Lecture 6 (Section 1.7)

Span

Linear independence

Span

- $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is the collection of vectors

$$\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n]\mathbf{x} \qquad (x_1, \ldots, x_n \in \mathbb{R})$$

- Solutions of $A\mathbf{x} = \mathbf{0}$

$$\mathbf{x} = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \ldots + s_m\mathbf{u}_m \qquad (s_1, \ldots, s_m \in \mathbb{R})$$

Span

Theorem

$\mathbf{v}_m \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{m-1}\} \quad \Leftrightarrow \quad \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{m-1}, \mathbf{v}_m\} = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{m-1}\}$

- $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{m-1}, \mathbf{v}_m\} \neq \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{m-1}\}$ if and only if $\mathbf{v}_m \notin \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{m-1}\}$ or equivalently,

$$x_1\mathbf{v}_1 + \ldots + x_{m-1}\mathbf{v}_{m-1} = \mathbf{v}_m$$

  has no solution

Linear independence

Definition

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly independent if

$$x_1\mathbf{v}_1 + \ldots + x_m\mathbf{v}_m = \mathbf{0}$$

has only the trivial solution.

- $A = [\mathbf{v}_1 \ldots \mathbf{v}_m]$
- The columns of A are linearly independent $\Leftrightarrow$

  $$A\mathbf{x} = \mathbf{0}$$

  has only the trivial solution

Theorems

Theorem 7

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly dependent if and only if at least one of the vectors is a linear combination of the others.

Theorem 7'

Let A be a $k \times n$ matrix. The columns of A are linearly independent $\Leftrightarrow$ A has n pivot columns.

Theorem 8

Any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ in $\mathbb{R}^k$ is linearly dependent if $m > k$.

Theorem 9

If one of the vectors in $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is zero, then the set is linearly dependent.