Lecture 17 (Sections 3.1, 3.2)

1. Determinants
2. Row and column operations

2×2 and 3×3 determinants

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- $\det A = a_{11}a_{22} - a_{12}a_{21}$
- A is invertible $\Leftrightarrow \det A \neq 0$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- $\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$
- A is invertible $\Leftrightarrow \det A \neq 0$
- $\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$

$$A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \quad A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \quad A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$n \times n$ determinants

Definition

Let A be an $n \times n$ matrix with entries a_{ik}.

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{k=1}^{n} (-1)^{1+k} a_{1k} \det A_{1k}$$

Cofactor expansion

Set $C_{ik} = (-1)^{i+k} \det A_{ik}$. Then for any row i

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \ldots + a_{in} C_{in}$$

and for any column k

$$\det A = a_{1k} C_{1k} + a_{2k} C_{2k} + \ldots + a_{nk} C_{nk}$$

- If A is triangluar, then $\det A = a_{11} a_{22} \ldots a_{nn}$

Row and column operations

Effect of row operations on determinant

Let A be a square matrix.

- If B is obtained from A by a row replacement, then $\det B = \det A$
- If B is obtained from A by a row interchange, then $\det B = -\det A$
- If B is obtained from A by a row scaling with factor α, then $\det B = \alpha \cdot \det A$

- Let U be an echelon form of A, and let the number of row interchanges used to produce U from A was i. Then

$$\det A = (-1)^i \det U = (-1)^i (\text{product of the diagonal entries of } U)$$

- A is invertible $\Leftrightarrow \det A \neq 0$
- $\det A^T = \det A$