Lecture 3 (Sections 1.3, 1.4)

Vector Equation

Vectors in $\mathbb{R}^n$

Linear Combinations

Matrix Equation

Vector Equation

$\mathbf{u}$ and $\mathbf{v}$ are (column) vectors (in $\mathbb{R}^2$), and c is a scalar (in $\mathbb{R}$)

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \qquad c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}, \qquad \mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Vector equation: Find numbers x_1 and x_2 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$$

For general case

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{k1} & a_{k2} & \ldots & a_{kn} & b_k \end{bmatrix}, \qquad \mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \ldots \\ a_{k1} \end{bmatrix}, \ldots, \mathbf{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \ldots \\ a_{kn} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_k \end{bmatrix}$$

Find scalars $x_1, x_2, \ldots, x_n$ such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

Vectors in $\mathbb{R}^n$

Properties of vector addition

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Properties of scalar multiplication

- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = cd\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

Linear Combinations

- $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^k$ vectors, $c_1, c_2, \ldots, c_n \in \mathbb{R}$ scalars
- Linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$:

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

  $c_1, c_2, \ldots, c_n$ are the weights
- The subset of $\mathbb{R}^k$ spanned (or generated) by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is the set of all possible linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. It is denoted by $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

Linear Combinations

The followings are equivalent

- Whether weights $x_1, x_2, \ldots, x_n$ exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

- Whether $\mathbf{b}$ is in $\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$
- Whether the linear system with the augmented matrix

$$[\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \ldots \;\; \mathbf{a}_n \;\; \mathbf{b}]$$

  is consistent

The followings are equivalent

- Find all possible weights $x_1, x_2, \ldots, x_n$ such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

- Solve the linear system whose augmented matrix is

$$[\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \ldots \;\; \mathbf{a}_n \;\; \mathbf{b}]$$

Matrix Equation

- Let A be the matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$
- Define the matrix-vector product as

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$$

- Then, $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$ can be written as

$$A\mathbf{x} = \mathbf{b}$$

- The unknown is $\mathbf{x}$, which is a vector