# Lecture 22: Section 6.1

1. Inner product, length, angle, and distance
2. Orthogonality

# Inner product, length, angle, and distance

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the inner product of $\mathbf{u}$ and $\mathbf{v}$, (aka dot- or scalar prod.), is

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = u_1 v_1 + \ldots + u_n v_n$$

Properties:

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}, \qquad (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}, \qquad (\alpha \mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v})$
- $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0 \Leftrightarrow \mathbf{u} = 0$

Length, angle, and distance

- Length (norm, magnitude) of $\mathbf{u}$: $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + \ldots + u_n^2}$
- Angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$: $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta$
- Distance between $\mathbf{u}$ and $\mathbf{v}$: $\mathrm{dist}\,(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$

Normalization

- $\|\alpha \mathbf{u}\| = |\alpha| \, \|\mathbf{u}\|$
- If $\mathbf{u} \neq \mathbf{0}$, the length of $\frac{1}{\|\mathbf{u}\|}\mathbf{u}$ is $1$

# Orthogonality

If $\mathbf{u} \cdot \mathbf{v} = 0$ for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we say they are orthogonal, and write $\mathbf{u} \perp \mathbf{v}$

The Pythagorean Theorem: $\mathbf{u} \perp \mathbf{v} \Leftrightarrow \|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$

Let $H$ be a subspace of $\mathbb{R}^n$

- If $\mathbf{z} \perp \mathbf{v}$ for all $\mathbf{v} \in H$, then we say $\mathbf{z} \in \mathbb{R}^n$ is orthogonal to $H$, and write $\mathbf{z} \perp H$
- $H^\perp = \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} \perp H\}$ (the orth. complement of $H$, a subspace of $\mathbb{R}^n$)
- Suppose $H = \mathrm{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Then $\mathbf{z} \perp \mathbf{v}_1, \ldots \mathbf{z} \perp \mathbf{v}_p \Leftrightarrow \mathbf{z} \perp H$

## A fundamental theorem

For any matrix $A$, it holds that

$$(\mathrm{Row}\, A)^\perp = \mathrm{Nul}\, A \qquad \text{and} \qquad (\mathrm{Col}\, A)^\perp = \mathrm{Nul}\, A^T$$

Let $H = \mathrm{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, and $P = [\mathbf{v}_1, \ldots, \mathbf{v}_p]$. We have $H = \mathrm{Col}\, P$.

$$H^\perp = (\mathrm{Col}\, P)^\perp = \mathrm{Nul}\, P^T$$