthogonal projection of $\mathbf{x}$ onto $\mathbf{u}$

$$\hat{\mathbf{x}} = \alpha \mathbf{u} \quad \text{with } \alpha = \frac{\mathbf{u} \cdot \mathbf{x}}{\mathbf{u} \cdot \mathbf{u}}, \qquad \text{or} \qquad \hat{\mathbf{x}} = \frac{\mathbf{u} \cdot \mathbf{x}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \mathbf{u} \frac{\mathbf{u}^T \mathbf{x}}{\mathbf{u}^T \mathbf{u}} = \frac{1}{\mathbf{u}^T \mathbf{u}} \mathbf{u} \mathbf{u}^T \mathbf{x}$$

Orthogonal matrices

Let $U = [\mathbf{u}_1 \dots \mathbf{u}_p]$

- $\mathcal{U} = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthonormal set $\quad\Leftrightarrow\quad U^T U = I$ (orthogonal matrix)
- $U^T U = I \quad\Leftrightarrow\quad U^{-1} = U^T$
- $U^T U = I \quad\Leftrightarrow\quad (U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

Spectral theorem for symmetric matrices

If A is real, symmetric, $n \times n$ matrix, then A has orthonormal set of n eigenvectors, i.e., $A = UDU^{-1}$ with diagonal matrix D and orthogonal matrix U.

- $A = A^T$, $A\mathbf{x} = \lambda\mathbf{x}$, $A\mathbf{y} = \mu\mathbf{y}$, $\lambda \neq \mu \quad\Rightarrow\quad \mathbf{x} \cdot \mathbf{y} = 0$
- A is orthogonally diagonalizable $\Leftrightarrow$ A is symmetric

Orthogonal diagonalization of symmetric matrices

Algorithm for orthogonally diagonalizing symmetric A:

- Solve the characteristic equation $\det(A - \lambda I) = 0$, to find the eigenvalues λ_k
- Find a basis $\mathcal{V}_k$ for the eigenspace $\mathrm{Nul}\,(A - \lambda_k I)$
- Orthonormalize $\mathcal{V}_k$, that is, modify $\mathcal{V}_k$ into an orthonormal basis $\mathcal{U}_k$
- Form the orthogonal matrix U from the vectors in $\mathcal{U}_1, \mathcal{U}_2, \ldots$, and form D from the eigenvalues
- $A = UDU^{-1} = UDU^T$

Spectral decomposition

$$A = UDU^T = [\mathbf{u}_1 \ldots \mathbf{u}_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} = [\lambda_1 \mathbf{u}_1 \ldots \lambda_n \mathbf{u}_n] \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

$$= \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \ldots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$$